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Necessity Measure Optimization in Linear Programming Problems with Interactive Fuzzy Numbers

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Abstract

In this paper, we treat fuzzy linear programming problems with uncertain parameters whose ranges are specified as fuzzy polytopes. The problem is formulated as a necessity measure optimization model. It is shown that the problem can be reduced to a semi-infinite programming problem and solved by a combination of a bisection method and a relaxation procedure. An algorithm in which the bisection method and the relaxation procedure converge simultaneously is proposed. A simple numerical example is given to illustrate the solution procedure.

Key Words: Possibilistic linear programming, necessity measure, interactive fuzzy numbers, semi-infinite programming, relaxation procedure, bisection method

1 Introduction

Fuzzy programming approach [6,12,15] is useful and efficient to treat a programming problem under uncertainty. While classical and stochastic programming approach may require a lot of cost to obtain the exact coefficient value or distribution, fuzzy programming approach does not (see Rommelfanger [13]). From this fact, fuzzy programming approach will be very advantageous when the coefficients are not known exactly but vaguely by human expertise.

Fuzzy programming has been developed under an implicit assumption that all uncertain coefficients are non-interactive one another. This assumption makes the reduced problem very tractable. The tractability can be seen as one of advantages of fuzzy programming approaches. However, it is observed that in a simple problem, such as a portfolio selection problem, solutions of models are often intuitively unacceptable because of the implicit assumption (see Inuiguchi and Tanino [9]). This implies that the non-interaction assumption is not sufficient to model the real world problem. In this sense, we should deal with fuzzy programming problems with interactive uncertain coefficients. However, unfortunately, the reduced problem usually becomes very difficult. Therefore, treatments of interactive uncertain coefficients without loss of tractability of the reduced problem are requested in the field of fuzzy programming. Several attempts have been done.

Rommelfanger and Kresztfalvi [14] proposed to use Yager's parameterized t-norm in order to control the spreads of fuzzy linear function values. The interaction among uncertain parameters is treated indirectly in these approaches. The parameter of Yager's t-norm should be selected for each objective function and for each constraint and the reduced problem is not always a linear programming problem. However, the selection of the parameter of Yager's t-norm will not be very easy.

Inuiguchi and Sakawa [8] treated a fuzzy linear programming with a quadratic membership function. Since quadratic membership function resembles to a multivariate normal distribution, they succeeded to show the equivalence between special models of stochastic linear programming problem and fuzzy linear programming problem. In this approach, though the fractile optimization problem [5] using a necessity measure can be reduced to a convex programming problem, the reduced problem should be solved by an iterative use of quadratic programming techniques.

Inuiguchi and Tanino [10] proposed scenario decomposed fuzzy numbers. In their approach, the interaction between uncertain parameters are expressed by fuzzy if-then rules. They showed that a fuzzy linear programming problem with scenario decomposed fuzzy numbers can be reduced to a linear programming problem.

Furthermore, Inuiguchi, Ramík and Tanino [7] proposed oblique fuzzy vectors. A non-singular matrix shows the interaction among uncertain parameters in an oblique fuzzy vector. It is shown that

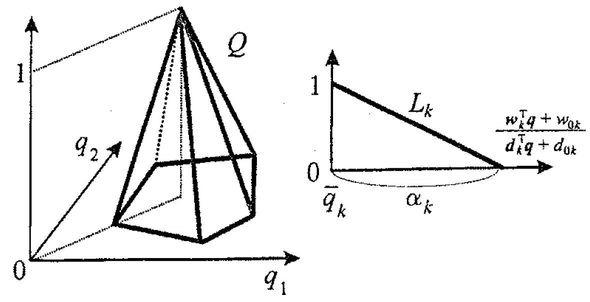

Figure 1: An example of fuzzy set Q

linear function values of oblique fuzzy vectors can be calculated easily. Owing to this property, fuzzy linear programming problems with oblique fuzzy vectors can be reduced to linear programming problems with a special structure. A solution algorithm utilizing the special structure has been proposed. Oblique fuzzy vectors are able to incorporate knowledge about linear function values of uncertain parameters. However, a non-singular matrix is not always sufficient to express the interaction among uncertain variables in real world problems.

Recently, Inuiguchi and Tanino [11] have introduced a fuzzy polytope to fuzzy linear programming problems. They have shown that the fractile optimization model using a necessity measure can be reduced to a semi-infinite linear programming problem. The reduced problem can be solved by a relaxation procedure. The fuzzy polytope can be constructed from the information about the linear fractional values of uncertain coefficients. Therefore, the fuzzy polytope will be useful when we know the vague values of sums of uncertain coefficients, ratios between two uncertain coefficients or more generally, the linear fractional values of uncertain coefficients.

In this paper, as a continuance of Inuiguchi and Tanino's paper [11] using a fuzzy polytope, we apply a necessity measure optimization model and propose a solution algorithm. In the next section, we describe the problem setting. Then we apply a necessity measure optimization model and show that the problem is reduced to a semi-infinite programming problem but not necessarily to be a linear one in Section 3. In Section 4, we propose a solution algorithm based on a relaxation procedure and a bisection method. In the algorithm, the relaxation procedure and the bisection method converge simultaneously. A simple numerical example is given to illustrate the solution procedure in Section 5. Finally, concluding remarks are given in the last section.

2 Problem Statement

In this paper, we treat the following linear programming problem with uncertain parameters:

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} & \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \lesssim_i b_i, \ i = 1, 2, \ldots, m, \end{array} \tag{1}$$

where $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{T}}$, $\boldsymbol{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in})^{\mathrm{T}}$, $i = 1, 2, \ldots, m$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)^{\mathrm{T}}$ are uncertain parameters. Let $\boldsymbol{q}^{\mathrm{T}} = (\boldsymbol{a}_1^{\mathrm{T}}, \boldsymbol{a}_2^{\mathrm{T}}, \ldots, \boldsymbol{a}_m^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}}, \boldsymbol{c}^{\mathrm{T}})$. We assume that some linear fractional function values $(\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{q} + w_{k0})/(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{q} + d_{k0})$, $k = 1, 2, \ldots, p$ $(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{q} + d_{k0} \neq 0)$ are vaguely known, where $\boldsymbol{w}_k$, $\boldsymbol{d}_k \in \mathbf{R}^{(mn+m+n)}$, $k = 1, 2, \ldots, p$ are constant vectors, and w_{k0}, $d_{k0} \in \mathbf{R}$, $k = 1, 2, \ldots, p$ are constants. Namely, we assume that the fuzzy boundary of linear fractional function values are known. Based on the linear fractional function value information we may construct an $(mn+m+n)$-dimensional fuzzy set $Q \subseteq \mathbf{R}^{(mn+m+n)}$ with the following membership function:

$$\mu_Q(\boldsymbol{q}) = \min_{k=1,2,\ldots,p} L_k \left(\frac{\dfrac{\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{q} + w_{0k}}{\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{q} + d_{0k}} - \bar{q}_k}{\alpha_k} \right), \tag{2}$$

where $L_k : \mathbf{R} \to [0, 1]$, $k = 1, 2, \ldots, p$ are reference functions, i.e., upper semi-continuous non-increasing functions such that $L_k(0) = 1$ and $\lim_{r \to +\infty} L_k(r) = 0$. $\bar{q}_k$ shows the most plausible value for the k-th linear fractional function value $(\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{q}+w_{0k})/(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{q}+d_{0k})$. α_k shows the spread, i.e., to what extent the linear fractional function value $(\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{q}+w_{0k})/(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{q}+d_{0k})$ possibly exceeds $\bar{q}_k$. When we know the maximum possible shortage of $(\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{q}+w_{0k})/(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{q}+d_{0k})$ from $\bar{q}_k$, by multiplying $\boldsymbol{w}_k$, w_{0k} and $\bar{q}_k$ by (-1), we can treat it as we know the maximum possible excess of $(-\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{q}-w_{0k})/(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{q}+d_{0k})$ from $-\bar{q}_k$. The fuzzy set Q is assumed to be bounded, i.e., h-level sets $[Q]_h=\{\boldsymbol{q}\mid\mu_Q(\boldsymbol{q})\geq h\}$ for all $h\in(0,1]$ are bounded. Moreover, without loss of generality, we assume that $\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{q}+d_{k0}>0$ for all possible $\boldsymbol{q}$. Let $L_k^*(h)=\sup\{r\mid L_k(r)\geq h\}$. Then, because of (2), we have

$$\begin{aligned}[Q]_h&=\{\boldsymbol{q}\mid \boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{q}+w_{0k}\leq(\bar{q}_k+\alpha_kL_k^*(h))(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{q}+d_{0k}),\ k=1,2,\ldots,p\}\\&=\{\boldsymbol{q}\mid \boldsymbol{wd}_k^*(h)^{\mathrm{T}}\boldsymbol{q}\leq -wd_{0k}^*(h),\ k=1,2,\ldots,p\}.\end{aligned}\tag{3}$$

Here, $\boldsymbol{wd}_k^*(h)=\boldsymbol{w}_k-(\bar{q}_k+\alpha_kL_k^*(h))\boldsymbol{d}_k$ and $wd_{0k}^*(h)=w_{0k}-(\bar{q}_k+\alpha_kL_k^*(h))d_{0k}$. Since $[Q]_h\subseteq\mathbf{R}^{(mn+m+n)}$ is bounded, from (3), we know that $p>(mn+m+n)$ and that $[Q]_h$ is a polytope for all $h\in(0,1]$. In what follows, we may use the closure of a strong h-level set $\mathrm{cl}(Q)_h=\mathrm{cl}\{\boldsymbol{q}\mid\mu_Q(\boldsymbol{q})>h\}$. In two dimensional case, such a fuzzy set Q is exemplified in Figure 1.

Let $L_k^{\#}(h)=\sup\{r\mid L_k(r)>h\}$. Then, in the same way as (3), we have

$$\mathrm{cl}(Q)_h=\{\boldsymbol{q}\mid \boldsymbol{wd}_k^{\#}(h)^{\mathrm{T}}\boldsymbol{q}\leq -wd_{0k}^{\#}(h),\ k=1,2,\ldots,p\},\tag{4}$$

where $\boldsymbol{wd}_k^{\#}(h)=\boldsymbol{w}_k-(\bar{q}_k+\alpha_kL_k^{\#}(h))\boldsymbol{d}_k$ and $wd_{0k}^{\#}(h)=w_{0k}-(\bar{q}_k+\alpha_kL_k^{\#}(h))d_{0k}$. As shown in (3), h-level sets of Q are polytopes. Thus, we may say that Q is a fuzzy polytope.

Notation '$f\lesssim_i g$' is a fuzzy version of '$f\leq g$' and stands for 'f is approximately smaller than or equal to g'. The subscript i implies that the elasticity of the fuzzy inequality may depend on the constraint. To treat such a fuzzy inequality, $\lesssim_i$ is regarded as a fuzzy binary relation. In this paper, we define $\lesssim_i$ by

$$\mu_{\lesssim_i}(f,g)=\nu_i(f-g),\tag{5}$$

where $\nu_i:\mathbf{R}\to[0,1]$ is an upper semi-continuous non-increasing function such that $\nu_i(r)=1$ for all $r\leq 0$ and $\lim_{r\to+\infty}\nu_i(r)=0$ (see Inuiguchi, et. al [5]). An h-level set of $\lesssim_i$ is obtained as

$$[\lesssim_i]_h=\{(f,g)\mid f-g\leq\nu_i^*(h)\},\tag{6}$$

where $\nu_i^*(h)=\sup\{r\mid\nu_i(r)\geq h\}$. Especially, when ν_i is defined by

$$\nu_i(r)=\begin{cases}1, & \text{if } r\leq 0,\\ 0, & \text{otherwise,}\end{cases}\tag{7}$$

$\lesssim_i$ is a conventional inequality relation $\leq$.

When L_k, $k=1,2,\ldots,p$ are defined by

$$L_k(r)=\begin{cases}1, & \text{if } r\leq 0,\\ 0, & \text{otherwise,}\end{cases}\tag{8}$$

Q is a conventional polytope. When $\boldsymbol{d}_k=\mathbf{0}$, $\boldsymbol{w}_k=\mathbf{e}_k$ (a unit vector whose k-th component is one), $d_{0k}=1$ and $w_{0k}=0$, Q is a vector of non-interactive fuzzy numbers. When L_k, $k=1,2,\ldots,p$ are defined by (8), $\boldsymbol{d}_k=\mathbf{0}$, $\boldsymbol{w}_k=\mathbf{e}_k$ (a unit vector whose k-th component is one), $d_{0k}=1$ and $w_{0k}=0$, Q is a vector of intervals. Thus, Problem (1) includes the conventional fuzzy linear programming problems and interval linear programming problems.

3 Formulation and reduction

In order to treat Problem (1), we should introduce some interpretation of the problem. Inuiguchi and Tanino [11] applied a fractile optimization model using a necessity to Problem (1) and reduced it to a semi-infinite linear programming problem. In this paper, we apply a necessity measure optimization model. The readers who are familiar with possibility theory [3] may have a question why a possibility measure is not treated but only a necessity measure. The reason is mainly a technical reason. Namely, the model with a possibility measure is reduced to a non-convex programming problem which is relatively hard to be solved. Moreover, the application of a possibility measure provides an uncertainty prone model so that the solution is risk-seeking. Because of this, solitary use of a possibility measure is not suitable for the robust planning, safety design, and so on. Nevertheless, the introduction of a possibility measure to our model is useful to express the decision maker's attitude and preference under uncertainty. Handling of a possibility measure in our problem will be one of the future topics.

A necessity measure of a fuzzy set $B \subseteq \Omega$ under a fuzzy set $A \subseteq \Omega$, i.e., $N_A(B)$ is defined by (see Dubois and Prade [2])

$$N_A(B) = \inf_r \max(1 - \mu_A(r), \mu_B(r)), \tag{9}$$

where μ_A and μ_B are membership functions of fuzzy sets A and B. Ω is a universal set. $N_A(B)$ evaluates to what extent the uncertain variable surely takes a value in a fuzzy set B under the information that the uncertain variable value is in a fuzzy set A. For the necessity measure, we have (see, Inuiguchi and Ichihashi [4])

$$N_A(B) \geq h \text{ if and only if } (A)_{1-h} \subseteq [B]_h. \tag{10}$$

When Ω is a metric space and B has an upper semi-continuous membership function μ_B, $[B]_h$, $h \in (0,1]$ are closed sets. Therefore, (10) is equivalent to

$$N_A(B) \geq h \text{ if and only if } \mathrm{cl}(A)_{1-h} \subseteq [B]_h. \tag{11}$$

Let us apply the necessity measure optimization model [5] to Problem (1). Then Problem (1) is formulated as

$$\begin{array}{ll} \text{maximize} & \mathrm{Nes}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \lesssim_0 \bar{z}), \\ \text{subject to} & \mathrm{Nes}(\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \lesssim_i b_i) \geq h^i, \ i = 1, 2, \ldots, m, \end{array} \tag{12}$$

or equivalently,

$$\begin{array}{ll} \text{maximize} & h, \\ \text{subject to} & \mathrm{Nes}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \lesssim_0 \bar{z}) \geq h, \\ & \mathrm{Nes}(\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \lesssim_i b_i) \geq h^i, \ i = 1, 2, \ldots, m, \end{array} \tag{13}$$

where $\bar{z}$ is a target value specified by the decision maker and $\lesssim_0$ is a fuzzy inequality relation defined by ν_0 in the same way as (5). Thus, "$\lesssim_0 \bar{z}$" corresponds to a fuzzy goal "approximately smaller than $\bar{z}$". $h^i \in (0,1]$, $i = 1, 2, \ldots, m$ are constants specified by the decision maker. 'Nes(C)' shows the necessity degree of the event that the condition C is satisfied. Since possible range of all uncertain parameters $\boldsymbol{a}_i$, $i = 1, 2, \ldots, m$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are jointly given by a fuzzy set Q. To evaluate all necessity degrees in Problem (12), we should consider events as a set or a fuzzy set of $\mathbf{R}^{(mn+m+n)}$. We define

$$\mu_{E_0(\boldsymbol{x})}(\boldsymbol{q}) = \sup \left\{ \mu_{\lesssim_0}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, z_0) \;\middle|\; \boldsymbol{q}^{\mathrm{T}} = \left(\boldsymbol{q}_{mn+m}^{\mathrm{T}}, \boldsymbol{c}^{\mathrm{T}}\right) \right\}, \tag{14}$$

$$\mu_{E_i(\boldsymbol{x})}(\boldsymbol{q}) = \sup \left\{ \mu_{\lesssim_i}(\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}, b_i) \;\middle|\; \boldsymbol{q}^{\mathrm{T}} = \left(\boldsymbol{q}_{n(i-1)}^{\mathrm{T}}, \boldsymbol{a}_i^{\mathrm{T}}, \boldsymbol{q}_{n(m-i)+(i-1)}^{\mathrm{T}}, b_i, \boldsymbol{q}_{m-i+n}^{\mathrm{T}}\right) \right\}, \ i = 1, 2, \ldots, m, \tag{15}$$

where $\boldsymbol{q}_s$ is a vector of $\mathbf{R}^s$ and $\mu_{E_i(\boldsymbol{x})}$ is a membership function of $E_i(\boldsymbol{x})$. Now, we can define

$$\mathrm{Nes}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq z) = N_Q(E_0(\boldsymbol{x})), \tag{16}$$

$$\mathrm{Nes}(\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \lesssim_i b_i) = N_Q(E_i(\boldsymbol{x})), \ i = 1, 2, \ldots, m. \tag{17}$$

Applying (11) to Problem (13) with the substitution of (16) and (17), we have

$$\begin{array}{ll} \text{maximize} & h, \\ \text{subject to} & \mathrm{cl}(Q)_{1-h} \subseteq [E_0(\boldsymbol{x})]_{h^0}, \\ & \mathrm{cl}(Q)_{1-h^i} \subseteq [E_i(\boldsymbol{x})]_{h^i}, \ i = 1, 2, \ldots, m. \end{array} \tag{18}$$

For the sake of convenience, we define $\boldsymbol{q}(\boldsymbol{a}_i, b_i) = (\boldsymbol{q}_{n(i-1)}^{\mathrm{T}}, \boldsymbol{a}_i^{\mathrm{T}}, \boldsymbol{q}_{n(m-i)+(i-1)}^{\mathrm{T}}, b_i, \boldsymbol{q}_{m-i+n}^{\mathrm{T}})^{\mathrm{T}}$. From (14), (15) and (6), we have

$$[E_0(\boldsymbol{x})]_h = \{\boldsymbol{q} \mid \boldsymbol{q}^{\mathrm{T}} = (\boldsymbol{q}_{mn+m}^{\mathrm{T}}, \boldsymbol{c}^{\mathrm{T}}),\ (\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \bar{z}) \in [\lesssim_0]_h\} = \{(\boldsymbol{q}_{mn+m}^{\mathrm{T}}, \boldsymbol{c}^{\mathrm{T}}) \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \bar{z} \leq \nu_0^*(h)\}, \tag{19}$$

$$[E_i(\boldsymbol{x})]_{h^i} = \{\boldsymbol{q} \mid \boldsymbol{q} = \boldsymbol{q}(\boldsymbol{a}_i, b_i),\ (\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}, b_i) \in [\lesssim_i]_{h^i}\} = \{\boldsymbol{q}(\boldsymbol{a}_i, b_i) \mid \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} - b_i \leq \nu_i^*(h^i)\}. \tag{20}$$

Then, by (14), (19) and (20), Problem (18) is reduced to

$$\begin{array}{ll} \text{maximize} & h, \\ \text{subject to} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \bar{z} \le \nu_0^*(h),\ \forall \boldsymbol{c} : \left(\boldsymbol{q}_{mn+m}^{\mathrm{T}}, \boldsymbol{c}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathrm{cl}(Q)_{1-h}, \\ & \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} - b_i \le \nu_i^*(h^i),\ \forall \left(\boldsymbol{a}_i^{\mathrm{T}}, b_i\right)^{\mathrm{T}} : \boldsymbol{q}(\boldsymbol{a}_i, b_i) \in \mathrm{cl}(Q)_{1-h^i},\ i = 1, 2, \ldots, m, \\ & h \in [0, 1]. \end{array} \tag{21}$$

This problem is a semi-infinite programming problem [1]. Since h is not a constant, Problem (21) does not possess the linearity. When h is fixed, the problem becomes a system of semi-infinite linear inequalities. The existence of a solution satisfying a system of semi-infinite linear inequalities can be examined by a relaxation procedure for solving a semi-infinite linear programming problem. From these facts, we can solve this problem by a bisection method together with a relaxation procedure. However, if we apply the relaxation procedure at each step of a bisection method, the combined solution algorithm will require a lot of computational effort.

4 A Solution Algorithm

We propose a solution algorithm such that the relaxation procedure and the bisection method converge simultaneously. To this end, we use a vector function $\boldsymbol{c}[I] : (0, 1] \to \mathbf{R}^n$ whose function value is defined by $\boldsymbol{c}$-value of an optimal solution to the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & \sum\limits_{k \in I} \delta_k, \\ \text{subject to} & \boldsymbol{w}\boldsymbol{d}_k^{\#}(1-h)^{\mathrm{T}} \left(\boldsymbol{q}_{mn+m}^{\mathrm{T}}, \boldsymbol{c}^{\mathrm{T}}\right)^{\mathrm{T}} + \delta_k = -wd_{0k}^{\#}(1-h),\ k = 1, 2, \ldots, p, \\ & \delta_k \ge 0,\ k = 1, 2, \ldots, p, \end{array} \tag{22}$$

where $I \subseteq \{1, 2, \ldots, p\}$ is an index set such that $|I| = mn + m + n$. Once Problem (22) is solved for a given I and h, we fix the value $\boldsymbol{c}[I](h)$ as the $\boldsymbol{c}$-value of the obtained optimal solution. This breaks the non-determinacy of $\boldsymbol{c}[I](h)$ due to the multiplicity of optimal solutions of Problem (22).

Now we are ready to describe the proposed solution algorithm.

Solution Algorithm

Step 1. Select $\boldsymbol{x}^0$ arbitrarily and set $s_i = 0$, $i = 0, 1, \ldots, m$, $h^{\mathrm{L}} = 0$, $h^{\mathrm{U}} = 1$ and $h = (h^{\mathrm{U}} + h^{\mathrm{L}})/2$.

Step 2. Solve a linear programming problem,

$$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^{0\mathrm{T}}\boldsymbol{c}, \\ \text{subject to} & \boldsymbol{w}\boldsymbol{d}_k^{\#}(1-h)^{\mathrm{T}} \left(\boldsymbol{q}_{mn+m}^{\mathrm{T}}, \boldsymbol{c}^{\mathrm{T}}\right)^{\mathrm{T}} \le -wd_{0k}^{\#}(1-h),\ k = 1, 2, \ldots, p. \end{array} \tag{23}$$

Let $\hat{\boldsymbol{c}}$ be $\boldsymbol{c}$-value of the obtained optimal solution to Problem (23) and $\hat{I}$ an index set of active constraints at the optimal solution. Similarly, for $i = 1, 2, \ldots, m$, solve a linear programming problem,

$$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^{0\mathrm{T}}\boldsymbol{a}_i - b_i, \\ \text{subject to} & \boldsymbol{w}\boldsymbol{d}_k^{\#}(1-h^i)^{\mathrm{T}}\boldsymbol{q}(\boldsymbol{a}_i, b_i) \le -wd_{0k}^{\#}(1-h^i),\ k = 1, 2, \ldots, p. \end{array} \tag{24}$$

Let $\hat{\boldsymbol{a}}_i$ and $\hat{b}_i$ be $\boldsymbol{a}_i$-value and b_i-value of the obtained optimal solution to Problem (24), respectively.

Step 3. For $i = 1, 2, \ldots, m$, if $\hat{\boldsymbol{a}}_i^{\mathrm{T}}\boldsymbol{x}^0 - \hat{b}_i > \nu_i^*(h^i)$, then update $s_i = s_i + 1$ and let $\boldsymbol{a}_{is_i} = \hat{\boldsymbol{a}}$ and $b_{is_i} = \hat{b}_i$. In this step, if some s_i, $i \in \{0, 1, \ldots, m\}$ is updated, then go to Step 6.

Step 4. If $\mu_{\underset{\sim}{<}_0}(\hat{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}^0, \bar{z}) \ge h$, then go to Step 5. Otherwise, update $h^{\mathrm{L}} = h$ and $h = (h^{\mathrm{U}} + h^{\mathrm{L}})/2$. Solve Problem (23). Update $\hat{\boldsymbol{c}}$ by the $\boldsymbol{c}$-value of the obtained optimal solution. Repeat Step 4.

Step 5. If $h^{\mathrm{U}} - h^{\mathrm{L}} < \varepsilon$, then terminate the algorithm. In this case, we obtained the optimal solution as $\boldsymbol{x}^0$. If $h^{\mathrm{U}} - h^{\mathrm{L}} \ge \varepsilon$ and $I_s \ne \hat{I}$ for all $s \le s_0$, then update $s_0 = s_0 + 1$ and define $I_{s_0} = \hat{I}$ and $\boldsymbol{c}[I_{s_0}](h) = \hat{\boldsymbol{c}}$.

Step 6. If $s_0 = 0$, then set $s_0 = s_0 + 1$, $I_{s_0} = \hat{I}$ and $\boldsymbol{c}[I_{s_0}](h) = \hat{\boldsymbol{c}}$. Solve a linear programming problem,

$$\begin{array}{ll} \text{minimize} & z, \\ \text{subject to} & \boldsymbol{c}[I_{l_0}](h)^{\mathrm{T}}\boldsymbol{x} \le z,\ l_0 = 1, 2, \ldots, s_0, \\ & \boldsymbol{a}_{il_i}^{\mathrm{T}}\boldsymbol{x} - b_{il_i} \le \nu_i^*(h^i),\ l_i = 1, 2, \ldots, s_i,\ i = 1, 2, \ldots, m. \end{array} \tag{25}$$

Table 1: Parameters of the fuzzy set

k	$\boldsymbol{w}_k$	w_{0k}	$\boldsymbol{d}_k$	d_{0k}	$\bar{q}_k$	α_k
1	$(0,0,0,-1)^{\mathrm{T}}$	0	$(0,1,0,0)^{\mathrm{T}}$	0	2	0.5
2	$(0,0,0,1)^{\mathrm{T}}$	0	$(0,1,0,0)^{\mathrm{T}}$	0	-2	0.5
3	$(0,0,0,-1)^{\mathrm{T}}$	0	$(0,0,1,0)^{\mathrm{T}}$	0	2	0.5
4	$(0,0,0,1)^{\mathrm{T}}$	0	$(0,0,-1,0)^{\mathrm{T}}$	0	-2	0.5
5	$(0,0,1,0)^{\mathrm{T}}$	0	$(0,0,0,0)^{\mathrm{T}}$	1	1	0.4
6	$(0,0,-1,0)^{\mathrm{T}}$	0	$(0,0,0,0)^{\mathrm{T}}$	1	-1	0.4
7	$(0,-1,0,0)^{\mathrm{T}}$	0	$(0,0,0,0)^{\mathrm{T}}$	1	-0.7	0.5
8	$(0,-3,0,-1)^{\mathrm{T}}$	0	$(0,0,1,0)^{\mathrm{T}}$	0	-1	0.6
9	$(-1,-1,1,-1)^{\mathrm{T}}$	0	$(0,0,0,0)^{\mathrm{T}}$	1	1	0.4
10	$(1,1,-1,1)^{\mathrm{T}}$	0	$(0,0,0,0)^{\mathrm{T}}$	1	-1	0.4

Let $(\boldsymbol{x}^{0\mathrm{T}}, z^0)^{\mathrm{T}}$ be an optimal solution if it exists. If Problem (25) is unbounded, let $z^0 = -\infty$ and $\boldsymbol{x}^0 = \hat{\boldsymbol{x}} + \lambda \boldsymbol{t}$ with a sufficiently large $\lambda > 0$, where $\hat{\boldsymbol{x}}$ is an obtained feasible solution and $\boldsymbol{t}$ is an obtained extreme ray. If Problem (25) is infeasible, then Problem (13) is infeasible, too, and the algorithm is terminated. If $\mu_{\lesssim_0}(z^0, \bar{z}) < h$, then update $h^{\mathrm{U}} = h$ and $h = (h^{\mathrm{U}} + h^{\mathrm{L}})/2$ and repeat Step 6, else return to Step 2.

The convergence of this algorithm can be shown as follows. Since we have $\mu_{\lesssim_0}(\hat{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}^0, \bar{z}) \leq 1$ and we update h by $h = (h^{\mathrm{U}} + h^{\mathrm{L}})/2$, Step 4 terminates in a finite number of repeats. Similarly, since we have $\mu_{\lesssim_0}(z^0, \bar{z}) \geq 0$ and we update h by $h = (h^{\mathrm{U}} + h^{\mathrm{L}})/2$, Step 6 terminates in a finite number of repeats. The linear programming problems at Step 2 always have optimal solutions because of the boundedness of Q. If Problem (13) is infeasible, the algorithm terminate at Step 6 in finite iterations because a new extreme point of a polytope $\mathrm{cl}(Q)_{h^i}$ for some constraint is added at each iteration.

In order to show the convergence of the proposed algorithm, it suffices to prove that one of the followings holds at each iteration when Problem (13) is feasible. This is because $\mathrm{cl}(Q)_h$ is a polytope:

(a) at least one of s_i, $i = 1, 2, \ldots, m$ increases,
(b) s_0 increases,
(c) $(h^{\mathrm{U}} - h^{\mathrm{L}})$ is reduced to less than the half.

We prove this. If Step 4 is skipped in the iteration, (a) holds. If h^{U} is updated at Step 6, obviously (c) is valid. If s_0 is updated at Step 5 or Step 6, directly (b) holds. Then we consider a case that Step 4 is not skipped, h^{U} is not updated at Step 6 and s_0 is not updated. Assume we come to Step 6 without any update among (a), (b) and (c) in this iteration. Since $\hat{\boldsymbol{c}}^{\mathrm{T}} \geq z^0$ holds at Step 6, we should have $\mu_{\lesssim_0}(\hat{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}^0, \bar{z}) \geq \mu_{\lesssim_0}(z^0, \bar{z}) \geq h > h^{\mathrm{L}}$. Therefore, h^{L} must have been updated before Step 6. This contradicts with the assumption. Hence, at least one of (a), (b) and (c) holds at each iteration.

5 Numerical Example

Example. Let us consider the following linear programming problem with interactive fuzzy parameters:

$$\begin{array}{rl} \text{maximize} & -2.5x_1 + c_2x_2, \\ \text{subject to} & 2.3x_1 + 0.8x_2 \lesssim_1 20, \\ & a_{21}x_1 + a_{22}x_2 \lesssim_2 14, \\ & a_{31}x_1 + 2x_2 \lesssim_3 24, \\ & x_1 \geq 0,\ x_2 \geq 0, \end{array} \tag{26}$$

where a_{21}, a_{22}, a_{31} and c_2 are uncertain parameters. For those parameters, we know $-c_2$ is about twice of a_{22} and also about twice of a_{31}, a_{31} is about 1, a_{22} is approximately greater than 0.7 and the ratio of the sum of $3a_{22}$ and c_2 to a_{31} is approximately greater than 1. Finally, the sum of $-a_{21}$, $-a_{22}$, a_{31} and $-c_2$ is about 1. In order to express those information, let us assume parameters $\boldsymbol{d}_k$, d_{0k}, $\boldsymbol{w}_k$, w_{0k}, $\bar{q}_k$ and α_k, $k = 1, 2, \ldots, 9$ are given as shown in Table 1. Since only 4 parameters are uncertain, we consider 4-dimensional fuzzy set in this example. We define the functions L_k, $k = 1, 2, \ldots, 9$ are same and defined as $L_k(r) = \max(1 - r, 0)$.

We assume that the functions ν_i, $i = 1, 2, 3$ are same and defined as

$$\nu_i(r) = \min(1, \max(1 - r/5, 0)),\ i = 1, 2, 3. \tag{27}$$

Let $h^i = 0.5$, $i = 1, 2, 3$, $\bar{z} = -22$ and $\nu_0(r) = \min(1, \max(1 - r/5, 0))$. We apply the necessity measure optimization model. We apply the proposed solution algorithm with an initial solution $x^0 = (3, 21)^{\mathrm{T}}$ and $\varepsilon = 10^{-4}$. In the 6th iteration, we obtain an optimal solution $x \approx (4.8019, 4.8019)^{\mathrm{T}}$ and $h \approx 0.4254$. A part of the solution procedure is shown in Table 2.

6 Concluding Remarks

In this paper, we treated a linear programming problem with a fuzzy polytope. The fuzzy polytope is obtained from vague knowledge on sums and ratios of uncertain variables, more generally, linear fractional function values of uncertain variables. The fuzzy polytope would be useful to represent interactive fuzzy numbers in real world problems. A necessity measure optimization model is discussed. It is shown that the problem is reduced to a semi-infinite programming problem and solved by a bisection method together with a relaxation procedure. A solution algorithm in which the bisection method and the relaxation procedure converge simultaneously is proposed.

The introduction of a possibility measure to the fuzzy programming problems treated in this paper is one of the future topics. Furthermore, a symmetric model [5] using a necessity measure is also a future topic.

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Table 2: A part of the solution procedure

Step 1. Set $\boldsymbol{x}^0 = (3,21)^{\mathrm{T}}$, $s_i = 0$, $i = 0,1,2,3$, $h^{\mathrm{U}} = 1$, $h^{\mathrm{L}} = 0$ and $h = 0.5$.

Step 2. Solve a linear programming problem,

$$\begin{aligned} &\text{maximize} \quad 21c_2, \\ &\text{subject to} \\ &\quad 1.75a_{22} \le -c_2 \le 2.25a_{22}, \\ &\quad 1.75a_{31} \le -c_2 \le 2.25a_{31}, \\ &\quad 0.8 \le a_{31} \le 1.2, \ a_{22} \ge 0.45, \\ &\quad -3a_{22} + 0.7a_{31} - c_2 \le 0, \\ &\quad 0.8 \le -a_{21} - a_{22} + a_{31} - c_2 \le 1.2. \end{aligned} \tag{$*$}$$

We obtain $c_2 = -1.4$ and $\hat{I} = \{4,6,8,10\}$. Under the same constraints, solve a linear programming problem with an objective function

$$\text{maximize} \quad 3a_{21} + 21a_{22}.$$

We obtain $a_{21} = 1.65435$ and $a_{22} = 1.67765$. Again, under the same constraints, solve a linear programming problem with an objective function

$$\text{maximize} \quad 3a_{31}.$$

We obtain $a_{31} = 1.24$.

Step 3. Since $\hat{\boldsymbol{a}}_1^{\mathrm{T}}\boldsymbol{x}^0 - \hat{b}_1 = 3.7 > \nu_1^*(h_1) = 2$, we set $s_1 = 1$ and $\boldsymbol{a}_{11} = (2.3, 0.8)^{\mathrm{T}}$. Since $\hat{\boldsymbol{a}}_2^{\mathrm{T}}\boldsymbol{x}^0 - \hat{b}_2 = 26.19 > \nu_2^*(h_2) = 2$, we set $s_2 = 1$ and $\boldsymbol{a}_{21} = (1.65435, 1.67765)^{\mathrm{T}}$. Since $\hat{\boldsymbol{a}}_3^{\mathrm{T}}\boldsymbol{x}^0 - \hat{b}_3 = 21.72 > \nu_3^*(h_3) = 2$, we set $s_3 = 1$ and $\boldsymbol{a}_{31} = (1.24, 2)^{\mathrm{T}}$. s_i, $i = 1,2,3$ are updated, then we go to Step 6.

Step 6. Since $s_0 = 0$, we set $s_0 = 1$. Let $I_1 = \{4,6,8,10\}$ and $c_2[I_1](0.5) = -1.4$. Solve a linear programming problem,

$$\begin{aligned} \text{minimize} \quad & z, \\ \text{subject to} \quad & -2.5x_1 - 1.4x_2 \le z, \\ & 2.3x_1 + 0.8x_2 \le 22, \\ & 1.65435x_1 + 1.67765x_2 \le 16, \\ & 1.24x_1 + 2x_2 \le 26, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

We obtain $\boldsymbol{x}^0 = (9.50976, 0.15948)^{\mathrm{T}}$ and $z^0 = -23.99762$. We have $\mu_{\underset{\sim}{\le}_0}(z^0, \bar{z}) = 1 > h = 0.5$ and return to Step 2.

Step 2. Under the constraints of $(*)$, solve a linear programming problem with an objective function,

$$\text{maximize} \quad 0.15948c_2,$$

We obtain $c_2 = -1.4$ and $\hat{I} = \{2,6,8,10\}$. Under the constraints of $(*)$, solve a linear programming problem with an objective function

$$\text{maximize} \quad 9.50976a_{21} + 0.15948a_{22}.$$

We obtain $a_{21} = 2.092$ and $a_{22} = 1.24$. Again, under the constraints of $(*)$, solve a linear programming problem with an objective function

$$\text{maximize} \quad 9.50976a_{31}.$$

We obtain $a_{31} = 1.24$.

Step 3. $\hat{\boldsymbol{a}}_1^{\mathrm{T}}\boldsymbol{x}^0 - \hat{b}_1 = \nu_1^*(h_1) = 2$. Since $\hat{\boldsymbol{a}}_2^{\mathrm{T}}\boldsymbol{x}^0 - \hat{b}_2 = 6.09215 > \nu_2^*(h_2) = 2$, we set $s_2 = 2$ and $\boldsymbol{a}_{21} = (2.092, 1.24)^{\mathrm{T}}$. $\hat{\boldsymbol{a}}_3^{\mathrm{T}}\boldsymbol{x}^0 - \hat{b}_3 = -11.8890 < \nu_3^*(h_3) = 2$. s_2 is updated, then we go to Step 6.

Step 6. Solve a linear programming problem,

$$\begin{aligned} \text{minimize} \quad & z, \\ \text{subject to} \quad & -2.5x_1 - 1.4x_2 \le z, \\ & 2.3x_1 + 0.8x_2 \le 22, \\ & 1.65435x_1 + 1.67765x_2 \le 16, \\ & 1.24x_1 + 2x_2 \le 26, \\ & 2.092x_1 + 1.24x_2 \le 16, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

We obtain $\boldsymbol{x}^0 = (7.64818, 0)^{\mathrm{T}}$ and $z^0 = -19.1205$. Since $\mu_{\underset{\sim}{\le}_0}(z^0, \bar{z}) = 0.42409 < h = 0.5$, we update $h^{\mathrm{U}} = 0.5$ and $h = 0.25$. Repeat Step 6.

Step 6. $c_2[I_1](0.25) = -1.6875$. Solve a linear programming problem,

$$\begin{aligned} \text{minimize} \quad & z, \\ \text{subject to} \quad & -2.5x_1 - 1.6875x_2 \le z, \\ & 2.3x_1 + 0.8x_2 \le 22, \\ & 1.65435x_1 + 1.67765x_2 \le 16, \\ & 1.24x_1 + 2x_2 \le 26, \\ & 2.092x_1 + 1.24x_2 \le 16, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

We obtain $\boldsymbol{x}^0 = (4.80192, 4.80192)^{\mathrm{T}}$ and $z^0 = -20.10804$. We have $\mu_{\underset{\sim}{\le}_0}(z^0, \bar{z}) = 0.62161 > h = 0.25$ and return to Step 2.

********** Continue **********