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Singular Limit Analysis to Higher Dimensional Patterns of a Chemotaxis Growth System

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1 Introduction

We consider the following model equations which describes the movement of the biological individuals by the diffusion and chemotaxis effects in [4, 5];

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = d_u \Delta u - \nabla(u \nabla \chi(v)) + f(u) \\ \\ \dfrac{\partial v}{\partial \tau} = d_v \Delta v + h(u,v) \end{cases} \qquad \tau > 0, \quad \mathbf{x} \in \mathbf{R}^N, \tag{1.1}$$

where $u(\tau, \mathbf{x})$ and $v(\tau, \mathbf{x})$ are respectively the population density and the concentration of chemotactic substance at time τ and position $\mathbf{x} \in \mathbf{R}^N$. d_u and d_v are diffusion rates of u and v. $\nabla \chi(v)$ is the velocity of the direct movement of u due to chemotaxis, which generally satisfies $\chi(v) > 0$ and $\chi'(v) \geq 0$ for $v > 0$. Here we specify the growth term $f(u)$ as $f(u) = (g(u) - \alpha)u$ where $g(u)$ is the growth rate with cooperation and competition effects and α is the degradation rate due to exterior forces such as predation or intoxication. Though the functional form of $f(u)$ is basically classified into several cases depending on $g(u)$ and α, we consider the cubic-like form, which has three roots 0, $\underline{u}$ and $\overline{u}$ of $f(u) = 0$. The term $h(u,v)$ in (1.1) is simply specified as $h(u,v) = \beta u - \gamma v$ with the production rate $\beta > 0$ and the degradation rate $\gamma > 0$.

In [4, 5], we studied (1.1) assuming the situation that the movement of individuals is mainly due to chemotaxis and that the chemotactic substance diffuses so fast compared with the migration of individuals which move by diffusion and chemotaxis, so we introduce a small parameter $\varepsilon > 0$. By using the suitable transformations [4], the equations (1.1) can be rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \varepsilon^2 \Delta u - \varepsilon k \nabla(u \nabla \chi(v)) + f(u) \\ \dfrac{\partial v}{\partial \tau} = \Delta v + u - \gamma v \end{cases} \qquad \tau > 0, \quad \mathbf{x} \in \mathbf{R}^N, \tag{1.2}$$

where k is a positive constant such that $\chi(v)$ is sutably normalized. As was stated above, $f(u)$ satisfies $f(0) = f(a) = f(1) = 0$ for some $0 < a < 1$, $f(u) < 0$ for $0 < u < a$, $f(u) > 0$ for $a < u < 1$ and $f'(0) < 0$, $f'(1) < 0$. Moreover, we assume $\int_0^1 f(u)du > 0$. The boundary and initial conditions are taken to be

$$\lim_{|\mathbf{x}| \to +\infty} (u(\tau, \mathbf{x}), v(\tau, \mathbf{x})) = (0, 0) \qquad \tau > 0 \tag{1.3}$$

and

$$(u(0, \mathbf{x}), v(0, \mathbf{x})) = (u_0(\mathbf{x}), v_0(\mathbf{x})) \qquad \mathbf{x} \in \mathbf{R}^N. \tag{1.4}$$

For (1.2), we show the existence of the nonnegative global solution in2–dimensional domain and the exponential atractor with finite dimension [8].

In [4, 5], the existence and numerical stability of the radially symmetric stationary solutions of (1.2) - (1.4) in $\mathbf{R}^N$ ($N = 1, 2$) are studied for small $\varepsilon > 0$. Moreover, we have the limiting system of (1.2) - (1.4) as $\varepsilon \downarrow 0$ and show that by solving it the stability of the stationary solutions is suggested for small $\varepsilon > 0$.

Since it seems to be different to do the numerical simulation of (1.2) - (1.4) in $\mathbf{R}^3$, we consider the limiting system and by solving this problem we suggest the existence of the realistic stationary patterns in this paper.

In Section 2, we introduce the limiting system as $\varepsilon \downarrow 0$. In Section 3, the existence of radially symmetric stationary solutions of the limiting system in $\mathbf{R}^3$ is shown. In Section 4, we consider the stability of the radially symmetric stationary solutions in $\mathbf{R}^3$ and show the dependency of the parameter k, the forms of $f(u)$ and $\chi(v)$. Through this paper, we treat with the specified forms of $\chi(v)$ and $f(u)$, that is, $\chi(v) = sv^2/(s^2 + v^2)$ and $f(u) = u(1-u)(u-0.1)$, $\varepsilon = 0.05, \gamma = 1$ for the numerical simulations.

2 Limiting System as $\varepsilon \downarrow 0$

In order to study the pattern-dynamics arising in solutions to (1.2) - (1.4) with small $\varepsilon > 0$, we derive the limiting system from (1.2) when $\varepsilon \downarrow 0$. To do it, we introduce the new time variable t with $\tau = t/\varepsilon$. Then (1.2) is rewritten as

$$\begin{cases} \varepsilon u_t = \varepsilon^2 \Delta u - \varepsilon k \nabla(u \nabla \chi(v)) + f(u) \\ \varepsilon v_t = \Delta v + u - \gamma v \end{cases} \qquad t > 0, \quad \mathbf{x} \in \mathbf{R}^N. \tag{2.1}$$

Using the well known two-timing methods, one can intuitively understand that the time evolution of the solution of (2.1) consists of two stages. In the first stage, the solution is approximately described by the following system:

$$\begin{cases} u_t = & \frac{1}{\varepsilon} f(u) \\ & & t > 0, \quad \mathbf{x} \in \mathbf{R}^N. \\ v_t = & \frac{1}{\varepsilon}\{\Delta v + u - \gamma v\} \end{cases} \tag{2.2}$$

Since the system for u is bistable from the assumption of $f(u)$, the solution $u(t, \mathbf{x})$ tends, in short time, to 0 in one region, say $\Omega_{0\varepsilon}$ where $0 \le u_0(\mathbf{x}) < a$, while it tends to 1 in the other region, say $\Omega_{1\varepsilon}$ where $a < u_0(\mathbf{x})$. This implies the occurrence of layer regions, say R_ε, which is the boundary between two regions $\Omega_{0\varepsilon}$ and $\Omega_{1\varepsilon}$, that is, $\mathbf{R}^N$ decomposes into $\mathbf{R}^N = \Omega_{0\varepsilon} \cup \Omega_{1\varepsilon} \cup R_\varepsilon$. In these two subregions, $\Omega_{0\varepsilon}$ and $\Omega_{1\varepsilon}$, the second variable v approximately satisfies the following stationary problems:

$$0 = \Delta v + g_i(v) \qquad \text{in } \Omega_{i\varepsilon} \ (i = 0, 1), \tag{2.3}$$

where $g_0(v) = -\gamma v$ and $g_1(v) = 1 - \gamma v$.

In the second stage, the solution is no longer described by (2.2), (2.3) so that the layer regions must change. This means that $\Omega_{0\varepsilon}$, $\Omega_{1\varepsilon}$ and R_ε vary as time goes on. We now assume the situation in the limit $\varepsilon \downarrow 0$ such that there is an $(N-1)$-dimensional hypersurface $\Gamma(t)$, which means the interface of u, in $\mathbf{R}^N$ such that $R_\varepsilon(t) \to \Gamma(t)$ holds as $\varepsilon \downarrow 0$, that is, $\mathbf{R}^N = \Omega_0(t) \cup \Omega_1(t) \cup \Gamma(t)$ where $\Omega_{i\varepsilon} \to \Omega_i(t) = \{\mathbf{x} \in \mathbf{R}^N, u(t, \mathbf{x}) = i\}$ $(i = 0, 1)$. Letting V^* be the normal velocity of the intrface $\Gamma(t)$, we can derive the equation to describe the dynamics of $\Gamma(t)$ as follows (see [10]):

$$\begin{cases} V^* = c^* + k\chi'(v)\frac{\partial v}{\partial n} - \varepsilon(N-1)\kappa + \varepsilon G & t > 0, \quad \mathbf{x} \in \Gamma(t), \\ 0 = \Delta v + g_i(v) & t > 0, \quad \mathbf{x} \in \Omega_i(t), \end{cases}$$

where n means the outerward unit normal vector from $\Omega_1(t)$ to $\Omega_0(t)$ on $\Gamma(t)$, κ is the mean curvature at the interface. Here, c^* is the velocity of the traveling front solution of the scalar bistable reaction diffusion equation (see [3]). Although $G = O(1)$ for small ε in general, we neglect this term in order to study the effect of the curvature to the motion of the interface at the first step. Therefore, the equation is rewritten as

$$\begin{cases} V^* = c^* + k\chi'(v)\frac{\partial v}{\partial n} - \varepsilon(N-1)\kappa & t > 0, \quad \mathbf{x} \in \Gamma(t), \\ 0 = \Delta v + g_i(v) & t > 0, \quad \mathbf{x} \in \Omega_i(t), \end{cases} \tag{2.4}$$

which we call the *singular limit system* or simply the *interface equation* of (2.1). The smoothness of v on the interface Γ is imposed to satisfy $v \in C^1$, that is,

$$v(t, \cdot) \in C^1(\mathbf{R}^N) \qquad t > 0. \tag{2.5}$$

It clearly shows that the dynamics of the interface is determined by three effects; the velocity of the 1-dimensional traveling front solution, the chemotactic effect due to the gradient of $\chi(v)$ and the geometric effect of the interface. Moreover, from (1.3), we assume that

$$\lim_{|\mathbf{x}|\to\infty} v(t,\mathbf{x}) = 0, \quad t > 0. \tag{2.6}$$

In the previous paper [4], we show the existence of radially symmetric stationary solutions $(u(r), v(r))$ of the interface equation (2.4)–(2.6) in $\mathbf{R}^N$ ($N = 1, 2, 3$) with $|\mathbf{x}| = r$ where the center and the interface locate at the origin and $r = \eta$, respectively. Moreover, the stability of these solutions was discussed for $N = 1, 2$.

Bonami et al. [1] treated with the case where the equation for v is stationary and the potentials of two equilibria $(0,0)$ and $(1, 1/\gamma)$ are almost all same, that is, c^*, effects of chemotaxis and curvature are same of order with respect to ε. In this situation, the solution of the interface equation is good approximation to one of the original reaction–diffusion equation.

3 Existence of the radially symmetric stationary solutions in $\mathbf{R}^3$

In this section, we consider the existence of a radially symmetric stationary solution of the interface eqaution (2.4)–(2.6). In order to show that, we first treat with the following problem:

$$\begin{cases} 0 = c^* + k\chi'(v)v_r - \dfrac{(N-1)\varepsilon}{r}, & r = \eta \\[2ex] 0 = v_{rr} + \dfrac{N-1}{r} v_r + g_i(v), & r \in \Omega_i, \quad (i = 0, 1) \\[2ex] v_r(0) = 0, \quad \lim\limits_{r\to\infty} v(r) = 0 & \text{and} \quad v \in C^1(\mathbf{R}_+), \end{cases} \tag{3.1}$$

where $|\mathbf{x}| = r$, $\Omega_1 = (0, \eta)$ and $\Omega_0 = (\eta, \infty)$.

Then, the solutions $(\eta, v(r; \eta))$ except for the first equations of (3.1) for $N = 3$ is described by

$$v(r;\eta) \equiv \begin{cases} \dfrac{1}{\gamma} + (\alpha - \dfrac{1}{\gamma}) \dfrac{\eta \sinh \sqrt{\gamma} r}{r \sinh \sqrt{\gamma}\eta} & r \in (0, \eta) \\[2ex] \dfrac{\alpha\eta}{r} e^{-\sqrt{\gamma}(r-\eta)} & r \in (\eta, \infty) \end{cases} \tag{3.2}$$

with $\alpha = v(\eta; \eta) = \eta \mathbf{K}_{\frac{1}{2}}(\sqrt{\gamma}\eta) \mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta)/\sqrt{\gamma}$. Substituting (3.2) into the first equation in (3.1), we obtain

$$c^* - k\chi' \left(\frac{\eta \mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta) \mathbf{K}_{\frac{1}{2}}(\sqrt{\gamma}\eta)}{\sqrt{\gamma}} \right) \eta \mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta) \mathbf{K}_{\frac{3}{2}}(\sqrt{\gamma}\eta) - \frac{2\varepsilon}{\eta} \equiv H(\eta, k, \varepsilon) = 0. \tag{3.3}$$

By using the solution η of $H(\eta, k, \varepsilon) = 0$, one easily finds that the solution of (3.1) is represented by $(\eta, v(r;\eta))$.

Theorem 1. [4] *Let $k^* > 0$ be a constant to satisfy $c^* - \frac{k^*}{2\sqrt{\gamma}}\chi'(\frac{1}{2\gamma}) = 0$. For fixed small $\varepsilon > 0$, there exists a constant $\overline{k}(\varepsilon)$ ($> k^*$) such that for $k^* < k < \overline{k}(\varepsilon)$ there are at least two solutions $(\overline{\eta}, v(r;\overline{\eta}))$ and $(\underline{\eta}, v(r;\underline{\eta}))$ such that $\overline{\eta} = O(1)$ and $\underline{\eta} = O(\varepsilon)$, for $0 < k < k^*$, there are at least one solution $(\underline{\eta}, v(r, \underline{\eta}))$ with $\underline{\eta} = O(\varepsilon)$ for $N = 3$, respectively.*

Letting $\eta = \eta(k)$ be a solution of (3.2), we define the pair of functions $(u^0(r), v^0(r))$ by

$$\begin{cases} u^0(r) = \begin{cases} 1 & r \in (0, \eta) \\ 0 & r \in (\eta, \infty), \end{cases} \\ \\ v^0(r) = v(r;\eta) \qquad r \in (0, \infty) \end{cases} \tag{3.4}$$

and call it a *radially symmetric stationary solution* of the interface equation (2.4)–(2.6) for $N = 3$, respectively.

Next, as $\chi(v) = sv^2/(s^2 + v^2)$, we draw numerically the global picture of radially symmetric stationary solutions of (3.1) for $N = 3$ when k is varied in Figure 1. In this case, there is a critical value $s^* > 0$ of a parameter s of $\chi(v)$ such that for (i) $0 < s < s^*$, there are three branches, while for (ii) $s^* < s$, there are two ones when k is varied. In Figure 2, the existence region of the solution is shown in the (k, s) - plane for $N = 3$.

On the other hands, by the numerical simulations it does not able to suggest which stationary solution in $\mathbf{R}^3$ is realistic till now. Therefore, from the theoretical view point, we consider the stability of the radially symmetric stationary solutions of the interface equation in the 3–dimensional domain in the next section. Moreover, it is shown that the stationary solutions $(\underline{\eta}, v(r;\underline{\eta}))$ are at least unstable with respect to the disturbances of radial direction.

4 Stability of the radially symmetric stationary solutions in $\mathbf{R}^3$

In this section, we show the stability of the radially symmetric stationary solution of (2.4)–(2.6) for $N = 3$, which satisfies $\eta = O(1)$ for small $\varepsilon > 0$.

To study the stability, we represent deformations of the interface $r = \eta$ by the polar coordinate $(r, \theta, \varphi) = (\eta + \zeta(t, \theta, \varphi), \theta, \varphi)$ with the azimuthal angle (θ, φ), where u takes 1 for $(r, \theta, \varphi) \in (0, \eta + \zeta(t, \theta, \varphi)) \times (0, \pi) \times (0, 2\pi)$, while u takes 0 for $(r, \theta, \varphi) \in (\eta + \zeta(t, \theta, \varphi), \infty) \times (0, \pi) \times (0, 2\pi)$. For $\Gamma = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$, it follows from the first equation in (2.4) that

$$\frac{r\sin\theta}{\sqrt{r^2\sin^2\theta + \zeta_\varphi^2 + \zeta_\theta^2\sin^2\theta}}\zeta_t = c + k\chi'(v)\frac{v_r r^2\sin^2\theta - v_\theta\sin^2\theta\zeta_\theta - v_\varphi\sin^3\theta\zeta_\varphi}{r\sin\theta\sqrt{r^2\sin^2\theta + \zeta_\varphi^2 + \zeta_\theta^2\sin^2\theta}} - 2\varepsilon\kappa + O(\zeta^2).$$

By using the balance of the above equation with respect to lower parts of ζ and their derivatives, it holds that

$$\begin{aligned}\zeta_t = \ & k\left\{\chi'(v_0)\left(v_r^{(1)}+v_r^{(2)}\right)+\chi''(v_0)v_{0r}\left(v^{(1)}+v^{(2)}\right)\right\} \\ & +2\varepsilon\left(\frac{\zeta}{\eta^2}+\frac{\zeta_{\theta\theta}}{2\eta^2}+\frac{\zeta_{\varphi\varphi}}{2\eta^2\sin^2\theta}+\frac{\zeta_\theta\cos\theta}{2\eta^2\sin\theta}\right)+O(\zeta^2).\end{aligned} \tag{4.1}$$

Defining the completely orthonormal system $\{Y_{\ell,m}(\theta,\varphi)\}$ on the sphere by

$$Y_{\ell,m}(\theta,\varphi)=\sqrt{\frac{(\ell-|m|)!(2\ell+1)}{(\ell+|m|)!}}P_\ell^m(\cos\theta)\exp(-im\varphi)$$

where $P_\ell^m(x)=(1-x^2)^{\frac{|m|}{2}}\dfrac{d^{|m|}}{dx^{|m|}}P_\ell(x)$ and $P_\ell(x)=\dfrac{1}{2^\ell\ell!}\dfrac{d^\ell}{dx^\ell}(x^2-1)^\ell$, we have

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi\varepsilon\left\{\frac{2\zeta}{\eta_0^2}+\frac{\zeta_{\theta\theta}}{\eta^2}+\frac{\zeta_{\varphi\varphi}}{\eta_0^2\sin^2\theta}+\frac{\zeta_\theta\cos\theta}{\eta_0^2\sin\theta}\right\}Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi$$

$$=-\frac{\varepsilon}{\eta^2}(\ell+2)(\ell-1)\zeta_{\ell,m}(t),$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi v^{(1)}Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi=\frac{d}{dr}v_0(\eta)\zeta_{\ell,m}(t),$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi v^{(2)}Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi=\eta\mathbf{I}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\zeta_{\ell,m}(t),$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi \frac{\partial}{\partial r}v^{(1)}Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi$$

$$=-\frac{1}{3}\left[\frac{1}{2}+\sqrt{\gamma}\eta\left(\mathbf{I}_{\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac{3}{2}}(\sqrt{\gamma}\eta)-2\mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac{5}{2}}(\sqrt{\gamma}\eta)\right)\right]\zeta_{\ell,m}(t),$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi \frac{\partial}{\partial r}v^{(2)}Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi$$

$$=\frac{1}{2\ell+1}\left[\frac{1}{2}+\sqrt{\gamma}\eta\left(\ell\mathbf{I}_{\ell-\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\ -\ (\ell+1)\mathbf{I}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac{3}{2}}(\sqrt{\gamma}\eta)\right]\zeta_{\ell,m}(t)\right.$$

where $\zeta_{\ell,m}(t)=\dfrac{1}{4\pi}\displaystyle\int_0^{2\pi}\int_0^\pi\zeta(\theta,\varphi,t)Y_{\ell,m}(\theta,\varphi)\sin\theta d\theta d\varphi$.

It follows from (4.1) that

$$\begin{aligned}
\frac{d}{dt}\zeta_{\ell,m} = \; & \Big\{k\chi''(v_0)v_{0r}\left[v_{0r}+\eta \mathbf{I}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\right] \\
& -k\chi'(v_0)\left[\frac{1}{3}\left[\frac{1}{2}+\sqrt{\gamma}\eta\left(\mathbf{I}_{\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac{3}{2}}(\sqrt{\gamma}\eta)-2\mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac{5}{2}}(\sqrt{\gamma}\eta)\right)\right]\right. \\
& -\frac{1}{2\ell+1}\left[\frac{1}{2}+\sqrt{\gamma}\eta\left(\ell \mathbf{I}_{\ell-\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\right.\right. \\
& \left.\left.\left. -\;(\ell+1)\mathbf{I}_{\ell+\frac{1}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\ell+\frac{3}{2}}(\sqrt{\gamma}\eta)\right)\right]\right] - \frac{\varepsilon}{\eta^2}(\ell+2)(\ell-1)\Big\}\zeta_{\ell,m}(t) \\
\equiv \; & F(\ell,k,\varepsilon)\zeta_{\ell,m}(t),
\end{aligned} \tag{4.2}$$

where $v_{0r} = -\eta \mathbf{I}_{\frac{3}{2}}(\sqrt{\gamma}\eta)\mathbf{K}_{\frac{3}{2}}(\sqrt{\gamma}\eta)$.

Definition (Linearlized stability of the stationary solution) *If $F(\ell,k,\varepsilon) < 0$ for all $\ell \in \mathrm{N}$ ($\ell > 1$), then the stationary solution $(\eta, v(r;\eta))$ is stable. If not, the solution is unstable.*

Remark 1. (i) $F(0,k,\varepsilon) = \frac{\partial}{\partial\eta}H(\eta;k,\varepsilon)$, *that is, the stability of the stationary solution under the radially symmetric disturbances is determined by the sign of* $\frac{\partial}{\partial\eta}H(\eta;k,\varepsilon)$.
(ii) $F(1,k,\varepsilon) = 0$, *that is, the stationary solution has phase shift free in* (2.4).

Next, we numerically treat with the functional form of $F(\ell,k,\varepsilon) = F(\ell,k,\varepsilon,s)$. In Figure 3, the curves of $F(\ell,k,\varepsilon,s) = 0$ for $\ell = 2,3,4$ is shown in the (k,s)-plane. For small $s > 0$, the solution is stable and with any fixed $s > 0$, the solution is so for large $k > 0$. Figure 4 show the form of $F(\ell,k,\varepsilon,s)$ for $s = 0.3, 0.5, 0.6, 1.0$ and $\ell = 2,3,4$. It is known that these above results are similar as that of the case for $N = 2$ in [4].

Proposition 1. (Asymptotic behavior of $F(\ell,k,\varepsilon)$) *It holds that*

$$\lim_{k\to k^*}\left\{F(\ell,k,\varepsilon)\overline{\eta}^2 + (\ell+2)(\ell-1)F^*(\varepsilon)\right\} = 0,$$

where $F^*(\varepsilon) = \varepsilon - k^*\chi''(\frac{1}{2\gamma})/(8\gamma^2)$.

Proof. Because of $\lim_{k\to k^*}\overline{\eta} = \infty$, we can prove this proposition from (4.2) by using the asymptotic behavior of the modified Bessel functions $\mathrm{I}_{\ell+\frac{1}{2}}(z)$ and $\mathrm{K}_{\ell+\frac{1}{2}}(z)$ as z tends to infinity.

Remark 2. *If $F^*(\varepsilon) > 0$, it follws from the proposition that for any integer $\ell > 1$, it holds $F(\ell,k,\varepsilon)\overline{\eta}^2 < 0$, that is, the stationary solution becomes stable as k tends to k^*.*

As k tends to k^*, it holds that if $F^*(\varepsilon) > 0$, then

$$0 > F(2,k,\varepsilon) > F(3,k,\varepsilon) > \cdots > F(\ell,k,\varepsilon) > F(\ell+1,k,\varepsilon) > \cdots,$$

if $F^*(\varepsilon) < 0$, *then*

$$0 < F(2,k,\varepsilon) < F(3,k,\varepsilon) < \cdots < F(\ell,k,\varepsilon) < F(\ell+1,k,\varepsilon) < \cdots.$$

For the numerical simulation, it holds that $F^*(\varepsilon) < 0$ *for* $0.98\cdots < s < 5.45\cdots$.

On the other hands, it is suggested that $F_3(\ell,k,\varepsilon) < 0$ for $\ell > 1$ as k tends to $\overline{k}(\varepsilon)$ in Figure 4. Since $\overline{k}(\varepsilon)$ is the turning point of the global branch of the stationary solution, we may assume that $\overline{\eta}(\varepsilon)$ becomes of order ε for small ε as k tends to $\overline{k}(\varepsilon)$ from Theorem 1. Then, we have

$$\frac{F(\ell,k,\varepsilon)}{\overline{\eta}(\varepsilon)} = -\frac{(\ell+2)(\ell-1)\mu}{\varepsilon} + O(1) \tag{4.3}$$

for some positive constant μ. Therefore, as k tends to $\overline{k}(\varepsilon)$, it follows from the (4.3) that

$$0 > F(2,k,\varepsilon) > F(3,k,\varepsilon) > \cdots > F(\ell,k,\varepsilon) > F(\ell+1,k,\varepsilon) > \cdots.$$

In this paper, we do not discuss the relation of the solutions between the interface equation (2.4) and the original reaction–diffusion equation (1.2). That is, the solution of (2.4) becomes the good approximation of the solution of (1.2). Moreover, there is the problem such that the asymptotic behavior of the critical eigenvalues of the linearized eigenvalue problem of (1.2) is represented by using $F(\ell,k,\varepsilon)$ (see [9]).

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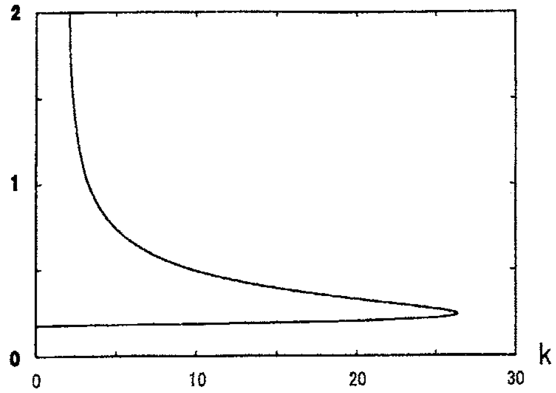
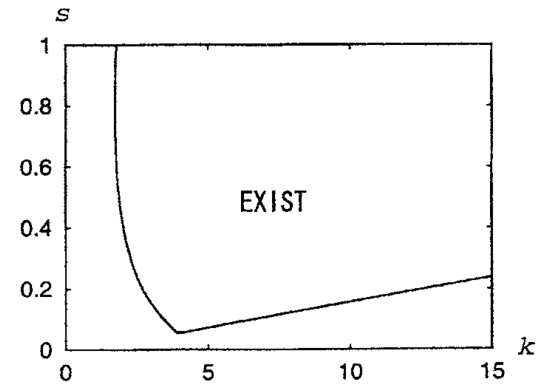


3-dim. symmetric stationary solution(s = 0.42)

Figure 1



Radially symmetric stationary solutions

Figure 2

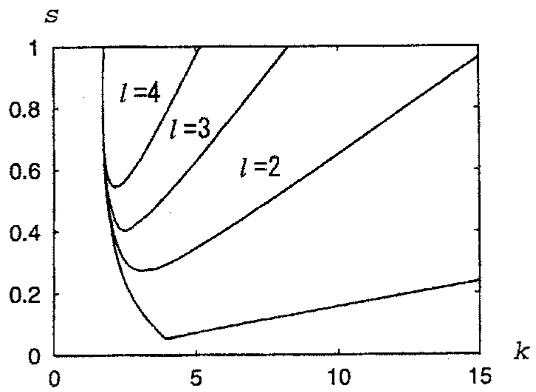


Figure 3.1

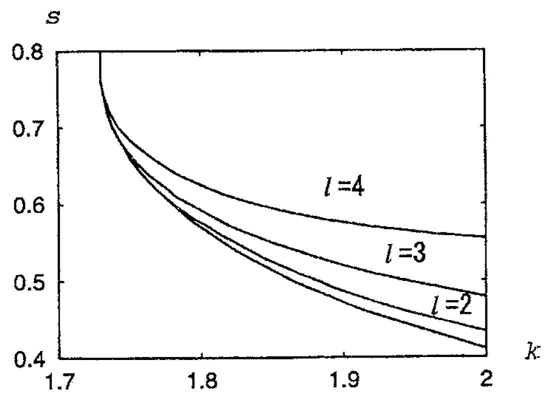


Figure 3.2

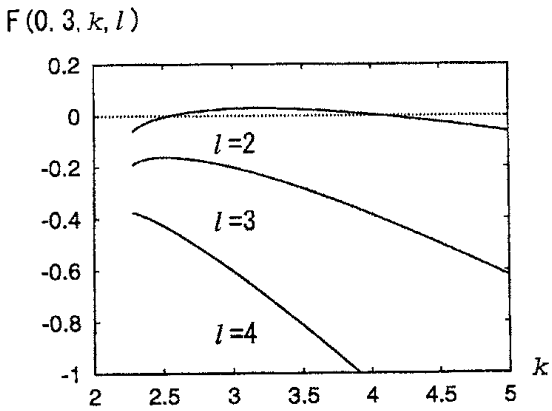


Figure 4.1

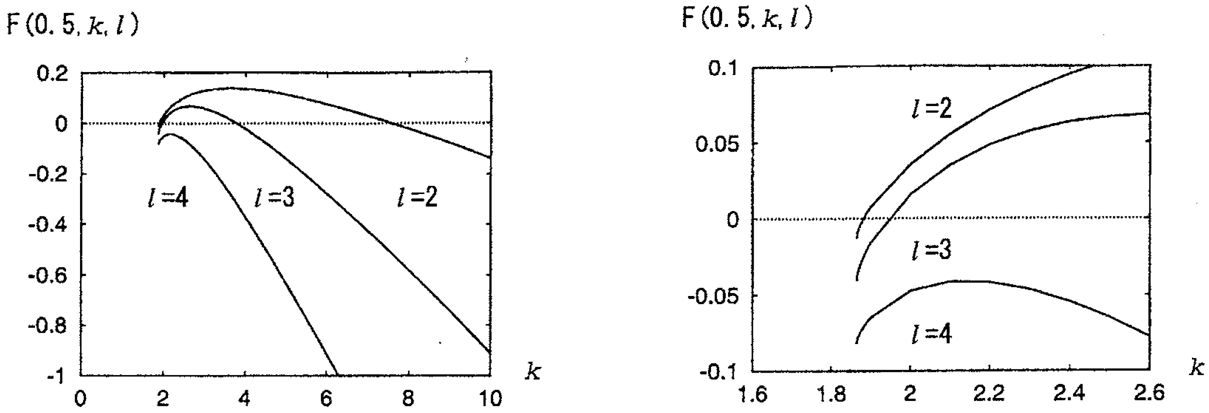


Figure 4. 2

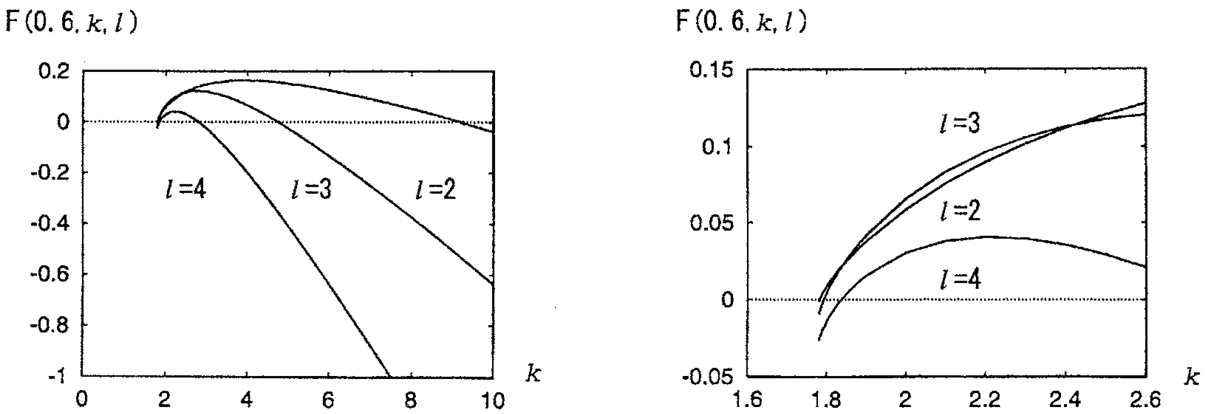


Figure 4. 3

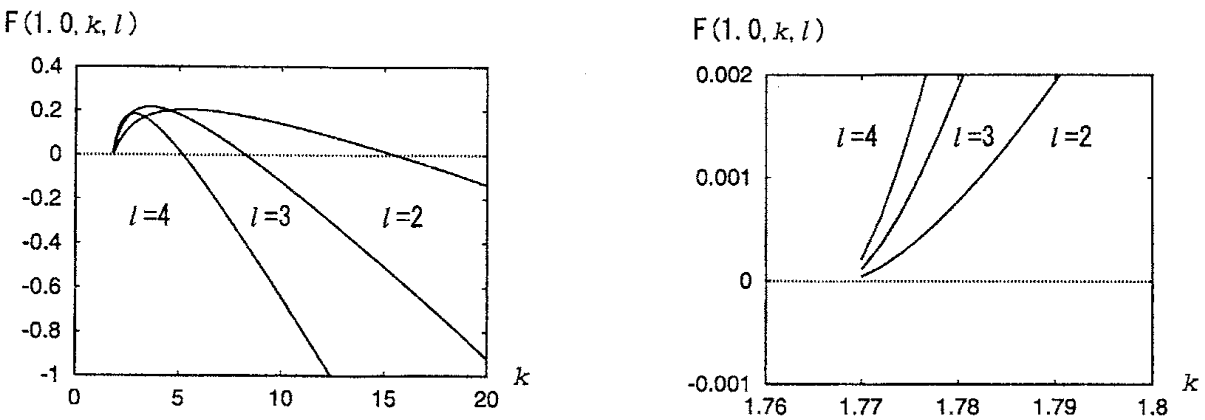


Figure 4. 4