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On Computing Sum of Roots with Positive Real Parts of Polynomials

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Abstract

In this paper we present a method to compute or estimate the sum of roots with positive real parts (SORPRP) of a polynomial, which is related to a certain index of stability in optimal control, without computing numerical values of the roots explicitly. The method is based on symbolic computations and enables us to deal with polynomials with parametric coefficients for their SORPRP. This leads to provide a novel systematic method to achieve optimal regulator design in control by combining with quantifier elimination. We show some experimental result for a typical class of plants to confirm the effectiveness of the proposed method.

1 Introduction

In control and system theory, investigating location of roots of the characteristic polynomial is one of important and fundamental topics related to the stability of feedback control systems. For example, in case of a typical feedback system with a plant $P(s) = \frac{n_q(s)}{d_q(s)}$ controlled by a controller $C(s) = \frac{n_c(s)}{d_c(s)}$ where $n_q(s), d_q(s), n_c(s), d_c(s) \in \mathbb{Q}[s]$, the stability of the system is described as follows: The feedback system is stable if and only if all of the roots of the *closed-loop characteristic polynomial* $g(s) = n_p n_c + d_p d_p$ locate within the left-half plane of the Gaussian plane. This is called *Hurwitz stability*. We may consider more general notion of stability, called $\mathcal{D}$-stability, which implies that all of the roots locate inside a restricted region $\mathcal{D}$ within the left-half plane of the Gaussian plane.

Control design problem is to find a controller $C(s)$ so that the system satisfies given specifications. As the controller $C(s, \mathbf{q})$ has fixed-structure with some parameters $\mathbf{q}$, what we have to do is to seek feasible controller parameters $\mathbf{q}$ which satisfies the specifications. For such problems, techniques in computer

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algebra have been successfully applied [9, 13, 1, 2]. Stability is the first necessary requirement for control system design. Assigning roots of a certain polynomial within a desired region is an essential problem for stability study. Root assignment problem for Hurwitz stability is to find controller parameters $\mathbf{q}$ so that the system is Hurwitz stable. This is easily verified by the well-known Routh-Hurwitz criterion. In the case of $\mathcal{D}$-stability, a wedge shape region or a circle is usually used as stability region $\mathcal{D}$. For root assignment problems with such stability regions, controller design problem is reduced to check a sign definite condition (SDC) $\forall z > 0,\ f(z) > 0$ where $f(z) \in \mathbb{Q}(\mathbf{q})[z]$, see [14, 12]. Applying real quantifier elimination (QE) to the sign definite condition, we can obtain possible regions of controller parameters $\mathbf{q}$ to meet $\mathcal{D}$-stability. For a sign definite condition we can utilize an efficient quantifier elimination algorithm specialized to SDC [1, 10]. These two controller synthesis methods with respect to stability are implemented as functions in a MATLAB toolbox for robust parametric control [3]

In this paper we focus on the *sum of roots with positive real parts* (SORPRP) of a given even polynomial, and provide another successful application of computer algebra to control design problem, where the SORPRP is related to certain index of stability in optimal control. We call the index "stability index". Here we compute or estimate the SORPRP without computing explicit numerical values of roots. Hence, we can handle polynomials with parametric coefficients for their SORPRP.

The key point of the method is that computing SORPRP is reduced to computation of the maximal real root of another univariate polynomial. Subsequently this enables us to achieve control system design with respect to SORPRP systematically. In fact, since the actual control design problems treated are recast as simple conditions on an univariate polynomial with parametric coefficients (one of them is a sign definite condition), we can utilize an efficient quantifier elimination algorithm using Sturm-Habicht sequence [1, 10]. The proposed method is applied to an even polynomial derived from "Linear Quadratic Regulator (LQR) problem" which is one of the main concerns in control theory.

2 SORPRP of even polynomials

First we consider an even polynomial $f(x)$ of degree $2m$ in $\mathbb{Q}[x]$ with non zero constant, and let $\alpha_1, \ldots, \alpha_m$ be roots of $f(x)$ with positive real parts and $\alpha_{m+1}, \ldots, \alpha_{2m}$ roots with negative real parts. We set $\Omega = \{\alpha_1, \ldots, \alpha_{2m}\}$. So,

$$f(x) = a_{2m}x^{2m} + a_{2m-2}x^{2m-2} + \cdots + a_2x^2 + a_0 \ = \ a_{2m}\prod_{i=1}^{2m}(x - \alpha_i),$$

where $a_{2k} \in \mathbb{Q}$ for $0 \le k \le m$, $a_{2m} \ne 0$ and $a_0 \ne 0$. Our first target is to compute $W = \alpha_1 + \ldots + \alpha_m$ without computing all α_i's. For simplicity, we call W the *SORPRP* of f. Since, for each non real root of $f(x)$, its complex conjugate has the same real part, we have the following:

Lemma 1
W is a real number.

2.1 Polynomial having SORPRP as its root

Let $B_{i_1,\ldots,i_m} = \alpha_{i_1} + \cdots + \alpha_{i_m}$ for $i_1 < \ldots < i_m$, and $\mathcal{B}$ the set of all distinct values of $B_{i_1,\ldots,i_m}$.

Definition 2
Gathering all sums $B_{i_1,\ldots,i_m}$, we can construct a polynomial $R_{m,f}(z)$ and its square-free part $\bar{R}_{m,f}(z)$, where z is a new variable:

$$R_{m,f}(z) = \prod_{i_1<\ldots<i_m} (z - B_{i_1,\ldots,i_m}), \quad \bar{R}_{m,f}(z) = \prod_{B\in\mathcal{B}} (z - B).$$

As there might be a case where $B_{i_1,\ldots,i_m}$ coincides with $B_{j_1,\ldots,j_m}$ for distinct $(i_1,\ldots,i_m)$ and $(j_1,\ldots,j_m)$, the square-free part $\bar{R}_{m,f}(z)$ might be smaller than $R_{m,f}(z)$. Since all $B_{i_1,\ldots,i_m}$ are algebraic number, it follows that $R_{m,f}(z)\in\mathbb{Q}[y]$ and so $\bar{R}_{m,f}(z)\in\mathbb{Q}[y]$. We may call $R_{m,f}(z)$ and $\bar{R}_{m,f}(z)$ the characteristic polynomial of sums of m roots, and the minimal polynomial of sums of m roots, respectively.

It is obvious that the SORPRP $W = \alpha_1 + \cdots + \alpha_m$ of $f(x)$ coincides with the maximal real root of $\bar{R}_{m,f}(z)$ $(R_{m,f}(z))$, since W is a real number. To compute $\bar{R}_{m,f}(z)$ and $R_{m,f}(z)$, we use the following triangular set related to *Cauchy moduli* [5] defined by $f(x)$.

Definition 3
Let D be an arbitrary computable integral domain and K its quotient field. For a polynomial $g(x)$ of degree n in $D[x]$, we define the following polynomials: $\{g_1(x_1), g_2(x_1,x_2),\ldots,g_n(x_1,\ldots,x_n)\}$, where $g_1(x_1) = g(x_1)$ and $g_i(x_1,\ldots,x_i)$ is the quotient of $g(x_i)$ divided by $(x_i - x_1)\cdots(x_i - x_{i-1})$ for each $i>1$. We note that $g_i(x_1,\ldots,x_i)\in D[x_1,\ldots,x_i]$ and $g_i(x_1,\ldots,x_i)$ coincides with the quotient of $g_{i-1}(x_1,\ldots,x_{i-2},x_i)$ divided by $x_i - x_{i-1}$. Here we call $\{g_1,\ldots,g_n\}$ the standard triangular set defined by $g(x)$, and also call $\{g_1,\ldots,g_k\}$ the k-th standard triangular set defined by $g(x)$.

It is well-known that $\{g_1,\ldots,g_k\}$ forms a Gröbner basis of the ideal $\langle g_1,\ldots,g_k\rangle$ generated by itself with respect to the lexicographic order $x_1<\ldots<x_k$ in $K[x_1,\ldots,x_k]$ and the set of all its zeros with multiplicities counted coincides with the set $\{(\beta_{i_1},\ldots,\beta_{i_k}) \mid i_1,\ldots,i_k\in\{1,\ldots,n\}$ are distinct to each other $\}$, where $\beta_1,\ldots,\beta_n$ are all roots of $g(x)$ in the algebraic closure of K. Thus, when $g(x)$ is square-free, $\langle g_1,\ldots,g_k\rangle$ is a radical ideal. We note that for each g_i its leading coefficient $lc(g_i)$ with respect to the order $<$ coincides with the leading coefficient $lc(g)$ of $g(x)$. Now let $\mathcal{F} = \{f_1(x_1),\ldots,f_m(x_1,\ldots,x_m)\}$ be the m-th standard triangular set defined by $f(x)$ in $\mathbb{Q}[x_1,\ldots,x_m]$. $R_{m,f}(z)$ can be computed by successive resultant computation and $\bar{R}_{m,f}(z)$ can be computed as the minimal polynomial of $z = x_1+\cdots+x_m$ modulo the ideal $\mathcal{I} = \langle\mathcal{F}\rangle$ (with square-free computation if necessary).

Computation of $R_{m,f}(z)$ via resultant Let $T_m(z) = z - (x_1+\cdots+x_m)$ and for each $k\le m$, we define T_k successively as follows:

$$T_{k-1}(z,x_1,\ldots,x_{k-1}) = res_{x_k}(f_k(x_1,\ldots,x_k), T_k(z,x_1,\ldots,x_k)),$$

where res_z means the resultant with respect to a variable z. Then $T_0(z)\in\mathbb{Q}[y]$ and $T_0(z)$ coincides with $a_{2m}^d R_{m,f}(z)$ for some positive integer d. This can be shown as follows: By construction of Sylvester matrices in resultant computation, it follows that the leading coefficient of T_i with respect to x_j, where T_i is considered as a univariate polynomial in x_j, is some powers of a_{2m} for each $j<i$, and the same for the leading coefficient of T_i with respect to z. Then, by the property of resultant, we have

$$\begin{aligned} T_0(z) &= a_{2m}^{d_1}\prod_{i=1}^{2m} T_1(z,\alpha_i), \\ T_1(z,\alpha_{i_1}) &= a_{2m}^{d_2}\prod_{i_2\neq i_1} T_2(z,\alpha_{i_1},\alpha_{i_2}), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} T_{m-1}(z,\alpha_{i_1},\ldots,\alpha_{i_{m-1}}) &= a_{2m}^{d_m} \prod_{i_m \neq i_1,\ldots,i_{m-1}} T_m(z,\alpha_{i_1},\ldots,\alpha_{i_m}), \\ T_m(z,\alpha_{i_1},\ldots,\alpha_{i_m}) &= z-(\alpha_{i_1}+\cdots+\alpha_{i_m}) \end{aligned}$$

where $i_1,\ldots,i_m$ are distinct to each other and each d_i is a positive integer. (See [8].) When $f(x) \in \mathbb{Z}[x]$, that is, all a_{2k} are integers, $T_0(z)$ belongs to $\mathbb{Z}[z]$, In order to avoid "coefficient growth" in resultant computation, we may apply factorization technique to each T_k or its factors for computing smaller factors of T_0. (See §4.2 for usage of factors.) We note that *multi-polynomial resultant* can be also applied for computing $T_0(z)$.

Computation of $\bar{R}_{m,f}(z)$ via minimal polynomial Let $z = x_1 + \cdots + x_m$ and $\mathcal{I} = \langle \mathcal{F} \rangle$ in $\mathbb{Q}[x_1,\ldots,x_m]$. Then, we consider a minimal polynomial $M(z)$ of z modulo $\mathcal{I}$, that is, $M(z)$ has the smallest degree among all polynomials $h(z)$ in $\mathbb{Q}[z]$ such that $h(x_1+\cdots+x_m)$ belongs to the ideal $\mathcal{I}$. Since the set of all zeros of $\mathcal{I}$ with multiplicities counted is $\{(\alpha_{i_1},\ldots,\alpha_{i_m}) \mid i_1,\ldots,i_m \in \{1,\ldots,2m\}$ are distinct to each other $\}$, it can be shown easily that $M(z)$ is a factor of $R_{m,f}(z)$ and has $\bar{R}_{m,f}(z)$ as its factor. (We may say that $M(z)$ stands between $R_{m,f}(z)$ and $\bar{R}_{m,f}(z)$.) Especially, when $f(x)$ is square-free, then $M(z)/lc(M(z))$ coincides with $\bar{R}_{m,f}(z)$. When $f(x) \in \mathbb{Z}[x]$, that is, all a_{2k} are integers, by removing denominators of coefficients appearing in $M(z)$, we may assume that $M(z)$ belongs to $\mathbb{Z}[z]$. Then the leading coefficient $lc(M)$ divides some power of a_{2m}, as $M(z)$ divides $T_0(z)$. As we already know the Gröbner basis $\{f_1,\ldots,f_m\}$ of $\mathcal{I}$, $M(z)$ can be computed rather easily.

2.2 Parametric case

Now we consider the case where each coefficient a_{2k} is some polynomial in parameters $\mathbf{p} = \{p_1,\ldots,p_t\}$. Thus, the even polynomial $f(x)$ is considered as a multivariate polynomial $f(x,\mathbf{p})$ in $\mathbb{Q}[x,\mathbf{p}]$. Setting $D = \mathbb{Q}[\mathbf{p}]$ and $K = \mathbb{Q}(\mathbf{p})$, we can compute the m-th standard triangular set

$$\mathcal{F} = \{f_1(x_1,\mathbf{p}),\ldots,f_m(x_1,\ldots,x_m,\mathbf{p})\}$$

in $D[x_1,\ldots,x_m]$. Then, as $lc(f_i) = a_{2m}(\mathbf{p})$ for each i, $\bar{\mathcal{F}} = \{f_1/a_{2m},\ldots,f_m/a_{2m}\}$ is the reduced Gröbner basis of $\langle \mathcal{F} \rangle$ in $K[x_1,\ldots,x_m]$. By $\mathcal{F}$, we can compute $T_0(z,\mathbf{p})$ by successive resultant computation and $M(z,\mathbf{p})$ as a minimal polynomial of z modulo the ideal $\langle \mathcal{F} \rangle$ in $K[x_1,\ldots,x_m]$. We note that using a block order $\{x_m > \ldots > x_1\} >> z$, $M(z,\mathbf{p})$ is found in a Gröbner basis of $\langle \mathcal{F} \cup \{z-(x_1+\cdots+x_m)\} \rangle$ in $K[x_1,\ldots,x_m,z]$. Then $T_0(z,\mathbf{p})$ belongs to $\mathbb{Q}[y,\mathbf{p}]$, and by removing denominators, we may assume that $M(z,\mathbf{p})$ also belongs to $\mathbb{Q}[y,\mathbf{p}]$. As $\bar{\mathcal{F}}$ is the reduced Gröbner basis of $\langle \mathcal{F} \rangle$ and the denominator coincides with $a_{2m}(\mathbf{p})$, the following holds. (See Exercises of Chapter 6.3 in [7].)

Theorem 4
For each $(c_1,\ldots,c_t) \in \mathbb{Q}^t$, consider the polynomial $f_c(z)$ obtained from $f(x,\mathbf{p})$ by substituting the parameters $(p_1,\ldots,p_t)$ with $(c_1,\ldots,c_t)$. If the leading coefficient $a_{2m}(c_1,\ldots,c_t)$ does not vanish, then $T_0(z,c_1,\ldots,c_t)$ coincides with $cR_{m,f_c}(z)$ for some non-zero constant c in $\mathbb{Q}$, and $M(z,c_1,\ldots,c_t)$ is a factor of $R_{m,f_c}(z)$ and has $\bar{R}_{m,f_c}(z)$ as its factor in $\mathbb{Q}[z]$.

By Theorem 4, we can handle the SORPRPs for polynomials with parametric coefficients. For the total computational efficiency, computing $M(z,\mathbf{p})$ is much better than computing $T_0(z,\mathbf{p})$ in many cases.

3 Formulation of Basic Problem

Here we explain the fundamental problem in this paper. We denote the polynomial obtained above ($T_0(z, \mathbf{p})$ or $M(z, \mathbf{p})$) by $\mathcal{R}(z)$. What we do after obtaining $\mathcal{R}(z)$ is the following:

Problem 1

Given a polynomial $\mathcal{R}(z)$ involving parameters $\mathbf{p}$ in coefficients, $\mathcal{R}(z) \in \mathbb{Q}(\mathbf{p})[z]$ and $M_1, M_2 \in \mathbb{Q}$ ($M_1 > M_2$). Then find feasible ranges of parameters $\mathbf{p}$ so that the maximal real root W of $\mathcal{R}(z)$ satisfies the following each requirement: (a) $W < M_1$, (b) $W > M_2$, and (c) $M_2 < W < M_1$. Here we exclude ranges where the leading coefficient of $\mathcal{R}(z)$ or its constant term vanishes.

In view of control theory the parameters $\mathbf{p}$ usually comes from controller or plant parameters of the control system to be designed, and the above three requirements are originated from control design specifications in terms of SORPRP. PROBLEM 1 is resolved by using quantifier elimination over the real closed field. Actually all of the requirements are reduced to simple first-order formulas for $\mathcal{R}(z) \in \mathbb{Q}(\mathbf{p})[z]$ as follows:

(a) $W < M_1$: This requirement is equivalent to the first-order formula $\forall z > M_1,\ \mathcal{R}(z) \neq 0$. This is so called a *sign definite condition* [1], hence we can solve it by an efficient quantifier elimination algorithm using Sturm-Habicht sequence [11, 6].

(b) $W > M_2$: This requirement is equivalent to the first-order formula: $\exists z > M_2,\ \mathcal{R}(z) = 0$. We can also solve it by an efficient quantifier elimination algorithm using Sturm-Habicht sequence [10].

(c) $M_2 < W < M_1$: This requirement is equivalent to the conjunction of (a) and (b), that is, $(\forall z > M_1, \mathcal{R}(z) \neq 0) \wedge (\exists z > M_2, \mathcal{R}(z) = 0)$. Hence, this is achieved by superposing both quantifier-free formulas obtained by performing quantifier elimination for (a) and (b).

4 LQR problem - control application

We here consider a typical optimal control problem named *Linear Quadratic Regulator (LQR) problem.* We will first briefly explain the problem in §5.1 and show some computational examples, by which we can confirm the effectiveness of our proposed method[1)].

Here we briefly explain about *Linear Quadratic Regulator (LQR) problem* (see [18] for more details) and introduce our target polynomial of which we want to estimate the SORPRP

Let us consider a linear time-invariant SISO (single-input single-output) system represented by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \\ y(t) &= cx(t), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^m$ is the state variable, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the output, $A \in \mathbb{R}^{m \times m}$ is the system matrix, $b \in \mathbb{R}^m$ is the input matrix, and $c^T \in \mathbb{R}^m$ is the output matrix. Then the LQR problem is to find a control input u which minimizes the cost function

$$J = \int_0^\infty (qy^2(t) + ru^2(t))dt, \tag{2}$$

1) All computations except quantifier elimination are done by using a computer algebra system *Risa/Asir*, see `http://www.math.kobe-u.ac.jp/Asir/asir.html`). All QE computations in this paper were carried out by QEPCAD, see `http://www.cs.usna.edu/ qepcad/B/QEPCAD.html`, since QEPCAD succeeded in achieving all of QE computations for our examples in a very small amount of time. For the larger sized problems, we may use an efficient QE algorithm based on Sturm-Habicht sequence [1, 10]. Some types of QE methods using Sturm-Habicht sequence are available in a Maple package SyNRAC [4, 17].

where $q > 0$ and $r > 0$ are called weights. If we take the lager value of q, we can get the faster response in general. On the other hands, the lager value of r is required when we have a severe restriction on the value of u, since r reflects the penalty on $u(t)$. Note that the ratio q/r plays an essential role for finding the optimal control input and determines the closed-loop poles.

Actually, it is well-known that the optimal closed-loop poles are determined by the corresponding polynomial given by

$$\varphi(s) = r \cdot d(s)d(-s) + q \cdot n(s)n(-s), \tag{3}$$

where $d(s)$ and $n(s)$ are the denominator and numerator of the transfer function of the plant (1) represented by $P(s) = c(sI - A)^{-1}b$. In other words, $P(s) = \frac{n(s)}{d(n)}$, where $d(s) := det(sI - A)$, $n(s) := c\, adj(sI - A)\, b$. Note that $deg(d(s)) = m$, $deg(n(s)) < m$ hold.

The polynomial $\varphi(s)$ is our target polynomial with $deg(\varphi(s)) = 2m$ and it is an even polynomial. It is strongly desired to establish a guiding principle to choose appropriate values of r and q or the ratio q/r, since the closed-loop poles are all the poles of $\varphi(s)$ which has negative real parts.

In the sequel we carry out an investigation of the weights r and q in terms of stability index, that is, the sum of roots with *negative* real parts (SORNRP) of $\varphi(s)$. We can attain this by just applying our method for SORPRP shown in the previous sections to $\mathcal{R}(-z)$, where the polynomial $\mathcal{R}(z)$ has SORPRP of $\varphi(s)$ as its root. Because, as $\psi(s)$ is even, the value of SORPRP coincides with the absolute value of SORPNP, and $R(-z)$ also has $-1\times$SORPNP as its maximal real root.

Particularly we study some behaviors of a parameter involving in the plant $P(s)$ and feasible bounds for SORPRP W versus the ratio of weights q/r or q with $r = 1$ under the specifications in §4. This kind of investigations is important in practice to see control performance limitations, since the stability index is one of appropriate measures for the quickness of feedback control systems.

4.1 A sample plant: 2nd-order system with time delay

Here we study the LQR problem for a class of typical second-order systems with time delay given by

$$P(s) = \frac{\omega_n^2 k e^{-Ls}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \simeq \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{1 - \frac{1}{2}Ls}{1 + \frac{1}{2}Ls},$$

where the exponential e^{-Ls} is transformed to a rational function by the Padé approximation. We consider the case where $k = 1$, $\zeta = 0.1$, $\omega_n = 30$ (kHz), and $r = 1$. Here, initially we assume that $L > 0$, $r, q > 0$.

Then the target even polynomial is expressed as

$$\begin{aligned} \varphi(s; q, L) &= d(s)d(-s) + q \cdot n(s)n(-s) \\ &= -25L^2s^6 + (-49L^2 + 100)s^4 + ((-25q - 25)L^2 + 196)s^2 + 100q + 100 \end{aligned}$$

We remark that the leading coefficient $-25L^2$ of $\varphi(s)$ never vanish as $L > 0$, and the constant term $100q + 100$ also never vanish as $q > 0$.

Let $\mathcal{I}_3$ be the ideal generated by the 3rd standard triangular set of $\varphi(s; q, L)$; $\{\varphi(x_1; q, L), \varphi_1(x_1, x_2; q, L), \varphi_2(x_1, x_2, x_3; q, L)\}$, where $\varphi_1(x_1, x_2; q, L)$ is the quotient of $\varphi(x_2, q, L)$ divided by $x_2 - x_1$ and $\varphi_2(x_1, x_2; x_3; q, L)$ is the quotient of $\varphi_1(x_3, q, L)$ divided by $(x_3 - x_1)(x_3 - x_2)$. Then we can obtained the following minimal polynomial in z of $x_1 + x_2 + x_3$ with respect to $\mathcal{I}_3$ immediately:

$$\mathcal{R}(z; q, L) = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3\mathcal{R}_4\mathcal{R}_5$$

where

$$
\begin{aligned}
\mathcal{R}_1 &= Lz + 2,\\
\mathcal{R}_2 &= Lz - 2,\\
\mathcal{R}_3 &= 625L^4z^4 - 5000L^3z^3 + (2450L^4 + 15000L^2)z^2 + (-9800L^3\\
&\quad -20000L)z + (-2500rqk2 - 99)L^4 + 9800L^2 + 10000,\\
\mathcal{R}_4 &= 625L^4z^4 + 5000L^3z^3 + (2450L^4 + 15000L^2)z^2 + (9800L^3\\
&\quad +20000L)z + (-2500rqk2 - 99)L^4 + 9800L^2 + 10000,\\
\mathcal{R}_5 &= -25z^4 - 49z^2 - 25rqk2 - 25.
\end{aligned}
$$

The maximal real root of $\mathcal{R}(z)$ coincides with the SORPRP W of $\varphi(s;q,L)$. Since we need to compute sum of roots with negative real parts in a sense of stability, we apply our method computing SORPRP to $\mathcal{R}(-z;q,L)$. But, it follows that $\mathcal{R}(-z;q,L) = \mathcal{R}(z;q,L)$.

Relationship between L and q : Here we consider the case where the bounds for the SORPRP are given, that is, M_1 and M_2 are fixed. Then we check the behavior of the plant parameter L versus a change of q. The possible regions of (L,q) to meet the specifications in the $L-q$ parameter space is obtained by applying quantifier elimination to $\mathcal{R}(z;q,L)$ as explained in §4.

(a) $W < M_1$: Let $M_1 = 500$, then the specification (a) is equivalent to the following first-order formula: $\forall z > 500,\ \mathcal{R}(z;q,L) \neq 0$. After performing quantifier elimination to this, we can obtain the following equivalent quantifier-free formula in (L,q) which describe feasible regions of (L,q) for (a):

$$
\begin{aligned}
&(q + 62500490001 >= 0 \wedge 250L - 1 \geq 0 \wedge\\
&2500L^4q - 39063112499901L^4 + 625004900000L^3 - 3750009800L^2 + 10000000L - 10000 \leq 0 \wedge\\
&2500L^4q - 39063112499901L^4 - 625004900000L^3 - 3750009800L^2 - 10000000L - 10000 \leq 0
\end{aligned}
$$

This is illustrated as a shaded region in Fig.1.

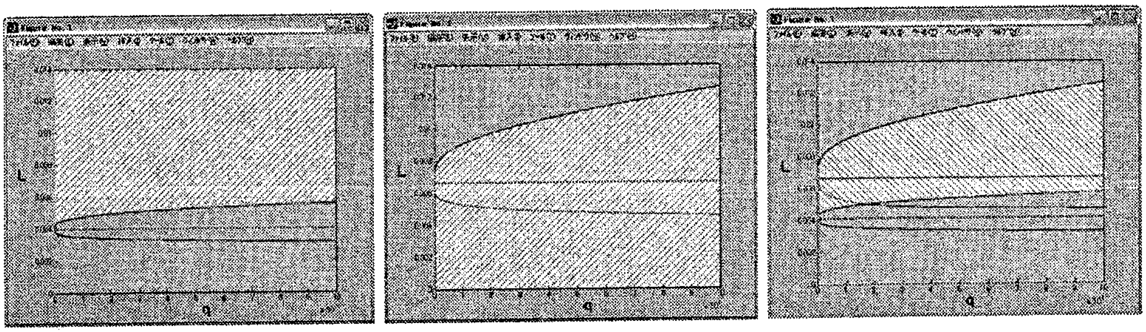

Figure 1: Feasible region of $L-q$ for (a) [Left] , (b) [Middle], and (a) $\wedge$ (b) [Right]

(b) $W > M_2$: Let $M_2 = 300$, then the specification (b) is equivalent to the following first-order formula: $\exists z > 300,\ \mathcal{R}(z;q,L) = 0$. After performing quantifier elimination to this, we can obtain a following equivalent quantifier-free formula in (L,q) which describe feasible regions of (L,q) for (b):

$$
\begin{aligned}
&(q + 8100176401 < 0) \vee (L > 0 \wedge 150L - 1 < 0) \vee\\
&(2500L^4q - 5062720499901L^4 - 135002940000L^3 - 1350009800L^2 - 6000000L - 10000 > 0) \vee\\
&(2500L^4q - 5062720499901L^4 + 135002940000L^3 - 1350009800L^2 + 6000000L - 10000 > 0)
\end{aligned}
$$

This is illustrated as a shaded region in Fig.2.

(c) $M_2 < W < M_1$: If $M_2 = 300, M_1 = 500$ for the requirement (c), the problem is recast as the following first-order formula: $(\forall z > 500,\ \mathcal{R}(z; q, L) \neq 0) \ \wedge \ (\exists z > 300,\ \mathcal{R}(z; q, L) = 0)$. A formula describing feasible regions of (L, q) for the requirement (c) can be obtained by superposing above two results for (a) and (b) in the parameter space $L - q$ as shown in Fig.3.

Control theoretical significance : Any system with parameter values of L and q within the feasible regions shown in Figs. 1, 2 and 3 meets the above requirements in terms of the magnitude of SORPRP. We can obtain the following knowledge from Fig. 3. The plant parameter L is restricted within an interval for a fixed value of q under the specification of $300 < W < 500$. The maximum and minimum edges of the feasible interval of L are monotonically increasing. Thus, for instance for the value of L around 0.01, q must be taken from the region which is larger than a certain value. We can obtain the exact threshold value easily since we have the feasible region as a semi-algebraic set by virtue of quantifier elimination. These greatly help control designers to choose appropriate value of the ratio of weights q/r for their control system more systematically.

5 Conclusion

In this paper we have presented a method to compute or estimate the sum of roots with positive real parts (SORPRP) of a polynomial with parametric coefficients based on symbolic and algebraic computations. Since the method does not compute explicit numerical values of the roots, we can treat polynomials with parametric coefficients for their SORPRP.

Combining the method with quantifier elimination, we succeeded in giving a novel systematic method for achieving optimal regulator design in control. In order to see its effectiveness and practicality, we made some experiments for a concrete example from optimal regulator control.

The method proposed here shall provide one of promising direction for an ad hoc part (i.e., choice of weights) of optimal regulator design that is one of the main concerns in control and gives another successful application of computer algebra to control design problem.

References

[1] H. Anai and S. Hara. Fixed-structure robust controller synthesis based on sign definite condition by a special quantifier elimination. In *Proceedings of American Control Conference 2000*, pages 1312–1316, 2000.

[2] H. Anai and S. Hara. Linear programming approach to robust controller design by a quantifier elimination. In *Proceedings of SICE Annual Conference 2002 (Osaka, Japan)*, pages 863–869, 2002.

[3] H. Anai, H. Yanami, K. Sakabe, and S. Hara. Fixed-structure robust controller synthesis based on symbolic-numeric computation: design algorithms with a CACSD toolbox (invited paper). In *Proceedings of CCA/ISIC/CACSD 2004 (Taipei, Taiwan)*, pages 1540–1545, 2004.

[4] H. Anai and H. Yanami. SyNRAC: A maple-package for solving real algebraic constraints. In *Proceedings of International Workshop on Computer Algebra Systems and their Applications (CASA) 2003 (Saint Petersburg, Russian Federation), P.M.A. Sloot et al. (Eds.): ICCS 2003, LNCS 2657*, pages 828–837. Springer, 2003.

[5] P. Aubry and A. Valibouze. Using galois ideals for computing relative resolvents. *Journal of Symbolic Computation*, 30:635–651, 2000.

[6] B. Caviness and J. Johnson, editors. *Quantifier Elimination and Cylindrical Algebraic Decomposition.* Texts and Monographs in Symbolic Computation. Springer, Berlin, Heidelberg, New York, 1998.

[7] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties and Algorithms.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, Berlin, Heidelberg, 1992.

[8] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry.* Graduate Texts in Mathematics **185**. Springer-Verlag, New York, Berlin, Heidelberg, 1998.

[9] P. Dorato, W. Yang, and C. Abdallah. Robust multi-objective feedback design by quantifier elimination. *J. Symb. Comp.* **24**, pages 153–159, 1997.

[10] L. González-Vega. A combinatorial algorithm solving some quantifier elimination problems. In B.F. Caviness and J.R. Johnson, editors, *Quantifier Elimination and Cylindrical Algebraic Decomposition*, Texts and Monographs in Symbolic Computation, pages 365–375. Springer, Wien, New York, 1998.

[11] L. González-Vega, T. Recio, H. Lombardi, and M.-F. Roy. Sturm-habicht sequences determinants and real roots of univariate polynomials. In B.F. Caviness and J.R. Johnson, editors, *Quantifier Elimination and Cylindrical Algebraic Decomposition*, Texts and Monographs in Symbolic Computation, pages 300–316. Springer, Wien, New York, 1998.

[12] S. Hara, T. Kimura, and R. Kondo. H_∞ control system design by a parameter space approach. In *Proceedings of MTNS-91*, pages 287–292, Kobe, Japan, 1991.

[13] M. Jirstrand. Nonlinear control system design by quantifier elimination. *Journal of Symbolic Computation*, 24(2):137–152, August 1997. Applications of quantifier elimination (Albuquerque, NM, 1995).

[14] T. Kimura and S. Hara. A robust control system design by a parameter space approach based on sign definition condition. In *Proceedings of KACC-91*, pages 1533–1538, Soul, Korea, 1991.

[15] B. Mishra. *Algorithmic Algebra.* Springer Verlag, 1993.

[16] T. Takagi. *Lecture of Algebra ('Daisugaku Kougi' in Japanese).* Kyoritsu Pub. Japan, 1930.

[17] H. Yanami and H. Anai. Development of synrac – formula description and new functions. In *Proceedings of International Workshop on Computer Algebra Systems and their Applications (CASA) 2004 : ICCS 2004, LNCS 3039*, pages 286–294. Springer, 2004.

[18] K Zhou, J.C Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, 1995.