数理解析研究所講究録 1474 巻 2006 年 144-153

A Class of Adjacency Matrices in a 3-node Network

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1 Introduction

Dynamics in networks are studied in various fields. For example, suppose network modeling of neurons with time delayed coupling. Some works have focused on the local stability or bifurcations of the equilibria [1–7]. Others dealt with global stability by the two mathematical techniques, monotone theory [8–10], Liapunov functionals [11–13] or the combination method of these two [14]. Another example is the network of oscillators with time-delayed coupling. Earl and Strogatz found a stability criterion for the synchronous state in networks of identical phase oscillators with delayed coupling [15]. The criterion is applied to many networks, so long as each oscillator has uniform insertion degree. That is, each oscillator receives signals from k others, where k is uniform for all oscillators.

In this paper, we study a 3-node network and propose a new subclassification of irreducible adjacency matrices. For 3-node networks, there are many types of topology where insertion degrees of individual elements are not necessarily uniform. Apart from the loops (that is, the input from itself), there exist 13 types of network topology. Milo *et al.* called them *network motifs* which are statistically significant subnetworks included by many real networks in their structure [16]. Including the loops the 3-node networks can be categorized into 86 types of network topology. It is convenient to classify these topologies. Generally, the concept of strongly connected graph is well-known, and there are 5 strongly connected graphs in the 13 types of network topology. Note that there is a one-to-one correspondence between labeled directed graphs with n vertices and $n \times n$ adjacency matrices [17, 18], and strongly connectivity of the graph is equivalent to irreducibility of the adjacency matrix [19]. Here the adjacency matrix $A = (a_{ij})$ of labeled directed graph G with n vertices is the $n \times n$ binary matrix in which $a_{ij} = 1$ if the directed path exists from the i-th vertex to the j-th vertex in G and $a_{ij} = 0$ otherwise.

In the next section we present a model of a 3-dimensional delayed differential system and give a result with respect to its convergent point. In §3 we give a definition of the subclassification based on the convergent point, and show that there exist 4 subclasses in the 5 strongly connected network topologies. One of the classes has two elements. In §4 we compare these two elements in terms of convergent speed and show that the more transmission paths, the faster the convergent speed, if and only if transmission delays exist.

2 Model and Convergent Point

We assume that each element tends to zero exponentially as $t \to \infty$ before coupling:

$$x_i'(t) = -\alpha_i x_i(t),$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $x_i'(t)$ is the derivative of $x_i(t)$ with respect to t and $\alpha_i > 0$ $(i = 1, 2, \ldots, n)$. We represent intersections between each elements using directed paths as follows. If i-th element receives the output from j-th element, then we add a directed path from j-th vertex to i-th vertex. Then an $n \times n$ adjacency matrix A is determined uniquely. Letting $B = (b_{ij})$ be the transpose of A and $f_i \colon \mathbb{R}^n \to \mathbb{R}$ be the transfer function, we can describe the system after coupling:

$$x_i'(t) = -\alpha_i x_i(t) + \sum_{j=1}^{n} \frac{b_{ij}}{\tilde{b}_i} f_j(x(t - \tau_j)), \tag{E0}$$

where $\tau_j \geq 0$ is a transmission delay of the output from the j-th element and $\tilde{b}_i := \sum_{j=1}^{n} b_{ij} \neq 0$ which implies that every element receives at least one signal. We also assume that each element tends to a nonzero constant which depends on both the topology of the network and the transmission delays not only the initial conditions after coupling. As one of the most simplest model achieving the above assumptions, we consider the following 3-dimensional equations with linear transmission functions:

$$x_i'(t) = \alpha_i \left[-x_i(t) + \frac{\sum_{j=1}^{3} b_{ij} x_j(t - \tau_j)}{\tilde{b}_i} \right] \tag{E}$$

for $t \geq 0$ with the initial conditions

$$x_i(s) = \varphi_i(s) \quad \text{for} \quad s \in [-\tau_i, 0], \tag{1}$$

where φ_i is continuous function from $[-\tau_i, 0]$ to $\mathbb{R}$ and $1/\alpha_i > 0$ is time constant$(i = 1, 2, 3)$.

It is easily seen that the straight line $\ell = \{(x_1, x_2, x_3) \colon x_1 = x_2 = x_3\}$ in the space $\mathbb{R}^3$ is a set of equilibria of Eq (E). And we can show that every solution of Eq. (E) converges to one of a point on ℓ as $t \to \infty$:

Theorem 1. (Convergent Point) If the matrix B is irreducible, then every solution $(x_1(t), x_2(t), x_3(t))$ of Eq. (E) with the initial conditions (1) converges to a point $(\tilde{x}, \tilde{x}, \tilde{x}) \in \ell$ as $t \to \infty$, where $\tilde{x}$ is given by

$$\tilde{x} = \frac{\sum_{i=1}^{3} w_i \left\{ \frac{\varphi_i(0)}{\alpha_i} + \int_{-\tau_i}^{0} \varphi_i(s) ds \right\}}{\sum_{i=1}^{3} w_i \left\{ \frac{1}{\alpha_i} + \tau_i \right\}}, \tag{2}$$

and

$$w_i = \tilde{b}_i (b_{i+1i} b_{i-1i} + b_{i+1i} b_{i-1i+1} + b_{i+1i-1} b_{i-1i}). \tag{3}$$

Here the indices $i = 1, 2, 3$ and are counted mod 3, that is, $b_{43} = b_{13}$, $b_{01} = b_{31}$, and so on (the proof is given in Appendix A).

We note that w_i $(i = 1, 2, 3)$ can be chosen positive because B is nonnegative and irreducible matrix. We also note that there exist some different matrices having the same ratio of $w_1 : w_2 : w_3$, that is, the convergent points of solutions starting at the same initial conditions are the same. For example, consider two types of connection topology given by $B = B_9$ or $B = B_{13}$:

$$B_9 = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}, \qquad B_{13} = \begin{pmatrix} 0&1&1 \\ 1&0&1 \\ 1&1&0 \end{pmatrix}.$$

If $B = B_9$, then $b_{ij} = 0$ for $j = i, i+1$ and $b_{ij} = 1$ for $j = i+2$, which imply $w_i = 1 \times (1 \times 0 + 1 \times 1 + 0 \times 1) = 1$ and $w_1 : w_2 : w_3 = 1 : 1 : 1$. If $B = B_{13}$, then $b_{ij} = 0$ for $j = i$ and $b_{ij} = 1$ for $j = i+1, i+2$, which imply $w_i = 2 \times (1 \times 1 + 1 \times 1 + 1 \times 1) = 6$ and $w_1 : w_2 : w_3 = 1 : 1 : 1$.

3 Class of Adjacency Matrices

In previous sections we have seen that there exist some different network topologies having a common convergent point. In this section we will classify the 3-node networks based on the convergent points. So we introduce the following new concept for nonnegative and irreducible matrices.

Definition 2. (Definition of Class $\mathcal{B}_w$) For an $n \times n$ nonnegative and irreducible matrix $B = (b_{ij})$ and a positive vector $w \in \mathbb{R}^n$, B is in a class $\mathcal{B}_w$ if w is a left eigenvector of $\hat{B} := \left(\frac{b_{ij}}{b_i}\right)$ associated with the eigenvalue 1.

We note that the matrix $\hat{B}$ is a stochastic matrix. Then $\hat{B}$ has the eigenvalue 1 and the spectral radius of $\hat{B}$ is 1 (cf. Theorem 15.7.1 in ref. 19). Furthermore, by Perron-Frobenius theorem (cf. Theorem 15.3.1 in ref. 19), the transpose of $\hat{B}$ has an eigenvalue 1 whose algebraic multiplicity is 1 and a positive eigenvector (i.e., $\hat{B}$ has a left eigenvector) w associated with the eigenvalue 1. We also note that when $n = 3$, the elements of $w = (w_1, w_2, w_3)$ satisfy Eq. (3) if and only if $B \in \mathcal{B}_w$.

In 3-node networks there are 13 types of network motif (cf. Fig. 1B in ref. 16). If we assume

$$\alpha_i = \alpha, \quad \tau_i = \tau, \quad (i = 1, 2, 3), \tag{4}$$

then we can only consider 13 types of matrix:

$$B_1 = \begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&1&0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix},\ B_3 = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix},\ B_4 = \begin{pmatrix} 0&1&1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix},\ B_5 = \begin{pmatrix} 0&1&1 \\ 0&0&0 \\ 0&1&0 \end{pmatrix},$$

$$B_6 = \begin{pmatrix} 0&1&1 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}, B_7 = \begin{pmatrix} 0&0&0 \\ 1&0&1 \\ 0&1&0 \end{pmatrix}, B_8 = \begin{pmatrix} 0&1&0 \\ 1&0&1 \\ 0&1&0 \end{pmatrix},\ B_9 = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix},\ B_{10} = \begin{pmatrix} 0&1&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix},$$

$$B_{11} = \begin{pmatrix} 0&1&1 \\ 1&0&1 \\ 0&0&0 \end{pmatrix},\ B_{12} = \begin{pmatrix} 0&1&1 \\ 1&0&1 \\ 0&1&0 \end{pmatrix},\ B_{13} = \begin{pmatrix} 0&1&1 \\ 1&0&1 \\ 1&1&0 \end{pmatrix}.$$

It is obvious that B_8, B_9, B_{10}, B_{12} and B_{13} are irreducible and the other 8 matrices are reducible. Classifying these 5 irreducible matrices by $\mathcal{B}_w$, we have the following:

- $B_8 \in \mathcal{B}_w$ for $w = (1, 2, 1)$.
- B_9 and $b_{13} \in \mathcal{B}_w$ for $w = (1, 1, 1)$.
- $B_{10} \in \mathcal{B}_w$ for $w = (2, 2, 1)$.
- $B_{12} \in \mathcal{B}_w$ for $w = (2, 4, 3)$.

It might be instinctively natural that the connection topologies given by B_9 and B_{13} have a common property such as the convergent points, because all nodes are connected symmetrically. But the insertion degree of each node are 1 and 2 in the case of B_9 and B_{13}, respectively. This means that there are more transmission paths of informations in Case B_{13} than in Case B_9 (see Fig. 1). Thus we can expect that the speed of convergence in the case of B_{13} is faster than in the case of B_9. In the next section we will discuss the matter in detail.

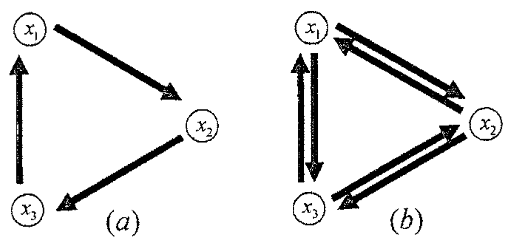

Figure 1: The graphs in Case B_9 (a), Case B_{13} (b).

4 Convergent Speed

In this section we assume (4). As we saw in §2, every solution of Eq. (E) tends to a point on the straight line ℓ in $\mathbb{R}^3$ as $t \to \infty$. In order to define the convergent speed to the line ℓ, let us review the characteristic roots of (E). The characteristic roots are solutions of the characteristic equation of Eq. (E) given by

$$p(\lambda) = \det[(\lambda + \alpha)I - \alpha e^{-\lambda\tau}\hat{B}] = 0, \tag{5}$$

and for a characteristic root λ of Eq. (E) there exists a nonzero vector $v \in \mathbb{R}^3$ such that $x(t) = e^{\lambda t}v$ is a solution of Eq. (E). In this paper we call the vector v *characteristic vector associated with* λ, to avoid the many preparations on the theory of linear delay differential equations given Chapter 7 in ref. 13.

Definition 3. (**Definition of Convergent Speed**) For a characteristic function $p(\lambda)$ of Eq. (E) define a value $s(p)$ as

$$s(p) = \max\{\Re\lambda\colon p(\lambda) = 0 \text{ and } \lambda \notin P_\ell\}.$$

Here $\lambda \in P_\ell$ means that the characteristic vector associated with λ is paralleled to the line ℓ. We call $-s(p)$ the convergent speed to the line ℓ. Let $p_i(\lambda)$ be the characteristic function of Eq. (E) when $B = B_i$. If $s(p_i) < s(p_j)$, then the convergent speed to the line

ℓ in the case $B = B_i$ is faster than in the case $B = B_j$, and we write $CaseB_j \prec CaseB_i$. If $s(p_i) = s(p_j)$, then we write $CaseB_i = CaseB_j$, which means Case B_i is equal to Case B_j in the convergent speed to the line ℓ.

Let us consider the network topologies given by the matrices B_9 and B_{13} in $\mathcal{B}_w$ for $w = (1, 1, 1)$. Although the convergent points in these two cases are the same as we saw in previous sections, we can find the difference in the convergent speed iff transmission delay is positive.

Theorem 4. (**Convergent Speed**) If $\tau = 0$, then $CaseB_9 = CaseB_{13}$. If $\tau > 0$, then $CaseB_9 \prec CaseB_{13}$ (the proof is given in Appendix B).

This result shows that the instinctive conjecture stated in the end of §3 is true iff the transmission delay τ is positive. Therefore, it is significant to consider transmission delays in the study of network dynamics.

5 Discussion

In this paper, we have given a detailed analysis of Eq. (E) with linear transmission functions. However, the transmission function is usually given by a sigmoid function in neural network models, for example $f_j(x) = \tanh(\beta_j x)$. Then, the linearized equations of (E0) near the origin are

$$x_i'(t) = -\alpha_i x_i(t) + \frac{\sum_{j=1}^{n} \beta_j b_{ij} x_j(t - \tau_j)}{\tilde{b}_i}. \tag{6}$$

Clearly Eq. (E) is a special case of Eq. (6) in which $\beta_i = 1$ for $i = 1, 2, 3$. But this case has been avoided in the analysis because the equilibria are not isolated and nonhyperbolic. In this paper we analyze such a special case, and find the convergent point. This result allows us to propose two new concepts, (i) the class of adjacency matrices and (ii) the convergent speed to the line ℓ. This analysis of the special case is crucial to the proposed new concepts, since the equilibrium point is uniquely found at origin if $\beta_i \neq 1$. It is interesting how these concepts play a role when $\beta_i \neq 1$.

In the *network motifs* proposed by Milo *et al.*, the loop (that is, the input from itself) is not considered [16]. However, it seems important to consider the loops mathematically. Actually, the results for local stability, bifurcations and sustained oscillations are reported in neural network models with the loops [9, 20, 21]. In our case if we consider the loops, then we can find 86 network motifs, 30 of them are irreducible. We also find that the number of elements of the class $\mathcal{B}_w$ for $w = (1, 2, 1)$ is four, for $w = (1, 1, 1)$ is six, for $w = (2, 2, 1)$ is four, and $w = (2, 4, 3)$ is two. It is interesting to compare the convergent speeds in these network motifs.

Finally we note that this analysis can be applicable to the networks with more than 4 nodes. How our new concepts function in such more complicated networks is an interesting open question.

A Proof of Teorem 2 (Convergent Point)

Consider the characteristic equation (5) of Eq. (E). Put $\frac{\lambda+\alpha}{\alpha}e^{\lambda\tau} = \mu$ in Eq. (5), then

$$p(\lambda) = (\alpha e^{-\lambda\tau})^3 \det\left[\mu I - \hat{B}\right] = 0.$$

Let μ_1, μ_2, μ_3 be the eigenvalues of $\hat{B}$, then

$$\det[\mu I - \hat{B}] = (\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3),$$

which implies

$$\lambda + \alpha = \mu_i \alpha e^{-\lambda\tau}, \quad i = 1, 2, 3. \tag{7}$$

Because of the fact stated in the paragraph under *Definition of Class* $\mathcal{B}_w$, we can put $\mu_1 = 1$ and find that $|\mu_i| \leq 1$. Assume that $\Re\lambda > 0$, then

$$|\lambda + \alpha| = \alpha|\mu_i||e^{-\lambda\tau}| \leq \alpha|e^{-\lambda\tau}| \leq \alpha.$$

This is impossible, and we obtain $\Re\lambda \leq 0$. Let $\lambda = i\omega$ ($\omega \in \mathbb{R}$). Then we can easily obtain $\omega = 0$ and $\mu_i = 1$. It is easy to check that $\lambda = 0$ is a simple root of Eq. (7) for $i = 1$. Since the algebraic multiplicity of $\mu_1 = 1$ is 1, $\lambda = 0$ is a simple root of Eq. (5). Therefore every solution $x(t) = (x_1(t), x_2(t), x_3(t))$ of Eq. (E) tends to one of equilibria, that is, there exists $\tilde{x} \in \mathbb{R}$ such that

$$\lim_{t\to\infty} x(t) = (\tilde{x}, \tilde{x}, \tilde{x}). \tag{8}$$

To obtain the convergent point, we consider the following functional W:

$$W(x(\cdot))(t) = \sum_{i=1}^{3} w_i \left\{ \frac{x_i(t)}{\alpha_i} + \int_{-\tau_i}^{0} x_i(t+s)ds \right\}$$

for a solution $x(t)$ of Eq. (E) with the initial conditions (1). The derivative of W along Eq. (E) is as follows:

$$\begin{aligned}
\dot{W}_{(E)}(x(\cdot))(t) &= \sum_{i=1}^{3} w_i \left\{ \frac{1}{\bar{b}_i} \sum_{j=1}^{3} b_{ij} x_j(t-\tau_j) - x_i(t-\tau_i) \right\} \\
&= \sum_{i=1}^{3} x_i(t-\tau_i) \left\{ -w_i + \sum_{j=1}^{3} \frac{b_{ji}}{\bar{b}_j} w_j \right\}.
\end{aligned}$$

As we stated in the end of the paragraph under *Definition of Class* $\mathcal{B}_w$, $w = (w_1, w_2, w_3)$ is an eigenvector of transpose of $\hat{B}$ associated with the eigenvalue $\mu_1 = 1$, if w satisfies Eq. (3). Hence we have

$$\dot{W}_{(E)}(x(\cdot)) = 0,$$

that is, $W(x(\cdot))$ is a conserved quantity, which yields

$$W(x(\cdot))(t) = W(x(\cdot))(0) \quad \text{for all } t \geq 0.$$

From the initial condition of $x(t)$ and Eq. (8),

$$\lim_{t\to\infty} W(x(\cdot))(t) = \sum_{i=1}^{3} w_i \left\{\frac{1}{\alpha_i} + \tau_i\right\} \tilde{x} = W(\varphi(\cdot))(0).$$

This completes the proof.

B Proof of Teorem 3 (Convergent Speed)

If $B = B_9$, then the eigenvalues of $\hat{B}$ are $1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}$ $(i^2 = -1)$. If $B = B_{13}$, 1 and double $-\frac{1}{2}$. Thus characteristic equations are given as follows. If $B = B_9$, then

$$p_9(\lambda) := q(\lambda)q_9^+(\lambda)q_9^-(\lambda) = 0,$$

where

$$q(\lambda) := \lambda + \alpha - \alpha e^{-\lambda\tau},$$
$$q_9^{\pm}(\lambda) := \lambda + \alpha - \alpha e^{-\lambda\tau} e^{\pm i\frac{2}{3}\pi}.$$

We note that it is enough to consider the roots of $q_9^+(\lambda) = 0$ because if λ_0 is a root of $q_9^+(\lambda) = 0$, then $\overline{\lambda_0}$ is a root of $q_9^-(\lambda) = 0$, and vice versa. Here $\bar{z}$ denote the complex conjugate of z for a complex number z. If $B = B_{13}$, then

$$p_{13}(\lambda) := q(\lambda)q_{13}(\lambda)^2,$$

where

$$q_{13}(\lambda) := \lambda + \alpha + \frac{1}{2}\alpha e^{-\lambda\tau}.$$

We consider the common factor $q(\lambda) = 0$ in two cases. Using Theorem 5 in ref. 22, we can find that any roots of $q(\lambda) = 0$ except for $\lambda = 0$ are nonreal and have negative real parts. Let λ be a root of $q(\lambda) = 0$ and v be the characteristic vector associated with λ. Then v is an eigenvector of $\hat{B}$ associated with the eigenvalue 1. In each case $v = k(1,1,1)$ for some $k \in \mathbb{R}$ and λ belongs to P_ℓ. On the other hand, let λ be a root of $q_9^+(\lambda) = 0$ (or $q_{13}(\lambda) = 0$) and v be the characteristic vector associated with λ. Then v is an eigenvector of $\hat{B}$ associated with the eigenvalue $e^{i\frac{2}{3}\pi}$ (or $-\frac{1}{2}$) and $\lambda \notin P_\ell$. Therefore the speeds of convergence to the line ℓ in Case B_9 and Case B_{13} are determined by the roots of $q_9^+(\lambda) = 0$ and $q_{13}(\lambda) = 0$, respectively.

If $\tau = 0$, it is easily seen that $s(p_9) = s(p_{13}) = -\frac{3}{2}\alpha$, which implies $CaseB_9 = CaseB_{13}$. In the following, we will show $s(p_{13}) < s(p_9)$ for all $\tau > 0$.

Substituting $\lambda\tau = u + iv$ to $q_9^+(\lambda) = 0$, we have

$$u + \alpha\tau = \alpha\tau e^{-u}\cos(-v + \frac{2}{3}\pi) \tag{9}$$

$$v = \alpha\tau e^{-u}\sin(-v + \frac{2}{3}\pi). \tag{10}$$

It is easy to see that Eqs. (9) and (10) are equivalent to

$$(u + \alpha\tau)^2 + v^2 = (\alpha\tau e^{-u})^2, \tag{9'}$$

$$u + \alpha\tau = v\cot(-v + \frac{2}{3}\pi) \tag{10'}$$

and
$$v\sin(-v + \frac{2}{3}\pi) > 0.$$

From Eqs. (9') and (10'), we have

$$v = \pm\phi(u), \quad \phi(u) := \sqrt{(\alpha\tau e^{-u})^2 - (u+\alpha\tau)^2},$$

and
$$u = \psi(v), \quad \psi(v) := -\alpha\tau + v\cot(-v + \frac{2}{3}\pi),$$

respectively. Analyzing the shapes of the graphs given by $v = \pm\phi(u)$ and $u = \psi(v)$ in u-v plane, we can find that the maximum of u satisfying Eqs. (9) and (10) are given by the intersection of these graphs for $v \in (0, 2\pi/3)$ (see Fig. 2). Representing the maximum of u as u^*, $s(p_9) = u^*/\tau$. We note that $u^* > -\alpha\tau - 1$, because the minimum of $\psi(v)$ for $v \in (0, 2\pi/3)$ is larger than $-\alpha\tau - 1$.

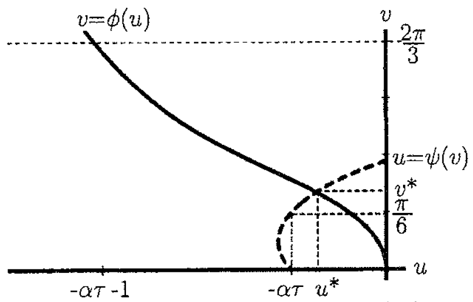

Figure 2: The graphs of Eqs. (9') (solid line) and (10') (dashed line), where $\alpha\tau = 0.5$.

Substituting $\lambda\tau = u + iv$ to $q_{13}(\lambda) = 0$, we have

$$u + \alpha\tau = -\frac{1}{2}\alpha\tau e^{-u}\cos(-v) \tag{11}$$

$$v = -\frac{1}{2}\alpha\tau e^{-u}\sin(-v). \tag{12}$$

When $v = 0$, Eq. (12) is satisfied. From Eq. (11), we obtain

$$h(u) = -\frac{1}{2}\alpha\tau e^{\alpha\tau}, \quad h(u) := (u+\alpha\tau)e^{u+\alpha\tau}. \tag{13}$$

When $v \neq 0$, it is easy to see that Eqs. (11) and (12) are equivalent to

$$(u+\alpha\tau)^2 + v^2 = (\frac{1}{2}\alpha\tau e^{-u})^2, \tag{11'}$$

$$u + \alpha\tau = v\cot(-v) \tag{12'}$$

and
$$v\sin v > 0.$$

From Eqs. (11') and (12'), we have

$$v = \pm\hat{\phi}(u), \quad \hat{\phi}(u) := \sqrt{(\frac{1}{2}\alpha\tau e^{-u})^2 - (u+\alpha\tau)^2},$$

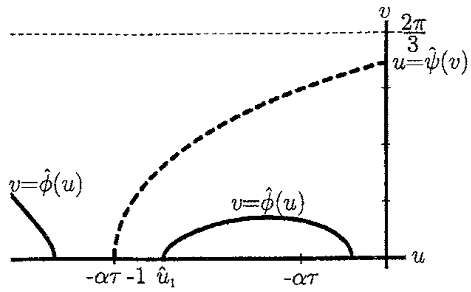

Figure 3: The graphs of Eqs. (11') (solid line) and (12') (dashed line) when $\alpha\tau e^{\alpha\tau} \leq 2/e$. $\hat{u}_1$ is one of the roots of Eq. (13).

and
$$u = \hat{\psi}(v), \quad \hat{\psi}(v) := -\alpha\tau + v\cot(-v).$$

When $\alpha\tau e^{\alpha\tau} \leq 2/e$, Eqs. (11') and (12') have no intersection for $u > -\alpha\tau - 1$ (see Fig. 3). On the other hand there exists a solution $\hat{u}_1 \in [-\alpha\tau - 1, -\alpha\tau)$ of (13). Then, $s(p_{13}) = \hat{u}_1/\tau$. If $u^* \geq -\alpha\tau$, then $s(p_{13}) < s(q_9)$, clearly. If $-\alpha\tau - 1 < u^* < -\alpha\tau$, then from (10') and $v^* \in (0, 2\pi/3)$,

$$0 < v^* < \frac{\pi}{6} \quad \text{and} -\frac{1}{2} < \cos(-v^* + \frac{2}{3}\pi) < 0,$$

where $v^* = \phi(u^*)$. Thus from (9) we have

$$-\frac{1}{2}\alpha\tau e^{\alpha\tau} < h(u^*) < 0,$$

which implies $\hat{u}_1 < u^* < -\alpha\tau$ and $s(p_{13}) < s(q_9)$.

When $\alpha\tau e^{\alpha\tau} > 2/e$, there is no solution of Eq. (13). We can easily see that the maximum of u satisfying Eqs. (11) and (12) are given by the intersection of the graphs of Eqs. (11') and (12') for $v \in (0, \pi)$ (see Fig. 4). Representing the maximum of u as $\hat{u}^*$, $s(p_{13}) = \hat{u}^*/\tau$. We will show $\hat{u}^* < u^*$ in the following. Assume this does not hold.

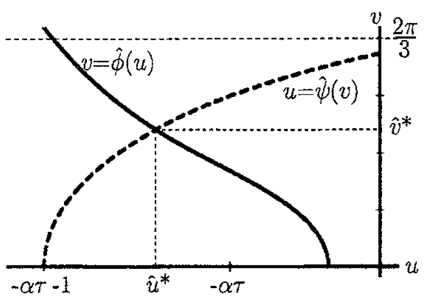

Figure 4: The graphs of Eqs. (11') (solid line) and (12') (dashed line) when $\alpha\tau e^{\alpha\tau} > 2/e$.

Put $v^* = \phi(u^*)$ and $\hat{v}^* = \hat{\phi}(\hat{u}^*)$. It is clear that $\hat{\phi}(\hat{u}^*) < \phi(\hat{u}^*)$. And $\phi(u)$ is monotone decreasing for $u \leq 0$ when $\alpha\tau e^{\alpha\tau} > 2/e$. Thus we have $\hat{v}^* < v^*$. We can easily see $\psi(v^*) > \hat{\psi}(v^*)$. Since $\hat{\psi}(v)$ is monotone increasing for $v > 0$, $\hat{\psi}(v^*) > \hat{\psi}(\hat{v}^*)$. Hence we obtain $u^* = \psi(v^*) > \hat{\psi}(\hat{v}^*) = \hat{u}^*$, which is a contradiction. Therefore $\hat{u}^* < u^*$ and $s(p_{13}) < s(p_9)$. This completes the proof.

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