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Theory of $f^{-\beta}$ law of the power spectrum in granular flows

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It is demonstrated that $f^{-\beta}$ law of the power spectrum with the frequency f and $\beta = 4/3$ in granular flows is produced by the emission of dispersive waves from the antikink of an congested domain. On the other hand, it is suggested that $1/f$ spectrum is the result of hydrodynamic back flow effects.

The frequency spectra obeying $1/f$ law are widely observed in nature.[1–10] Although to know the mechanism of $1/f$ spectra is one of important problems in science, we still do not have any unified views to explain the mechanism. In some cases, $1/f$ spectra are confused with $f^{-\beta}$ with $1 < \beta \leq 2$, but $1/f$ is special, because it may be related to the divergence of the relaxation time. Indeed, the spectrum $\beta = 2$ is nothing but the relaxation process without correlation, and the spectrum with $\beta = 3/2$ can be produced by the diffusion process of structural materials.[11–13]

One of typical situations to appear $f^{-\beta}$ spectra with $\beta \simeq 1$ is granular flows. The control of granular flows[14–18] is important for technical point of views, but large and long lived fluctuations make it difficult to control. It is known that granular flows in a pipe have the power spectra obeying $f^{-\beta}$ law[19]. Several years ago, Moriyama *et al.*[20] have confirmed that granular pipe flow in the air should have the spectrum with $\beta = 4/3$. We also expect that the power spectrum obeying $f^{-4/3}$ law is universal for dissipative flows in the coexistence of congested-flow and sparse-flow.[14, 20–23] This law is robust in the experiments of granular flows, which can be observed without tuning of a suitable set of parameters[20, 23], and is believed to be universal for dissipative flows such as traffic flow[25]. The mechanism to produce $f^{-4/3}$ law has now well been understood.[14, 20, 24] Similar behavior has been observed in a wide range of parameters in a simulation of surface flow of the sand pile.[10]

However, granular flows in liquids show different aspect from those in the air. For example, Nakahara and Isoda[26] have suggested that β is smaller than unity in granular flows in liquids. Moriyama *et al.*[27] demonstrate that the power spectra in the water obey $1/f$ law. Actually their evaluated exponent β is $\beta = 0.95 \pm 0.05$. Quite recently, Awazu and Matsushita[28] have demonstrated that power spectra obeying $1/f$ could be observed in granular flow in the air when the cock is fully closed to inhibit the flow of the air.

The puropose of this paper is to combine two aspects : The first part consists of recent understanding of $f^{-4/3}$ law as the result of the decay of a kink or an antikink.[24] The second part is to discuss the possibility of $1/f$ spectrum in granular flow as the result of back flow effects.

I. THEORY OF $f^{-4/3}$ LAW

This section is basically the same as that presented in ref.[24]. In this section we use the angular frequency $\omega = 2\pi f$ instead of f for the simplicity of the notation.

A. Main idea

In order to proceed the analysis we should recall that all of one-dimensional models for traffic and granular flows in weakly unstable regions can be described by trains of quasi-solitons stabilized by small dissipations.[14, 29–31] In general, a dilute region is connected with a congested region by asymmetric interfaces[14, 30, 31] which may be characterized by the soliton equation.[29] We call a front interface the kink and a backward interface the antikink. In general, the antikink (or the kink) exists in linearly unstable region in the perturbation from the uniform state, and the kink (or the antikink) exists in the linearly stable region.[30] Therefore, the antikink emits dispersive waves backward and they are caught by the next domain. In the simplest situation, we can ignore the widths of the kinks and antikinks which may be much smaller than the typical domain size.

From the observation of experiments for power spectra, the formation process of domains may not be important but be important to consider the emission of dispersive waves from an antikink. Thus, we ignore the formation of a congested domain but focus on the decay process of the domain. We also map the model onto a one-dimensional space, where the position fixed in an experimental system is denoted by x and the system size is L and the boundaries are located at $x = \pm L/2$. For simplicity, we place a detector to measure the power spectrum at $x = 0$, *i.e.* the center of the system. Let us introduce the normalized packing fraction $\phi(x,t) \equiv n(x,t)/n_0(t)$ where $n(x,t)$ and $n_0(t)$ are the density at (x,t) and the saturated density which depends on time, respectively.

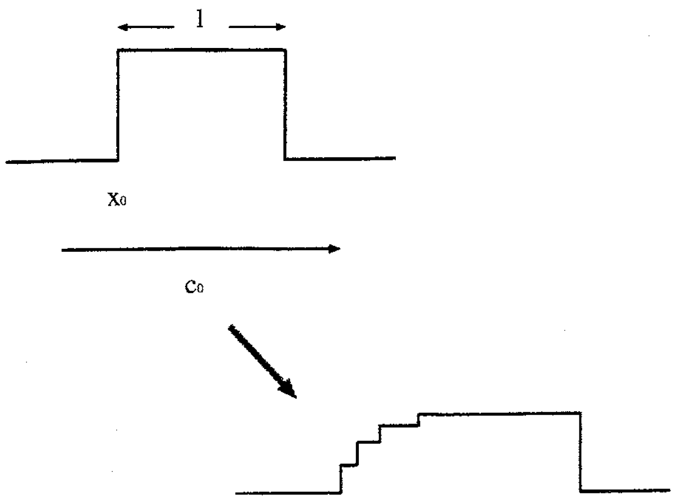

FIG. 1: A schematic picture of the propagation of a domain (its size l) with the speed c_0. The above figure represents an idealistic domain at $t = 0$ and the bottom figure represents a decayed domain due to the dispersive wave emitted from the antikink whose initial position is x_0.

If we assume that an idealistic congested domain exists in the system at time $t = 0$, the packing fraction is given by $\phi(x, t = 0) = 1$ between $x = x_0$ and $x = x_0 + l$, and $\phi(x, 0) = 0$ for otherwise, where l and x_0 are the size of the domain and the position of an antikink at $t = 0$, respectively. The equivalent expression is

$$\begin{aligned}\phi(x,0) &= \frac{l}{L}+\sum_{n=1}^{\infty}\frac{\cos\frac{2n\pi x}{L}}{n\pi}\{\sin\frac{2n\pi(x_0+l)}{L}-\sin\frac{2n\pi x_0}{L}\}\\ &-\sum_{n=1}^{\infty}\frac{\sin\frac{2n\pi x}{L}}{n\pi}\{\cos\frac{2n\pi(x_0+l)}{L}-\cos\frac{2n\pi x_0}{L}\}.\end{aligned} \tag{1}$$

On the other hand, the antikink is assumed to be unstable because of the dispersion of propagating velocity, though we can ignore such effects for the stable kink. Thus, we assume that the time dependence of $\phi(x, t)$ can be described by

$$\begin{aligned}\phi(x,t) &= \frac{l}{L}+\sum_{n=1}^{\infty}\frac{\cos\frac{2n\pi x}{L}}{n\pi}\{\sin\frac{2n\pi(x_0+l+c_0t)}{L}-\sin\frac{2n\pi(x_0+c_0t(1-(\frac{2n\pi\xi}{L})^2))}{L}\}\\ &-\sum_{n=1}^{\infty}\frac{\sin\frac{2n\pi x}{L}}{n\pi}\{\cos\frac{2n\pi(x_0+l+c_0t)}{L}-\cos\frac{2n\pi(x_0+c_0t(1-(\frac{2n\pi\xi}{L})^2))}{L}\},\end{aligned} \tag{2}$$

where c_0 is the propagating speed of the kink and ξ is the characteristic length of the dispersion relation. Equation (4) is the expression that the kink whose position is $x_0 + l + c_0t$ propagates with the constant speed c_0, while the dispersion of the propagating velocity of the antikink whose position may be $x_0 + c_0t(1 - (\frac{2n\pi\xi}{L})^2))$ makes unable to keep its shape (see Fig.1). It should be noted that $\int_{-L/2}^{L/2} dx\phi(x, t)$ is not conserved because the phase speed of the antikink is smaller than c_0, but $n_0(t)\int_{-L/2}^{L/2} dx\phi(x, t)$ should be conserved in our picture. However, the correction from $n_0(t)$ is not important, because $n_0(t)$ which is determined by the conservation law causes only the correction of the magnitude of the spectrum.

Thus, the time evolution of $\phi(0, t)$ at the observation point is given by

$$\phi(0,t)=\frac{l}{L}+\sum_{n=1}^{\infty}\frac{1}{n\pi}\{\sin\frac{2n\pi(x_0+l+c_0t)}{L}-\sin\frac{2n\pi(x_0+c_0t(1-(\frac{2n\pi\xi}{L})^2))}{L}\}. \tag{3}$$

With the aid of Wiener-Khinchin theorem, the power spectrum $I(\omega)$ and the auto-correlation function $C(t)$ can be written as

$$I(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} C(t), \qquad C(t) \equiv < \phi(0,0)\phi(0,t) >, \tag{4}$$

where the ensemble average in eq.(4) is interpreted as the average by the initial position of the antikink x_0. Because the domain propagates with c_0 if we neglect the dispersion, the existence probability of domains should be uniform except for the boundary regions. Thus, we may assume the probability distribution function $P(x_0) = 1/L$ and $C(t) = \frac{1}{L}\int_{-L/2}^{L/2} dx_0 \phi(0,0)\phi(0,t)$.

Before we proceed the analysis, let us summarize critical remarks on our approach. First, Wiener-Khinchin theorem requires that the system is in a statistically stationary state, but the decay process of a congested domain is not stationary. Since the stationary state is achieved by the supply of particles from the adjacent domain, we need to take into account the import and the export of particles between adjacent domains for the precise analysis. In addition, the assumption to contain only one domain in a system is unrealistic. Therefore, it may be appropriate to replace the system size L by the average distance between adjacent domains. Nevertheless, our simplification is useful to capture the essence of physical origin of $\omega^{-4/3}$ law.

B. Calculation of the power spectrum

In this section, let us evaluate $C(t)$ and $I(\omega)$. We note that some of expressions are complicated which are presented in Appendix of ref.[24].

Substituting eqs.(18) and (3) into eq.(4) we obtain

$$C(t) = \frac{l^2}{L^2} + J_0(t) + J_1(t) + J_2(t), \tag{5}$$

where

$$J_0(t) = \sum_{n=1}^{\infty} \frac{1}{2\pi^2 n^2}\{\cos\frac{2\pi n c_0 t}{L} - \cos\frac{2n\pi}{L}(l + c_0 t)\} \tag{6}$$

$$J_1(t) = -\sum_{n=1}^{\infty} \frac{1}{2n^2\pi^2}\{1 - \cos\left[\frac{2\pi n}{L}c_0 t\right]\cos\left[(\frac{2\pi n}{L})^3\xi^2 c_0 t\right] - \sin\left[\frac{2\pi n}{L}c_0 t\right]\sin\left[(\frac{2\pi n}{L})^3\xi^2 c_0 t\right]\} \tag{7}$$

$$\begin{aligned} J_2(t) &= \sum_{n=1}^{\infty} \frac{1}{2n^2\pi^2}\{1 - \cos\left[\frac{2\pi n}{L}(l - c_0 t)\right]\cos\left[(\frac{2\pi n}{L})^3\xi^2 c_0 t\right] \\ &\quad + \sin\left[\frac{2\pi n}{L}(l - c_0 t)\right]\sin\left[(\frac{2\pi n}{L})^3\xi^2 c_0 t\right]\}. \end{aligned} \tag{8}$$

Here, $J_0(t)$ in eq.(5) can be calculated as

$$J_0(t) = \frac{l}{2L}(1 - \frac{l}{L}) - \frac{l c_0 t}{L^2}, \tag{9}$$

where we use the formula $\sum_{n=1}^{\infty}\cos nx/n^2 = \pi^2/6 - \pi x/2 + x^2/4$. Thus, $I_0(\omega) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt e^{i\omega t}(l^2/L^2 + J_0(t))$ becomes

$$I_0(\omega) = \sqrt{2\pi}\frac{l(L+l)}{2L^2}\delta(\omega) + \sqrt{\frac{2}{\pi}}\frac{l c_0}{L^2}\omega^{-2}. \tag{10}$$

The evaluations of $J_1(t)$ and $J_2(t)$ are nontrivial. When we assume $c_0 t \ll L$ the summation in $J_1(t)$ can be replaced by the integral. From the expansion by $c_0 t/\xi$ we obtain

$$\begin{aligned} J_1(t) &\simeq -\frac{1}{3\pi L}(\xi^2 c_0 t)^{1/3}[\int_0^{\infty} dz \frac{1-\cos z}{z^{4/3}} - \left(\frac{c_0 t}{\xi}\right)^{2/3}\int_0^{\infty} dz \frac{\sin z}{z}] \\ &= -\frac{1}{3L\Gamma(4/3)}(\xi^2 c_0 t)^{1/3} + \frac{c_0 t}{6L}, \end{aligned} \tag{11}$$

where we use $\int_0^\infty dz(1-\cos z)/z^{4/3} = \pi/\Gamma(4/3)$ and $\int_0^\infty dz \sin z/z = \pi/2$ with the Gamma function $\Gamma(z)$. The corresponding Fourier transform of $J_1(t)$ is thus given by

$$I_1(\omega) = \frac{\sqrt{2}}{6\sqrt{\pi}L}(\xi^2 c_0)^{1/3}\omega^{-4/3} - \frac{\sqrt{2}c_0}{6\sqrt{\pi}L}\omega^{-2}. \tag{12}$$

On the other hand, for $l \gg c_0 t$, $J_2(t)$ can be evaluated as

$$\begin{aligned} J_2(t) &\simeq \sum_{n=1}^{\infty}\frac{1}{2\pi^2 n^2}\{1-\cos\frac{2\pi nl}{L}\cos(\frac{2\pi n}{L})^3\xi^2 c_0 t + \sin\frac{2\pi nl}{L}\sin(\frac{2\pi n}{L})^3\xi^2 c_0 t \\ &\quad -\frac{2\pi n}{L}c_0 t\cos\frac{2\pi nl}{L}\sin[\left(\frac{2\pi n}{L}\right)^3\xi^2 c_0 t]\} \\ &\simeq \frac{l}{\pi L}J_{21}(t) - \frac{c_0 t}{\pi L}J_{22}(t). \end{aligned} \tag{13}$$

The explicit expressions for $J_{21}(t)$ and $J_{22}(t)$ are complicated and not important for our purpose (see Appendix).

In the limit of $\omega \to 0$, $I_2(\omega)$, the Fourier transoform of $J_2(t)$, is dominated by $I_{21}(\omega)$ as

$$I_2(\omega) \simeq \frac{l}{\pi L}I_{21}(\omega) \to -\frac{\sqrt{2}}{6\sqrt{\pi}L}(\xi^2 c_0)^{1/3}\omega^{-4/3}, \tag{14}$$

where $I_{21}(\omega)$ is the Fourier transform of $J_{21}(t)$, and its explicit expression is given by eq.(??) in Appendix. It is notable that this asymptotic expression of $I_2(\omega)$ is canceled with the term proportional to $\omega^{-4/3}$ in $I_1(\omega)$. That is, the spectrum obeying $\omega^{-4/3}$ disappears and $I(\omega) \sim \omega^{-2}$ in the limit of $\omega \to 0$.

On the other hand, though $I_{21}(\omega)$ is singular in the limit of $\omega \to \infty$, $I_{21}(\omega)$ is regular for enough large ω. In fact, one can obtain the analytic expansion of $J_{21}(t)$ near $bt = 0.001$ as $2J_{21}(t)/\pi \simeq 1 + \alpha(bt - 0.001) + O((bt-0.001)^2) \simeq 1 + \alpha bt + \cdots$ with $\alpha = 0.000229538$. If we replace $J_{21}(t)$ by this approximate function, we obtain the approximate Fourier transform

$$I_{21}(\omega) \sim \frac{\pi^{3/2}}{\sqrt{2}}\delta(\omega) - \sqrt{\frac{\pi}{2}}\alpha b\omega^{-2} \tag{15}$$

for large ω.

Thus, we obtain the power spectrum $I(\omega) = I_0(\omega) + I_1(\omega) + I_2(\omega)$ as

$$I(\omega) = \frac{\sqrt{2}}{6\sqrt{\pi}L}(\xi^2 c_0)^{1/3}\omega^{-4/3} - \frac{\sqrt{2}c_0}{6\sqrt{\pi}L}(1-\frac{6l}{L})\omega^{-2} + \frac{l}{\pi L}I_{21}(\omega) \tag{16}$$

for $\omega \neq 0$. For large ω, $I(\omega)$ is dominated by the term proportional to $\omega^{-4/3}$ as

$$I(\omega) \to \frac{\sqrt{2}}{6\sqrt{\pi}L}(\xi^2 c_0)^{1/3}\omega^{-4/3}. \tag{17}$$

Thus, we derive the spectrum obeying $\omega^{-4/3}$. Figure 2 shows the comparison of eq.(16) with eq.(17), where we can see the tail obeying $\omega^{-4/3}$ for large ω, while eq.(16) seems to obey ω^{-2} for small ω. It is obvious that both expressions (16) and (17) become identical for larger ω.

II. POSSIBILITY OF $1/f$ SPECTRUM AS THE HYDRODYNAMIC EFFECT

The spectra obeying $1/f$ law for granular flows in pipe may be from the hydrodynamic effect. Thus, the origin of $1/f$ law is completely different from that stated for $f^{-4/3}$ law. The idea of $1/f$ spectrum based on the hydrodynamic effect has been suggested by Agu *et al.*[9], but their calculation is only applicable to the relaxation from the initial condition, while actual particles forget the information of the initial condition. In addition, their treatment is incorrect, because the sudden change of the velocity from zero to $\boldsymbol{u}_0$ at $t = 0$ introduces an extra term which was not considered by them.[32] Namely, the term proportional to $\int_{-\infty}^{t} dt'\dot{\boldsymbol{u}}(t')/\sqrt{t-t'}$ is not reduced to $\int_0^\infty dt'\dot{\boldsymbol{u}}(t')/\sqrt{t-t'}$ but $\int_0^\infty dt'\dot{\boldsymbol{u}}(t')/\sqrt{t-t'} + 1/\sqrt{t}\boldsymbol{u}_0$. Thus, the authocorrelation function does not have the singularity proportional to

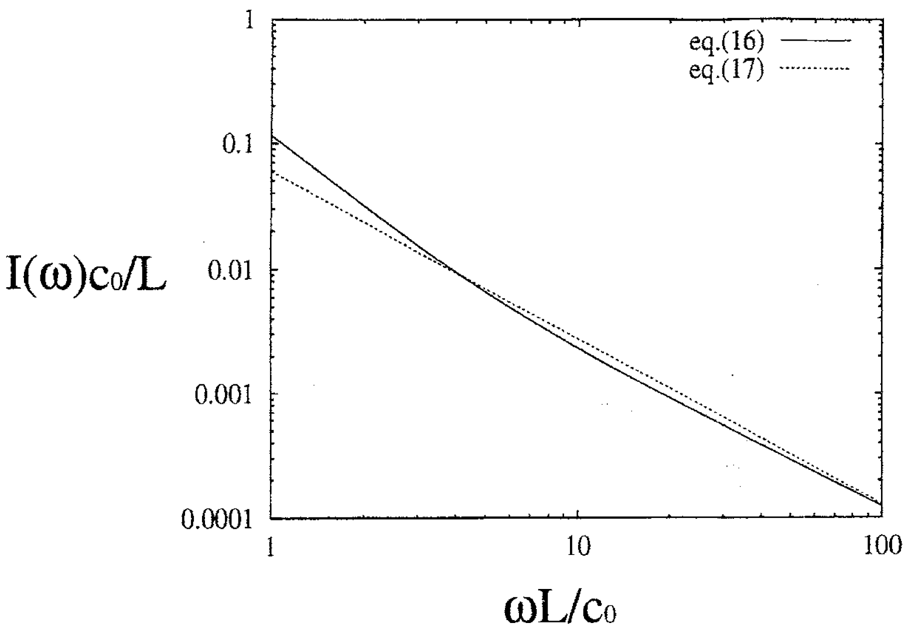

FIG. 2: Log-log plots of eqs.(16) and (17) as the frequency spectra. We adopt the parameters $l/L = \xi/L = 0.3$ and $Lb/c_0 = L/l = 10/3$. .

$1/\sqrt{t}$ for short t as suggested by Agu *et al.*[9] but it has the form $1 - b\sqrt{t}$ with a constant b for short time behavior and has the long time tail $t^{-3/2}$ for large t.[32] Therefore, a particle in fluid itself cannot produce $1/f$ spectrum. On the other hand, if there is the Brownian force acting the particle, the correlation of the random force should satisfies the fluctuation-dissipation relation (FDR), and thus, it leads to the well-known long time tail.[33]

However, granular particles in fluid can collide each other. Thus, their idea may be applicable, because the collisions can be regarded as the random noise without satisfying FDR and can destroy the singularity at the initial instance. In this letter, we try to pursue the above simple idea to explain $1/f$ law observed in granular flows. The argument is like a rough sketch to support this idea, but it might be followed by the more precise analysis.

Let us consider the equation of motion of a spherical particle suspended in the fluid. The particle α with the mass m obeys

$$m\frac{d\boldsymbol{u}^\alpha}{dt} = \boldsymbol{F}_h^\alpha + \boldsymbol{F}_c^\alpha + \boldsymbol{F}_g^\alpha + \boldsymbol{F}_b^\alpha \tag{18}$$

where $\boldsymbol{u}^\alpha$ is the velocity of particle α. The forces α $\boldsymbol{F}_h^\alpha$, $\boldsymbol{F}_c^\alpha$, $\boldsymbol{F}_g^\alpha$ and $\boldsymbol{F}_b^\alpha$ are respectively the hydrodynamic force, the collisional force among particles, the external force such as the gravity and the thermal Brownian force. Amongst them the Brownian force is not important for granular particles, because particles are enough large not to be agitated by the thermal noise.

The most difficult part to treat is the hydrodynamic force. Even when we restrict our interest to the case of the linearized Navier-Stokes equation, we must solve a moving boundary value problem which can be performed only by computational method. To treat this part approximately, we assume that the density of suspended particles is not high and the back flow effect induced by many-body motion is not important. In this situation, the balance between the statistical averaged hydrodynamic force and the gravity $\boldsymbol{F}_g = -F_g\hat{z}$ with the unit normal vector parallel to the vertical direction $\hat{z}$ leads to a constant sedimentation as

$$F_g = < R > U_s \tag{19}$$

where U_s and $< R >$ are the sedimentation speed and the statistical averaged resistance matrix. It is known that U_s

decreases with the density, and Batchelor[34] obtained

$$U_s = U_0(1 - 6.55\phi + \cdots), \quad U_0 \equiv \frac{2a^2(\rho_p - \rho_f)g}{9\eta} \tag{20}$$

for mono-disperse random suspensions, where ϕ, a, ρ_p, ρ_f, η and g are the volume fraction, the radius of the particle, the density of the particle, the density of fluid, the viscosity, and the gravitational acceleration, respectively. It should be noted that explicit formula[35–37] for high density suspensions may be valid only in limited situations, because homogeneous states cannot be maintained for the high density case. Based on the same level approximation, the stationary part of the hydrodynamic force can be renormalized into one-body and it may be written as[38, 39]

$$\boldsymbol{F}_h = -< R > \boldsymbol{u} - \frac{2}{3}\pi\rho a^3\dot{\boldsymbol{u}} - 6a^2(\pi\eta\rho)^{1/2}\int_{-\infty}^{t}\frac{\dot{\boldsymbol{u}}(s)}{\sqrt{t-s}}ds. \tag{21}$$

Here we omit the label α because this expression is valid for any particle if we adopt this approximation that the many-body effects are absorbed in the resistance matrix. We have to stress that this approximation is inconsistent because we neglect the time delayed effect for the resistance matrix caused by the back-flow effect.

Thus, the equation of motion of a tagged particle (without label α) may be written as

$$m^*\frac{d\boldsymbol{u}}{dt} = -6\pi\tilde{\eta}a(\boldsymbol{u} - U_s) - 6a^2\sqrt{\pi\eta\rho_f}\int_{-\infty}^{t}\frac{\dot{\boldsymbol{u}}(s)}{\sqrt{t-s}}ds + \boldsymbol{F}_c, \tag{22}$$

where $m^* = m + \frac{2}{3}\pi\rho_f a^3$ and $\tilde{\eta} = \eta(1 + 6.55\phi + \cdots)$. We should note that the equation of motion in the experimental frame is similar to the optimal velocity model of traffic flow[14], because the sedimentation rate U_s is a function of the local density and thus the first term of the right hand side represents the relaxation to the sedimentation rate. As suggested in the previous paper, the model without $\dot{\boldsymbol{u}}$ produces $f^{-4/3}$ spectrum.

However, if the density fluctuation is not large, U_s can be replaced by the spatial average $\bar{U}_s$ of U_s. Then, the term $-6\pi\tilde{\eta}a(\boldsymbol{u} - U_s)$ in (22) can be represented by $-6\pi\tilde{\eta}a\boldsymbol{u}$ in the co-moving frame with the speed $\boldsymbol{U}_s$. The change of the frame does not affect the shape of power spectrum except for the emergence of characteristic peaks. When we use the Laplace transform $\hat{h}(\omega) = \int_0^\infty dte^{-i\omega t}h(t)$, $\int_{-\infty}^\infty \theta(t)h(t)$ with $\theta(t) = 1$ for $t \geq 0$ and $\theta(t) = 0$ for otherwise, eq.(22) in the co-moving frame becomes

$$i\omega\hat{\boldsymbol{u}} \tag{23}$$

As indicated by McLennan[32], this equation is replaced by

$$\begin{aligned} m^*\frac{d\boldsymbol{u}}{dt} &= -6\pi\tilde{\eta}a\boldsymbol{u} - 6a^2\sqrt{\pi\eta\rho_f}/\sqrt{t}\boldsymbol{u}(0) \\ &\quad -6a^2\sqrt{\pi\eta\rho_f}\int_0^t\frac{\dot{\boldsymbol{u}}(s)}{\sqrt{t-s}}ds + \boldsymbol{F}_c, \end{aligned} \tag{24}$$

When we use the Laplace transform $\hat{h}(\omega) = \int_0^\infty dte^{-i\omega t}h(t)$, $\int_{-\infty}^\infty \theta(t)h(t)$ with $\theta(t) = 1$ for $t \geq 0$ and $\theta(t) = 0$ for otherwise, eq.(24) becomes

$$i\omega\hat{\boldsymbol{u}} - \boldsymbol{u}(0)(1 + \frac{\alpha}{\sqrt{i\omega}}) = -\gamma(\omega)\hat{\boldsymbol{u}} + \frac{\hat{\boldsymbol{F}}_c}{m^*} \tag{25}$$

where $\alpha = 6\pi^{3/2}a^2\sqrt{\eta\rho_f}$, $\hat{\boldsymbol{u}}$ and $\boldsymbol{F}_c$ are the Laplace transform of $\boldsymbol{u}$ and $\boldsymbol{F}_c$, respectively. Here $\gamma(\omega)$ is given by

$$\gamma(\omega) = \frac{1}{m^*}(6\pi\tilde{\eta}a + 6\pi\rho_f a^2\sqrt{i\nu\omega}) \tag{26}$$

with $\nu = \eta/\rho_f$. Since the particle feels the consecutive agitation from $\boldsymbol{F}_W$, the initial condition is not important.

Equation (22) still contains the collisional force which has not been given explicitly. We can imagine that this force can play a role of the random force for the motion of a tagged particle. In fact, we can derive the diffusion equation from Boltzmann equation for dilute tagged particles distributed in the bath particles.[41] Let us regard our system as a mixture of heavy particles (suspensions) and light particles (solvents). The light particles are thermalized and the collisions between light particles and the time evolution of the probability distribution function by the collisions between the heavy particles can be treated as the Fokker-Planck equation. The collisions between heavy particles can be described by Boltzmann equation. We assume that each collision is elastic which does not loose any essence of physics.[40] Then, Boltzmann equation for the probability distribution function $f_T(\boldsymbol{r}, \boldsymbol{u}, t)$ for the tagged particle may be written as

$$\begin{aligned} & i\omega f_T + (\boldsymbol{u}\cdot\nabla) f_T + \gamma(\omega)\frac{\partial}{\partial \boldsymbol{u}}\cdot \boldsymbol{u}\partial f_T \\ = & \int d\boldsymbol{u}_1 \int d\hat{\Omega}\sigma u_r (f_T' f_{b,1}' - f_T f_{b,1}) \end{aligned} \tag{27}$$

in the Laplace transform of time, where $u_r = |\boldsymbol{u} - \boldsymbol{u}_1|$ and $d\Omega$ is an element of solid angle. the unit vector. We assume that the velocities of particles are changed from $(\boldsymbol{u}, \boldsymbol{u}_1)$ to $(\boldsymbol{u}', \boldsymbol{u}_1')$ at a collision. Here we use the abbreviations $f_T' = f_T(\boldsymbol{u}')$ and $f_{b,1}' = f_b(\boldsymbol{u}_1')$. In eq.(27) we may assume that the bath particles are in equilibrium and the distribution function obeys $f_b = n\phi_{MB}(u)$ with $\phi_{MB} = (m/2\pi T_g)^{3/2}\exp(-mu^2/2T_g)$ with the granular temperature T_g. We also stress that the diffusion term in the Fokker-Planck equation can be ignored, because the thermal agitation of large particles is small. Thus, the collisions between heavy particles and light particles are absorbed in the drag force from the fluid.

As demonstrated by Dorfman[41], we can derive the diffusion equation from (27) without the drag force, though his calculation includes some mistakes. If we integrate (27) over $\boldsymbol{u}$, the contribution from the collisional integral disappears, then we obtain

$$i\omega P(\boldsymbol{r}, \omega) + \nabla\cdot \boldsymbol{J} = 0 \tag{28}$$

where $P = \frac{1}{n}\int d\boldsymbol{u} f_T(\boldsymbol{u}, \boldsymbol{r}, t)$ and $\boldsymbol{J} = \frac{1}{n}\int d\boldsymbol{u}\boldsymbol{u} f_T(\boldsymbol{r}, \boldsymbol{u}, \omega)$. We should note that the collision integral and the drag term do not have any explicit contribution to eq.(28) but the indirect contributions through the change of $f_T(\boldsymbol{r}, \boldsymbol{u}, t)$. Assuming the small deviation of f_T from the equlibrium state, *i.e.*, $f_T = nP(\boldsymbol{r}, t)\phi_{MB}(1 + \epsilon\Phi_1(\boldsymbol{r}, \boldsymbol{u}, t) + \cdots)$ with the aid of the scaling $i\omega \to \epsilon^2 i\omega$ and $\nabla \to \epsilon\nabla$ with small ϵ, we may obtain

$$\begin{aligned} & \phi_{MB}(\boldsymbol{u}\cdot\nabla)P + \gamma(\omega)P\frac{\partial}{\partial \boldsymbol{u}}\cdot(\boldsymbol{u}\phi_{MB}\Phi_1) \\ = & n_b P\int d\boldsymbol{u}_1 \int d\hat{\Omega}\sigma u_r \phi_{MB}(u_1)\phi_{MB}(u) \\ & \times(\Phi_1(\boldsymbol{u}') - \Phi(\boldsymbol{u})) \end{aligned} \tag{29}$$

in the first order μ. Now, let us adopt the lowest order Sonine approximation, i.e. $\Phi_1 = \boldsymbol{a}\cdot\boldsymbol{u}$. Then, multiplying $\boldsymbol{u}$ by eq.(29) and integating over $\boldsymbol{u}$, we obtain

$$\begin{aligned} & \int d\boldsymbol{u}\boldsymbol{u}\phi_{MB}(\boldsymbol{u}\cdot\nabla)P + \gamma(\omega)P\int d\boldsymbol{u}\boldsymbol{u}\frac{\partial}{\partial \boldsymbol{u}}\cdot(\boldsymbol{u}\phi_{MB}\Phi_1) \\ = & nP\int d\boldsymbol{u}\int d\boldsymbol{u}_1\int d\hat{\Omega}\sigma u_r\phi_{MB}(u_1)\phi_{MB}(u) \\ & \times \boldsymbol{u}\boldsymbol{a}\cdot(\boldsymbol{u}' - \boldsymbol{u}). \end{aligned} \tag{30}$$

The first term of the left hand side (LHS) of (30) is reduced to $(T/m)\nabla P$, and the second term of LHS becomes $-\gamma(\omega)(T/m)\boldsymbol{a}P$. The right hand side of eq.(30) can be evaluated as $-64/(9\sqrt{\pi})\boldsymbol{a}n(T/m)^{3/2}d^2P$ from the straight-foward calculation parallel to that by Dorfman.[41] Thus, we obtain

$$\boldsymbol{a} = -b(\omega)\frac{\nabla P}{P} \tag{31}$$

where $b = (64nd^2/9 - \gamma(\omega)\sqrt{\pi m/T})^{-1}\sqrt{\pi m/T}$. Substituting this result into the definition of the current $\boldsymbol{J}$, we obtain

$$\boldsymbol{J} = -D\nabla P(\boldsymbol{r}, t), \qquad D = \frac{T}{m}b(\omega). \tag{32}$$

Thus, the collisions of particles produces the diffusion as expected.

It is apparent that the diffusion can be reproduced by the random noise. Thus, we may replace $\boldsymbol{F}_c$ by the random white noise $\boldsymbol{F}_W$ whose i-th component satisfies

$$< F_W^i(t) >= 0, \quad < F_W^i(t)F_W^j(t') >= 2D\delta_{ij}\delta(t - t') \tag{33}$$

for the time scale larger than the mean collision interval. This is an bold simplification, but the essential role of collisions which is the diffusion of the tagged particle is kept by this replacement.

Since the particle feels the consecutive agitation from $\boldsymbol{F}_W$, the initial condition is not important. The solution of eq.(25) with the replacement of $\boldsymbol{F}_c$ by $\boldsymbol{F}_W$ may be given by

$$\hat{\boldsymbol{u}} = \frac{1}{i\omega + \gamma(\omega)} \frac{\hat{\boldsymbol{F}}_W}{m^*}. \tag{34}$$

We should note again that $\boldsymbol{F}_W$ satisfies eq.(33), while the usual suspension agitated by Brownian force $\boldsymbol{F}_b$ must satisfies FDR $< \hat{F}_b^i(\omega)\hat{F}_b^j(\omega') >= 2(2\pi)^3 m^* \gamma(\omega) k_B T \delta(\omega + \omega')$.

Thus, we obtain

$$< |\hat{u}(\omega)|^2 >= \frac{2(2\pi)^3 D}{|m^* i\omega + 6a^2(i\rho_f \eta\omega)^{1/2} + 6\pi\tilde{\eta}a|^2}. \tag{35}$$

This suggests that the spectrum tends to be a constant (white) for small ω, while decay $1/\omega^2$ for large ω. However, in the intermediate $\omega = 2\pi f$, the second term of the denominator may play the important role to behave as $1/f$ law. We also note that this spectrum can be used for the larger frecuency than the collision frequency.

Although the relation between the density correlation and the velocity correlation is not clear, we expect that the both hydrodynamic fields behave with the same time scale. Actually, in the steady state of shear flow problem, it is known that the velocity gradient can be represented as an explicit function of the density field.

Let us check the self-consistency of two assumptions used to observe $1/f$ law in eq.(35): First, we assume the collision frequency is higher than the observed scale. Second, we need to confirm the existence of the middle range to have $1/f$ law. To verify both assumptions, we have to estimate three characteristic frequencies. The collision frequency ν_c may be evaluated by $\sqrt{2}\bar{u}nd^2$ from the elementary kinetic theory, where the characteristic speed $\bar{u}$ may be given by U_0. On the other hand, the cut-off frequency ν_W between the white spectrum and $1/f$ may be given by $\nu_W = \pi^2\eta/(2\pi\rho_f a^2)$, and the cut-off frequency between the Lorentzian and $1/f$ is given by$\nu_L = 81\rho_f\eta/(8\pi^3(\rho_p - \rho_f)^2 a^2)$. Let us use water and the particles with the radius 1 mm and the density 3 g/cm^3 with the volume fraction $\phi = 0.1$ ($\eta \sim 10^{-3}$Pa·s). In this case it is not easy to observe $1/f$ spectrum in our analysis.

In our treatment we introduce at least two uncontrollable approximations separately: (i) we absorb the many-body hydrodynamic effect into one-body motion with the renormalization of the resistance matrix, (ii) the collision between particles is treated as the random noise, because it plays a role of diffusion term in the real space dynamics. It is also questionable that we have assumed that the system is in nearly equilibrium. In our treatment, the hydrodynamic interaction and the collisions are separable, but it is not true. The system for dense suspensions cannot be uniform because of the hydrodynamic interactions. For nonuniform suspensions, we may observe other spectra like $f^{-4/3}$ law. To check $1/f$ law in the simulation is not easy, because we have to solve the moving boundary problem with keeping time derivative of Navier-Stokes equation. Therefore, we could not observe $1/f$ law in the simulation of Stokesian dynamics.[40] We believe that the origin of $1/f$ law in granular flows in a pipe is different from others, *e.g.* the surface flow on a sand pile.[10]

III. CONCLUDING REMARKS

A. About $f^{-4/3}$ law

It should be noted that the actual process includes many other factors for larger ω and smaller ω. In experiments, $I(\omega)$ decays exponentially for larger ω, because the initial state is not in an idealistic domain as we have assumed here. To reproduce the full shape of spectrum we need to contain the formation process of domains for our analysis. As stated in the last part of section II, to think of the formation process of domains is also important to justify to use both Wiener-Khinchin theorem and the Fourier transform itself. Thus, to analyze the process in the balance between the formation of domains and the decay of domains is an important future problem to be solved. We also indicate that the mutual interaction between domains is important. Thus, it may be appropriate that the system size L we used may be regarded as the average distance between adjacent domains. To include the interaction between domains is also an important point to be improved.

It is interesting that the above defects in our analysis as well as the correction of the magnitude in the observation from $n_0(t)$ are related to the conservation law of macroscopic quantities. It is obvious that these processes should be considered for the precise theory. Nevertheless, we believe that our picture presented here captures the essence of physics and clarify the mechanism of emergence of $\omega^{-4/3}$ law. This success may be from the fact that $\omega^{-4/3}$ is obtained from the short time behavior (large ω behavior). Thus, the long time processes related to the conservatio law are not important to obtain $\omega^{-4/3}$ law.

Before we conclude our paper, let us comment on the spectra obeying $1/\omega$-like law in granular flows. For example, Nakahara and Isoda[26] have demonstrated that the behaviors of the power spectrum of granular flows in liquids are different from those in the air. In particular, Moriyama *et al.*[27] suggest that $I(\omega) \sim \omega^{-\beta}$ with $\beta = 0.95 \pm 0.05$ for granular flows in the water.

B. About $1/f$ law

In conclusion, in this paper, we have demonstrated that the main process to produce $f^{-4/3}$ law is the emission of the dispersive wave from an antikink. This result is universal when isolated congested domains exist in a dissipative flow. Through the analysis, we have revised the previous uncertain picture.

We also have demonstrated that $1/f$ law in the power spectrum of the velocity field can be produced by the random collisions between particles and the hydrodynamic delay effect.

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