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A KINETIC APPROACH TO A COMPARISON THEOREM FOR DEGENERATE PARABOLIC EQUATIONS

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1. Statement of the Result.

Let Ω be an open bounded subset of $\mathbb{R}^d$ and $T \in (0, +\infty]$. Let Q denote the set $(0,T) \times \Omega$, $\partial\Omega$ the boundary of Ω, $\mathbf{n}(\bar{x})$ the outerward unit normal to Ω at a point $\bar{x} \in \partial\Omega$ and Σ the set $(0,T) \times \partial\Omega$. We consider the following parabolic-hyperbolic problem:

$$\partial_t u + \operatorname{div} A(u) - \Delta\beta(u) = 0 \quad \text{in} \quad Q \tag{1.1}$$

with the initial condition:

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

and the boundary condition:

$$u(t,x) = u_b(t,x), \quad (t,x) \in \Sigma, \tag{1.3}$$

where the flux function A belongs to $C^1(\mathbb{R})$ and the function β is nondecreasing and Lipschitz continuous. This monotonicity assumption of β allows us some degenerate diffusion cases which appear in many interesting models, for example, filtration problems in porous media [2,5,8].

In the nondegenerate case (in which the function β is strictly increasing), the problem (1.1) is of parabolic type and hence the existence and uniqueness of solutions are well known. In the case where $\beta' \equiv 0$, the problem (1.1) being a nonlinear hyperbolic problem, the uniqueness of weak solutions is not ensured, and one must consider a notion of entropy solution, relying on the notion of boundary entropy-flux pairs to recover uniqueness (see [11,16]). When β is merely a nondecreasing function, in the case of homogeneous boundary data, i.e., $u_b \equiv 0$, Carrillo [3] succeeded in proving the uniqueness of entropy solutions by mainly using the dedoubling variable technique developed by Kružkov [11]. The equivalence of entropy solutions and weak solutions is also considered in [10]. In the case of nonhomogeneous boundary data existence and uniqueness of entropy solutions to (1.1)-(1.3) have been proved in [1,14,15]. The method used there is also the dedoubling variable technique.

On the other hand Perthame [12,17] proved the uniquness of entropy solutions to the Cauchy problem of the conservation law (in which $\beta' \equiv 0$ and $\Omega = \mathbb{R}^d$) by using the kinetic formulation which is introduced by Lions,Perthame and Tadmor [12], without relying on the dedoubling variable technique. Imbert and Vovelle [9] developed analogous techniques for conservation laws with boundary conditions, proved the Comparison Theorem for entropy sub- and supersolutions, and applied their results to the BGK-like model. This technique was also applied in [6] to study the parabolic approximation of a multidimensional conservation law with initial and boundary conditions.

The purpose of this note is to give a comparison result for their sub- and supersolutions by using kinetic techniques. Although the L^1 contractivity and, therefore, uniquness of entropy weak solutions has been obtained, it would seem that any comparison theorem for those solutions is not proven.

According to [14] we introduce the definition of entropy sub- and supersolution.

Define

$$\operatorname{sgn}^+(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r \le 0, \end{cases} \qquad \text{and} \qquad \operatorname{sgn}^-(r) = \begin{cases} -1 & \text{if } r < 0, \\ 0 & \text{if } r \ge 0, \end{cases}$$

and $r^\pm = \operatorname{sgn}^\pm(r) r$.

Definition 1.1. *A function u of $L^1(Q)$ is said to be a weak solution of the problem (1.1) - (1.3) if it satisfies: $\beta(u) - \beta(u_b) \in L^2(0,T; H_0^1(\Omega))$, $A(u) \in L^1(Q)^d$ and*

$$\int_Q u\varphi_t + (A(u) - \nabla\beta(u)) \cdot \nabla\varphi dx dt + \int_\Omega u_0 \varphi(0,x) dx = 0 \tag{1.4}$$

for any $\varphi \in C_c^\infty([0.T) \times \Omega)$.

Definition 1.2. *Let $u \in L^\infty(Q)$. u is said to be an entropy subsolution of (1.1) - (1.3) if it is a weak solution and satisfies:*

$$\int_Q (u-\kappa)^+\partial_t\varphi + (\mathcal{F}^+(u,\kappa) - \nabla(\beta(u)-\beta(\kappa))^+)\cdot\nabla\varphi dxdt$$
$$+\int_\Omega (u_0-\kappa)^+\varphi(0,x)dx + M\int_\Sigma (u_b-\kappa)^+\varphi d\sigma dt \geq 0 \tag{1.5}$$

for any $\kappa \in \mathbb{R}$ and any $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)^+$ such that $\operatorname{sgn}^+(\beta(u_b) - \beta(\kappa))\varphi = 0$ a.e. on Σ.

u is said to be an entropy supersolution if (1.7) is replaced by

$$\int_Q (u-\kappa)^-\partial_t\varphi + (\mathcal{F}^-(u,\kappa) - \nabla(\beta(u)-\beta(\kappa))^-)\cdot\nabla\varphi dxdt$$
$$+\int_\Omega (u_0-\kappa)^-\varphi(0,x)dx + M\int_\Sigma (u_b-\kappa)^-\varphi d\sigma dt \geq 0 \tag{1.6}$$

for any $\kappa \in \mathbb{R}$ and any $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)^+$ such that $\operatorname{sgn}^-(\beta(u_b) - \beta(\kappa))\varphi = 0$ a.e. on Σ.Here $C_c^\infty([0,T)\times\mathbb{R}^d)^+$ is the set of nonnegative functions in $C_c^\infty([0,T)\times\mathbb{R}^d)$.

We also set

$$M = \sup\{|A'(r)|; |r| \leq \max\{\|u_0\|_{L^\infty(\Omega)}, \|u_b\|_{L^\infty(\Sigma)}\} \tag{1.7}$$

and

$$L = \max_{1\leq i\leq N} \|\Delta_{\bar{x}} h_i(\overline{T_i x})\|_{L^\infty(\Sigma_{\lambda_i})}. \tag{1.8}$$

We are now in a position to state the main theorem which obviously extends the L^1 contractive property for entropy solutions

Theorem *Assume that the following conditions hold:*

(A1) *Ω is a bounded open subset of $\mathbb{R}^d$ whose boundary $\partial\Omega$ is C^2, $A \in C^1(\mathbb{R},\mathbb{R})$ and $\beta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz continuous function.*

(A2) *$u_0 \in L^\infty(\Omega)$ and $u_b \in L^\infty(\Sigma)$.*

Let $u \in L^\infty(Q)$ be an entropy subsolution of (1.1) - (1.3) with data (u_0, u_b) and let $\tilde{u}$ be an entropy supersolution of (1.1) - (1.3) with data $(\tilde{u}_0, \tilde{u}_b)$. Then we have

$$\frac{1}{T}\int_0^T\int_\Omega (u(t,x)-\tilde{u}(t,x))^+dxdt$$
$$\leq \int_\Omega (u_0(x)-\tilde{u}_0(x))^+dx + M\int_0^T\int_{\partial\Omega}(u_b(t,x)-\tilde{u}_b(t,x))^+d\sigma dt$$
$$+\frac{L}{2}\int_0^T\int_{\partial\Omega}(\beta(u_b(t,x)) - \beta(\tilde{u}_b(t,x)))^+d\sigma dt. \tag{1.9}$$

2. Sketch of Proof.

The semi-Kružkov entropies are the convex functions defined by

$$\eta_k^{\pm}(r) = (r-k)^{\pm}, \qquad k \in \mathbb{R},$$

while the corresponding entropy flux are the function defined by

$$\mathcal{F}^{\pm}(r,k) = \operatorname{sgn}^{\pm}(r-k)(A(r)-A(k)).$$

For a function $u \in L^{\infty}(Q)$ and $\xi \in \mathbb{R}$ we set

$$f_{\pm}(t,x,\xi) = \operatorname{sgn}^{\pm}(u(t,x)-\xi).$$

We assume that Ω is a C^2 bounded open subset in $\mathbb{R}^d$. Thus, we can find a finite open cover $\{B_i\}_{i=0}^N$ of $\overline{\Omega}$ and a partition of unity $\{\lambda_i\}_{i=0}^N$ on $\overline{\Omega}$ subordinate to $\{B_i\}_{i=0}^N$ such that, for $i \geq 1$, up to a change of coordinates represented by an orthogonal matrix T_i, the set $\Omega \cap B_i$ is the epigraph of a C^2 function $h_i : \mathbb{R}^{d-1} \to \mathbb{R}$, that is to say:

$$\Omega_{\lambda_i} \cap B_i = \{x \in B_i; (T_i x)_d > h_i(\overline{T_i x})\}$$

and

$$\partial\Omega_{\lambda_i} = \partial\Omega \cap B_i = \{x \in B_i; (T_i x)_d = h_i(\overline{T_i x}),$$

where $x = (\bar{x}, x_d) \in \mathbb{R}^d$ and $\bar{x} = (x_1, \cdots, x_{d-1})$. For simplicity we will drop the index i and we suppose that the change of coordinates is trivial:$Y_i = Id$. We also write $Q_\lambda = (0,T) \times \Omega_\lambda$, $\Sigma_\lambda = (0,T) \times \partial\Omega_\lambda$, $\Pi_\lambda = \{\bar{x}; x \in \operatorname{supp}(\lambda) \cap \Omega\}$ and $\Theta_\lambda = (0,T) \times \Pi_\lambda$. We denote by $\mathbf{n}(\bar{x})$ the outward unit normal to Ω_λ at a point $(\bar{x}, h(\bar{x}))$ of $\partial\Omega_\lambda$ and by $d\sigma(\bar{x})$ the $(d-1)$- dimensional area element in $\partial\Omega_\lambda$:

$$\mathbf{n}(\bar{x}) = (1 + |\nabla_{\bar{x}} h(\bar{x})|^2)^{-1/2}(\nabla_{\bar{x}} h(\bar{x}), -1),$$
$$d\sigma(\bar{x}) = (1 + |\nabla_{\bar{x}} h(\bar{x})|^2)^{1/2} d\bar{x}.$$

To regularize the functions, for small $\rho, s > 0$ let us consider a smooth function $\theta_{\rho,s} : \mathbb{R} \to \mathbb{R}^+$ such that $\operatorname{supp} \theta_{\rho,s} \subset [\rho s/2, (1+\rho)s]$, $\theta_{\rho,s}(r) = s^{-1}$ for $\rho s \leq r \leq s$ and $\int_{\mathbb{R}} \theta_{\rho,s}(r)dr = 1$. Then, for $\nu > 0$ and $\epsilon = (\epsilon_1, \cdots, \epsilon_d) \in (\mathbb{R}^+)^d$, we set $\gamma_{\rho,\epsilon}(x) = \prod_{i=1}^d \theta_{\rho,\epsilon_i}(x_i)$ and $\gamma_{\rho,\nu,\epsilon}(t,x) = \theta_{\rho,\nu}(t)\gamma_{\rho,\epsilon}(x)$.

For simplicity, we wii also use the following notations:

$$\mathbf{n}_1 = \sqrt{1 + |\nabla_{\bar{x}} h(\bar{x})|^2} \quad \mathbf{n},$$
$$\bar{x}_r = (\bar{x}, h(\bar{x}) + r) \qquad \text{for } \bar{x} = (x_1, \cdots, x_{d-1}),$$

ψ^λ stands for $\psi\lambda$ and $\overline{\psi}$ denotes the restriction of ψ to $\Sigma \times \mathbb{R}_\xi$, i.e., $\overline{\psi}(t,\bar{x},\xi) = \psi(t,\bar{x},h(\bar{x},\xi))$, where ψ is a function on $[0,T) \times \mathbb{R}^{d+1}$ and λ is an element of the partition of unity $\{\lambda_i\}_{i=0}^N$. Moreover we set $s \vee t = \max\{s,t\}$ and $s \wedge t = \min\{s,t\}$.

The proof of the theorem will follow from the following three lemmas whose proofs will be given in the forthcoming paper.

Lemma 2.1. *Let u be an entropy subsolution with data (u_0, u_b) and let λ be an element of the partition of unity $\{\lambda_i\}_{i=0}^N$. Then we have:*
(a) There exists $f_+^{\tau_0} \in L^\infty(\Omega \times \mathbb{R})$ such that

$$\lim_{s\to+0} \int_{\Omega\times\mathbb{R}} \left[\frac{1}{s}\int_0^s f_+(t,x,\xi)dt\right] \phi \, dxd\xi = \int_{\Omega\times\mathbb{R}} f_+^{\tau_0}\phi \, dxd\xi \tag{2.1}$$

for any $\phi \in C_c^\infty(\Omega \times \mathbb{R})$.
(b) For any $\psi \in C_c^\infty([0,T) \times \mathbb{R}^{d+1})^+$ and any weak cluster point f_+^τ of $\frac{1}{s}\int_0^s f_+(t,\bar{x}_r,\xi)dr$ as $s \to +0$ in $L^\infty(\Theta_\lambda \times \mathbb{R})$, we have*

$$\begin{aligned}
&\int_{Q_\lambda\times\mathbb{R}} (f_+(\partial_t + a\cdot\nabla)\psi^\lambda - \beta'\nabla f_+ \cdot \nabla\psi^\lambda)dtdxd\xi \\
&+ \int_{\Omega\times\mathbb{R}} f_+^{\tau_0}\psi^\lambda(0,x)dxd\xi + \int_{\Theta_\lambda\times\mathbb{R}} \beta'(\nabla_{\bar{x}}h(\bar{x})\cdot\nabla_{\bar{x}}f_+^b)\overline{\psi^\lambda}\, dtd\bar{x}d\xi \\
&+ \int_{\Theta_\lambda\times\mathbb{R}} (-\mathbf{n}_1\cdot a)f_+^\tau\overline{\psi^\lambda}dtd\bar{x}d\xi \\
&\geq \int_{Q_\lambda\times\mathbb{R}} \partial_\xi\psi^\lambda d(m_+ + n_+).
\end{aligned} \tag{2.2}$$

Lemma 2.2. *There exist families of probability measures $\{\nu_x^{\tau_0}\}_{x\in\Omega}$ and $\{\tilde{\nu}_x^{\tau_0}\}_{x\in\Omega}$ on $\mathbb{R}_\xi$, called Young measuers, supported in $(-\infty, \|u\|_{L^\infty}]$ and $[-\|\tilde{u}\|_{L^\infty}, \infty))$, respectively, and nonnegative functions $m_+^0(x,\xi)$ and $\tilde{m}_-^0(x,\xi))$ defined on $\Omega \times \mathbb{R}_\xi$ such that*

$$\begin{gathered}
m_+^0, \tilde{m}_-^0 \in C(\mathbb{R}_\xi; w\text{-}\mathcal{M}^+(\Omega)), \\
\lim_{\xi\to\infty} m_+^0(x,\xi) = \lim_{\xi\to-\infty} \tilde{m}_-^0(x,\xi) = 0 \quad \text{for a.e. } x \in \Omega, \\
f_+^{\tau_0}(x,\xi) = \nu_x^{\tau_0}([\xi,\infty)) = \partial_\xi m_+^0(x,\xi) + \operatorname{sgn}^+(u_0(x) - \xi)
\end{gathered} \tag{2.3}$$

and

$$\tilde{f}_-^{\tau_0}(x,\xi) = -\tilde{\nu}_x^{\tau_0}((-\infty,\xi]) = \partial_\xi \tilde{m}_-^0(x,\xi) + \operatorname{sgn}^-(\tilde{u}_0(x) - \xi)).$$

Lemma 2.3. *Let λ be an element of the partition of unity $\{\lambda_i\}_{i=0}^N$ and let f_+^τ and $\tilde{f}_-^\tau$ be weak* cluster point of $\frac{1}{s}\int_0^s f_+(t,\bar{x}_r,\xi)dr$ and $\frac{1}{s}\int_0^s \tilde{f}_-(t,\bar{x}_r,\xi)dr$, respectively ,as $s \to +0$, in $L^\infty(\Theta_\lambda \times \mathbb{R})$. There exist Young measures $\{\nu_{t,y}^\tau\}_{(t,y)\in\Sigma}$ and $\{\tilde{\nu}_{t,y}^\tau\}_{(t,y)\in\Sigma})$ on $\mathbb{R}_\xi$, supported in $(-\infty, \|u\|_{L^\infty}]$ and $[-\|\tilde{u}\|_{L^\infty}, \infty))$, respectively, and nonnegative functions $m_+^b(t,y,\xi)$ and $\tilde{m}_-^b(t,y,\xi))$ defined on $\Sigma \times \mathbb{R}_\xi$ such that*

$$\lim_{\xi\to\infty} m_+^b(t,y,\xi) = \lim_{\xi\to-\infty} \tilde{m}_-^b(t,y,\xi) = 0 \quad \textit{for a.e. } (t,y) \in \Sigma.,$$

$$f_+^\tau(t,y,\xi) = \nu_{t,y}^\tau([\xi,\infty)), \quad \tilde{f}_-^\tau = -\tilde{\nu}_{t,y}^\tau((-\infty,\xi]),$$

$$(-a\cdot\mathbf{n}_1)f_+^\tau = \partial_\xi m_+^b + M\mathrm{sgn}^+(u_b - \xi) \tag{2.4}$$

$$(-a\cdot\mathbf{n}_1)\tilde{f}_-^\tau = \partial_\xi \tilde{m}_-^b + M\mathrm{sgn}^-(\tilde{u}_b - \xi),$$

$$\int_{\Theta_\lambda} m_+^b(t,\bar{x}_0,\xi)\overline{\varphi}^\lambda(t,\bar{x}_0)dtd\bar{x} \geq 0 \tag{2.5}$$

for any $\overline{\varphi} \in C(\Sigma)^+$ satisfying $\mathrm{sgn}^+(\beta(u_b) - \beta(\xi))\overline{\varphi} = 0$ a.e. on Σ and

$$\int_{\Theta_\lambda} \tilde{m}_-^b(t,\bar{x}_0,\xi)\overline{\varphi}^\lambda(t,\bar{x}_0)dtd\bar{x} \geq 0$$

for any $\overline{\varphi} \in C(\Sigma)^+$ satisfying $\mathrm{sgn}^-(\beta(\tilde{u}_b) - \beta(\xi))\overline{\varphi} = 0$ a.e. on Σ.

We continue the proof of Theorem. Let f_+, n_+ and m_+ be the functions defined for u as above. $f_+^{\tau_0}$ denotes the time kinetic traces and f_+^τ a cluster point of space kinetic traces associated with u. The corresponding ones associated with $\tilde{u}$ will be denoted by $\tilde{f}_-, \tilde{n}_-, \tilde{m}_-, \tilde{f}_-^{\tau_0}$ and $\tilde{f}_-^\tau$, respectively. We set for $(t,\bar{x},\xi) \in \Theta_\lambda \times \mathbb{R}$,

$$F_+(t,\bar{x},\xi) = -\mathbf{n}_1(\bar{x}_0)\cdot a(\xi)f_+^\tau(t,\bar{x}_0,\xi) + \beta'(\xi)\nabla_{\bar{x}}h(\bar{x})\cdot\nabla_{\bar{x}}f_+^b(t,\bar{x}_0,\xi)$$

and

$$\tilde{F}_-(t,\bar{x},\xi) = -\mathbf{n}_1(\bar{x}_0)\cdot a(\xi)\tilde{f}_-^\tau(t,\bar{x}_0,\xi) + \beta'(\xi)\nabla_{\bar{x}}h(\bar{x})\cdot\nabla_{\bar{x}}\tilde{f}_-^b(t,\bar{x}_0,\xi)$$

where $\tilde{f}_-^b = \mathrm{sgn}^-(\tilde{u}_b - \xi)$. For $\rho,\nu \in \mathbb{R}_+$ and $\varepsilon = (\bar{\varepsilon},\varepsilon_d) \in {\mathbb{R}_+}^d$, set

$$f_+^{\rho,\nu,\varepsilon} = (f_+ \times \mathbf{1}_{Q_\lambda}) * \gamma_{\rho,\nu,\varepsilon}, \quad f_+^{\tau_0\rho,\varepsilon} = (f_+^{\tau_0} \times \mathbf{1}_{\Omega_\lambda}) * \gamma_{\rho,\varepsilon},$$
$$F_+^{\rho,\nu,\varepsilon} = (F_+ \times \mathbf{1}_{\Sigma_\lambda}) * \gamma_{\rho,\nu,\varepsilon}, \quad m_+^{\rho,\nu,\varepsilon} = (m_+ \times \mathbf{1}_{Q_\lambda}) * \gamma_{\rho,\nu,\varepsilon}$$
$$\text{and} \quad n_+^{\rho,\nu,\varepsilon} = (n_+ \times \mathbf{1}_{Q_\lambda}) * \gamma_{\rho,\nu,\varepsilon}.$$

As for $\tilde{f}_-, \tilde{f}_-^{\tau_0}, \tilde{F}_-$, etc., their regularizations $\tilde{f}_-^{\eta,\mu,\delta}, \tilde{f}_-^{\tau_0\eta,\delta}, \tilde{F}_-^{\eta,\mu,\delta}$, etc. are similarly defined in the same manner as above, but with different parameters η, μ, δ. Let $\psi \in C_c^\infty([0,T) \times \mathbb{R}^{d+1})^+$ and apply (2.2) in Lemma 2.1 to the test function $\psi^\lambda * \check{\gamma}_{\rho,\nu,\varepsilon}$, where $\check{\gamma}_{\rho,\nu,\varepsilon}$ is defined by $\check{\gamma}_{\rho,\nu,\varepsilon}(t,x,\xi) = \gamma_{\rho,\nu,\varepsilon}(-t,-x,-\xi)$:

$$\begin{aligned}\int_{\mathbb{R}^{d+2}} \Big(f_+^{\rho,\nu,\varepsilon}(\partial_t + a\cdot\nabla)\psi^\lambda - \beta'\nabla f_+^{\rho,\nu,\varepsilon}\cdot\nabla\psi^\lambda \\ +(f_+^{\tau_0\rho,\varepsilon}\theta_{\rho,\nu} + F_+^{\rho,\nu,\varepsilon})\psi^\lambda\Big)\, d\xi dt dx \\ \geq \int_{\mathbb{R}^{d+2}} \partial_\xi\psi^\lambda d(m_+^{\rho,\nu,\varepsilon} + n_+^{\rho,\nu,\varepsilon}).\end{aligned} \tag{2.6}$$

On the other hand we can regularize the equation satisfied by $\tilde{f}_-$ by the same method and obtain for same ψ's,

$$\begin{aligned}-\int_{\mathbb{R}^{d+2}} \Big(\tilde{f}_-^{\eta,\mu,\delta}(\partial_t + a\cdot\nabla)\psi^\lambda + \beta'\nabla \tilde{f}_-^{\eta,\mu,\delta}\cdot\nabla\psi^\lambda \\ +(\tilde{f}_-^{\tau_0\eta,\delta}\theta_{\eta,\mu} + \tilde{F}_-^{\eta,\mu,\delta})\psi^\lambda\Big)\, d\xi dt dx \\ \geq -\int_{\mathbb{R}^{d+2}} \partial_\xi\psi^\lambda d(\tilde{m}_-^{\eta,\mu,\delta} + \tilde{n}_-^{\eta,\mu,\delta}).\end{aligned} \tag{2.7}$$

Now let us fix a test function $\varphi(t,x) \in C_c^\infty([0,T)\times\mathbb{R}^d)^+$. Apply (2.6) to $\psi = -\tilde{f}_-^{\eta,\mu,\delta}(t,x,\xi)\varphi(t,x)$ and (2.7) to $\psi = f_+^{\rho,\nu,\varepsilon}(t,x,\xi)\varphi(t,x)$, and add the two equations together. After integrating by parts the left hand side of the resultant inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{d+2}} \Big(-f_+^{\rho,\nu,\varepsilon}\tilde{f}_-^{\eta,\mu,\delta}(\partial_t + a\cdot\nabla + \beta'\Delta + 2\beta'\nabla f_+^{\rho,\nu,\varepsilon}\cdot\nabla\tilde{f}_-^{\eta,\mu,\delta}\Big)\varphi^\lambda d\xi dt dx \\ &-\int_{\mathbb{R}^{d+2}} \Big(f_+^{\tau_0\rho,\varepsilon}\theta_{\rho,\nu}\tilde{f}_-^{\eta,\mu,\delta} + \tilde{f}_-^{\tau_0\eta,\delta}\theta_{\eta,\mu}f_+^{\rho,\nu,\varepsilon} \\ &\qquad +F_+^{\rho,\nu,\varepsilon}\tilde{f}_-^{\eta,\mu,\delta} + \tilde{F}_-^{\eta,\mu,\delta}f_+^{\rho,\nu,\varepsilon}\Big)\varphi^\lambda d\xi dt dx \\ &\geq -\int_{\mathbb{R}^{d+2}} \partial_\xi\tilde{f}_-^{\eta,\mu,\delta}\varphi^\lambda d(m_+^{\rho,\nu,\varepsilon} + n_+^{\rho,\nu,\varepsilon}) - \int_{\mathbb{R}^{d+2}} \partial_\xi f_+^{\rho,\nu,\varepsilon}\varphi^\lambda d(\tilde{m}_+^{\eta,\mu,\delta} + \tilde{n}_+^{\eta,\mu,\delta})\end{aligned}$$

Notice that if $\xi \in F$, then $f_+(t,x,\xi) = \mathrm{sgn}^+(\beta(u(t,x)) - \beta(\xi))$ and hence $\nabla f_+^{\rho,\nu,\varepsilon} = [\delta(\xi-u)\nabla\beta(u)]^{\rho,\nu,\varepsilon} \equiv \delta(\xi-u)\times \mathbf{1}_Q] * \gamma_{\rho,\nu,\varepsilon}$. Similarly, we have $\nabla\tilde{f}_-^{\eta,\mu,\delta} = [\delta(\xi-\tilde{u})\nabla\beta(\tilde{u})]^{\eta,\mu,\delta}$. On the other hand, it is easy to see that $\partial_\xi f_+^{\rho,nu,\varepsilon} = -\delta(\xi-u)^{\rho,\nu,\varepsilon} \equiv -[\delta(\xi-u)\times\mathbf{1}_Q] * \gamma_{\rho,\nu,\varepsilon}$ and $\partial_\xi\tilde{f}^{\eta,\mu,\delta} = -\delta(\xi-\tilde{u})^{\eta,\mu,\delta}$. Noting also that m_+ and $\tilde{m}_-$ are nonnegative measures, we have

$$\int_{\mathbb{R}^{d+2}} \Big(-f_+^{\rho,\nu,\varepsilon}\tilde f_-^{\eta,\mu,\delta}(\partial_t + a\cdot\nabla + \beta'\Delta$$
$$+2\beta'[\delta(\xi-u)\nabla\beta(u)]^{\rho,\nu,\varepsilon}[\delta(\xi-\tilde u)\nabla\beta(\tilde u)]^{\eta,\mu,\delta}\Big)\,\varphi^\lambda d\xi dt dx$$
$$-\int_{\mathbb{R}^{d+2}} \Big(f_+^{\tau_0\rho,\varepsilon}\theta_{\rho,\nu}\tilde f_-^{\eta,\mu,\delta} + \tilde f_-^{\tau_0\eta,\delta}\theta_{\eta,\mu}f_+^{\rho,\nu,\varepsilon}$$
$$+F_+^{\rho,\nu,\varepsilon}\tilde f_-^{\eta,\mu,\delta} + \tilde F_-^{\eta,\mu,\delta}f_+^{\rho,\nu,\varepsilon}\Big)\,\varphi^\lambda d\xi dt dx$$
$$\geq \int_{\mathbb{R}^{d+2}} \delta(\xi-\tilde u)^{\eta,\mu,\delta}\varphi^\lambda dn_+^{\rho,\nu,\varepsilon} + \int_{\mathbb{R}^{d+2}} \delta(\xi-u)^{\rho,\nu,\varepsilon}\varphi^\lambda d\tilde n_-^{\eta,\mu,\delta}$$

Let successively $\eta, \mu, \bar\delta$ and δ_d go to $+0$:

$$\int_{Q_\lambda\times\mathbb{R}} \Big(-f_+^{\rho,\nu,\varepsilon}\tilde f_-(\partial_t + a\cdot\nabla + \beta'\Delta$$
$$+2\beta'[\delta(\xi-u)\nabla\beta(u)]^{\rho,\nu,\varepsilon}\delta(\xi-\tilde u)\nabla\beta(\tilde u)\Big)\,\varphi^\lambda d\xi dt dx$$
$$-\int_{Q_\lambda\times\mathbb{R}} \Big(f_+^{\tau_0\rho,\varepsilon}\theta_{\rho,\nu}\tilde f_- + F_+^{\rho,\nu,\varepsilon}\tilde f_-\Big)\,\varphi^\lambda d\xi dt dx$$
$$\geq \int_{Q_\lambda\times\mathbb{R}} \delta(\xi-\tilde u)\varphi^\lambda dn_+^{\rho,\nu,\varepsilon} + \int_{Q_\lambda\times\mathbb{R}} \delta(\xi-u)^{\rho,\nu,\varepsilon}\varphi^\lambda d\tilde n_-.$$

Here we used the fact that regularized functions equal zero at $t=0$ and at the boundary. Then, let successively $\rho, \nu, \bar\varepsilon$ and ε_d go to $+0$ and use (2.2) in Lemma 2.1 to obtain

$$\int_{Q_\lambda\times\mathbb{R}} \Big(-f_+\tilde f_-(\partial_t + a\cdot\nabla + \beta'\Delta \tag{2.8}$$
$$+2\beta'\delta(\xi-u)\delta(\xi-\tilde u)\nabla\beta(u)\cdot\nabla\beta(\tilde u)\Big)\,\varphi^\lambda d\xi dt dx$$
$$-\int_{\Omega_\lambda\times\mathbb{R}} f_+^{\tau_0}\tilde f_-^{\tau_0}\varphi^\lambda(0,\cdot)d\xi dx + \int_{\Sigma_\lambda\times\mathbb{R}} \Big((\mathbf{n}_1\cdot a)f_+^\tau\tilde f_-^\tau$$
$$-\beta'(\nabla_{\bar x}h\cdot\nabla_{\bar x}f_+^b)\tilde f_-^b\Big)\,\overline{\varphi}^\lambda d\xi dt d\bar x$$
$$\geq \int_{Q_\lambda\times\mathbb{R}} \delta(\xi-\tilde u)\varphi^\lambda dn_+ + \int_{Q_\lambda\times\mathbb{R}} \delta(\xi-u)\varphi^\lambda d\tilde n_-.$$

Next, let successively $\rho, \nu, \bar\varepsilon$ and ε_d go to $+0$ and then let successively $\eta, \mu, \bar\delta$ and δ_d go to $+0$: For any weak* cluster point $\tilde f_-^\tau$ and for some weak* cluster point f_+^τ, we have

$$\begin{aligned}\int_{Q_\lambda\times\mathbb{R}} \Big(&-f_+\tilde f_-(\partial_t + a\cdot\nabla + \beta'\Delta \\ &+2\beta'\delta(\xi-u)\delta(\xi-\tilde u)\nabla\beta(u)\cdot\nabla\beta(\tilde u)\Big)\varphi^\lambda d\xi dt dx \\ &- \int_{\Omega_\lambda\times\mathbb{R}} f_+^{\tau_0}\tilde f_-^{\tau_0}\varphi^\lambda(0,\cdot)d\xi dx \\ &+ \int_{\Sigma_\lambda\times\mathbb{R}}\Big((\mathbf{n}_1\cdot a)f_+^\tau\tilde f_-^\tau - \beta'(\nabla_{\bar x}h\cdot\nabla_{\bar x}\tilde f_-^b)f_+^b\Big)\overline{\varphi}^\lambda d\xi dt d\bar x \\ &\geq \int_{Q_\lambda\times\mathbb{R}}\delta(\xi-\tilde u)\varphi^\lambda dn_+ + \int_{Q_\lambda\times\mathbb{R}}\delta(\xi-u)\varphi^\lambda d\tilde n_-. \end{aligned} \tag{2.9}$$

Adding (2.8) and (2.9) yields

$$\begin{aligned}\int_{Q_\lambda\times\mathbb{R}} \Big(&-f_+\tilde f_-(\partial_t + a\cdot\nabla + \beta'\Delta \\ &+2\beta'\delta(\xi-u)\delta(\xi-\tilde u)\nabla\beta(u)\cdot\nabla\beta(\tilde u)\Big)\varphi^\lambda d\xi dt dx \\ &- \int_{\Omega_\lambda\times\mathbb{R}} f_+^{\tau_0}\tilde f_-^{\tau_0}\varphi^\lambda(0,\cdot)d\xi dx \\ &+ \int_{\Sigma_\lambda\times\mathbb{R}}\Big((\mathbf{n}_1\cdot a)f_+^\tau\tilde f_-^\tau - \frac{1}{2}\beta'\nabla_{\bar x}h\cdot\nabla_{\bar x}(f_+^b\tilde f_-^b)\Big)\overline{\varphi}^\lambda d\xi dt d\bar x \\ &\geq \int_{Q_\lambda\times\mathbb{R}}(\delta(\xi-\tilde u)n_+ + \delta(\xi-u)\tilde n_-)\varphi^\lambda d\xi dt dx. \end{aligned}$$

for some weak* cluster points f_+^τ and $\tilde f_-^\tau$. Since

$$\begin{aligned}&2\beta'(\xi)\delta(\xi-u)\delta(\xi-\tilde u)\nabla\beta(u)\cdot\nabla\beta(\tilde u) \\ &\leq \mathbf{1}_F(\xi)\delta(\xi-u)\delta(\xi-\tilde u)(|\nabla\beta(u)|^2 + |\nabla\beta(\tilde u)|^2) \\ &= \delta(\xi-\tilde u)n_+(t,x,\xi) + \delta(\xi-u)\tilde n_-(t,x,\xi), \end{aligned}$$

we arrive at

$$\begin{aligned}&-\int_{Q_\lambda\times\mathbb{R}} f_+\tilde f_-(\partial_t + a\cdot\nabla + \beta'\Delta)\varphi^\lambda d\xi dt dx \\ &\geq \int_{\Omega_\lambda\times\mathbb{R}} f_+^{\tau_0}\tilde f_-^{\tau_0}\varphi^\lambda(0,\cdot)d\xi dx - \int_{\Sigma_\lambda\times\mathbb{R}}\Big((\mathbf{n}_1\cdot a)f_+^\tau\tilde f_-^\tau \\ &\qquad -\frac{1}{2}\beta'\nabla_{\bar x}h\cdot\nabla_{\bar x}(f_+^b\tilde f_-^b)\Big)\overline{\varphi}^\lambda d\xi dt d\bar x. \end{aligned} \tag{2.10}$$

We compute each term of (2.26). Firstly,

$$
\begin{aligned}
&-\int_{Q_\lambda\times\mathbb{R}} f_+\tilde{f}_-(\partial_t + a\cdot\nabla + \beta'\Delta)\varphi^\lambda d\xi dtdx \qquad (2.11)\\
&= \int_{Q_\lambda}\left((u-\tilde{u})^+ + \mathcal{F}^+(u,\tilde{u})\nabla\varphi^\lambda + (\beta(u)-\beta(\tilde{u}))^+\Delta\varphi^\lambda\right) dtdx.
\end{aligned}
$$

Secondly, by virtue of Lemma 2.2 and by using integration by parts one can calculate:

$$
\begin{aligned}
&\int_{\mathbb{R}} f_+^{\tau_0}\tilde{f}_-^{\tau_0}d\xi\\
&= \int_{-\infty}^{\tilde{u}_0}\nu_x^{\tau_0}([\xi,\infty))\partial_\xi\tilde{m}_-^0 d\xi - \int_{\tilde{u}_0}^{u_0\vee\tilde{u}_0}\nu_x^{\tau_0}([\xi,\infty))\tilde{\nu}_x^{\tau_0}((-\infty,\xi])d\xi\\
&\qquad\qquad - \int_{u_0\vee\tilde{u}_0}^{\infty}\partial_\xi m_+^0\tilde{\nu}_x^{\tau_0}((-\infty,\xi])d\xi\\
&= \nu_x^{\tau_0}([\tilde{u}_0,\infty))\tilde{m}_-^0(\cdot,\tilde{u}_0) + \int_{-\infty}^{\tilde{u}_0}\tilde{m}_-^0 d\nu_x^{\tau_0} - \int_{\tilde{u}_0}^{u_0\vee\tilde{u}_0}\nu_x^{\tau_0}([\xi,\infty))\tilde{\nu}_x^{\tau_0}((-\infty,\xi])d\xi\\
&\qquad\qquad + m_+^0(\cdot,u_0\vee\tilde{u}_0)\tilde{\nu}_x^{\tau_0}((-\infty,u_0\vee\tilde{u}_0]) + \int_{u_0\vee\tilde{u}_0}^{\infty} m_+^0 d\tilde{\nu}_+^{\tau_0}\\
&\geq -\int_{\tilde{u}_0}^{u_0\vee\tilde{u}_0} d\xi = -(u-\tilde{u}_0)^+.
\end{aligned}
$$

Here we used the fact that $d\nu_x^{\tau_0}([\xi,\infty))/d\xi = -d\nu_x^{\tau_0}(\xi)$ and $d\tilde{\nu}_-^{\tau_0}((-\infty,\xi])/d\xi = d\tilde{\nu}_-^{\tau_0}(\xi)$. Thus we have

$$
\int_{\Omega_\lambda\times\mathbb{R}} f_+^{\tau_0}\tilde{f}_-^{\tau_0}\varphi^\lambda(0,\cdot)d\xi dx \geq -\int_{\Omega_\lambda}(u_0-\tilde{u}_0)^+\varphi^\lambda(0,\cdot)dx. \qquad (2.12)
$$

Finally, we calculate analogously the boundary term by using Lemma 2.4:

$$
\begin{aligned}
&\int_{\mathbb{R}}(\mathbf{n}_1\cdot a)f_+^\tau\tilde{f}_-^\tau d\xi\\
&= -\int_{-\infty}^{\tilde{u}_b}\partial_\xi\tilde{m}_-^b\nu_{t,y}^\tau([\xi,\infty))d\xi - \int_{\tilde{u}_b}^{u_b\vee\tilde{u}_b}(\mathbf{n}_1\cdot a)\nu_{t,y}^\tau([\xi,\infty))\tilde{\nu}_{t,y}^\tau((-\infty,\xi])d\xi\\
&+ \int_{u_b\vee\tilde{u}_b}^{\infty}\partial_\xi m_+^b\nu_{t,y}^\tau((-\infty,\xi])d\xi\\
&\leq M\int_{\tilde{u}_b}^{u_b\vee\tilde{u}_b} d\xi = M(u_b-\tilde{u}_b)^+,
\end{aligned}
$$

where y stands for $\bar{x}_0$ and we used the fact that $d\nu_{t,y}^\tau([\xi,\infty))/d\xi = -d\nu_{t,y}^\tau(\xi)$ and $d\tilde{\nu}_{t,y}^\tau((-\infty,\xi])/d\xi = d\tilde{\nu}_{t,y}^\tau(\xi)$ as well as the fact that $m_+^b \geq$

0 for $\xi \geq u_b$ and $\tilde{m}_-^b \geq 0$ for $\xi \leq \tilde{u}_b$ by virtue of (2.21) and the corresponding inequality associated wiyh $\tilde{u}$, respectively. This implies

$$\int_{\Sigma_\lambda \times \mathbb{R}} (\mathbf{n}_1 \cdot a) f_+^\tau \tilde{f}_-^\tau \overline{\varphi}^\lambda d\xi dt d\bar{x} \leq M \int_{\Sigma_\lambda} (u_b - \tilde{u}_b)^+ \overline{\varphi}^\lambda dt d\bar{x}. \tag{2.13}$$

Moreover

$$\begin{aligned} &\int_{\Sigma_\lambda \times \mathbb{R}} \beta'(\xi) \nabla_{\bar{x}} h(\bar{x}) \cdot \nabla_{\bar{x}} (f_+^b \tilde{f}_-^b) \overline{\varphi}^\lambda d\xi dt d\bar{x} \qquad (2.14) \\ &= - \int_{\Sigma_\lambda \times \mathbb{R}} \beta'(\xi) \operatorname{div}_{\bar{x}} (\overline{\varphi}^\lambda \nabla_{\bar{x}} h(\bar{x})) f_+^b \tilde{f}_-^b d\xi dt d\bar{x} \\ &= - \int_{\Sigma_\lambda} (\beta(u_b) - \beta(\tilde{u}_b))^+ (\Delta_{\bar{x}} h(\bar{x}) \overline{\varphi}^\lambda + \nabla_{\bar{x}} h(\bar{x}) \cdot \nabla_{\bar{x}} \overline{\varphi}^\lambda) dt d\bar{x} \\ &\geq \int_{\Sigma_\lambda} (\beta(u_b) - \beta(\tilde{u}_b))^+ (-L \overline{\varphi}^\lambda + \nabla_{\bar{x}} h(\bar{x}) \cdot \nabla_{\bar{x}} \overline{\varphi}^\lambda) dt d\bar{x}. \end{aligned}$$

Combining (2.10) with (2.11) through (2.14) and choosing appropriate test functions φ's, we arrive at the estimate

$$\begin{aligned} &\frac{1}{T} \int_{Q_\lambda} (u - \tilde{u})^+ dt dx \\ &\leq \int_{\Omega_\lambda} (u_0 - \tilde{u}_0)^+ dx + M \int_{\Sigma_\lambda} (u_b - \tilde{u}_b)^+ dt d\bar{x} + \frac{L}{2} \int_{\Sigma_\lambda} (\beta(u_b) - \beta(\tilde{u}_b))^+ dt d\bar{x}. \end{aligned}$$

By summing over $i = 0, 1, \cdots, N$, we obtain the desired estimate (1.9) and the proof of Theorem is complete.

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