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DYNAMICAL SYSTEM AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR FOREST KINEMATIC MODEL

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Abstract. We are concerned with a forest kinematic model presented by Kuzunetsov et al. [3]. In this paper, we will construct global solutions and construct a dynamical system determined from the model equations. We introduce three kinds of ω-limit sets, namely, $\omega(U_0) \subset L^2\text{-}\omega(U_0) \subset \mathrm{w}^*\text{-}\omega(U_0)$, for each point U_0. Using a Lyapunov function, we will then investigate basic properties of these ω-limit sets. Especially it shall be shown that $L^2\text{-}\omega(U_0)$ consists of stationary solutions alone.

1. Introduction

We study the initial-boundary values problem for a parabolic-ordinary system

$$
(1.1)\quad \begin{cases}
\dfrac{\partial u}{\partial t} = \beta\delta w - \gamma(v)u - fu & \text{in} \quad \Omega \times (0,\infty),\\
\dfrac{\partial v}{\partial t} = fu - hv & \text{in} \quad \Omega \times (0,\infty),\\
\dfrac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in} \quad \Omega \times (0,\infty),\\
\dfrac{\partial w}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,\infty),\\
u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & \text{in} \quad \Omega.
\end{cases}
$$

This system has been introduced by Kuzunetsov et al. [3] in order to describe the kinetics of forest from the viewpoint of the age structure. For simplicity they consider a prototype ecosystem of a mono-species and with only two age classes in a two-dimensional domain Ω.

The unknown functions $u(x,t)$ and $v(x,t)$ denote the tree densities of young and old age classes, respectively, at a position $x \in \Omega$ and at time $t \in [0,\infty)$. The third unknown function $w(x,t)$ denotes the density of seeds in the air at $x \in \Omega$ and $t \in [0,\infty)$. The third equation describes the kinetics of seeds; $d > 0$ is a diffusion constant of seeds, and $\alpha > 0$ and $\beta > 0$ are seed production and seed deposition rates respectively. While the first and second equations describe the growth of young and old trees respectively; $0 < \delta \leq 1$ is a seed establishment rate, $\gamma(v) > 0$ is a mortality of young trees which is allowed to depend on the old-tree density v, $f > 0$ is an aging rate, and $h > 0$ is a mortality of old trees.

On w, the Neumann boundary conditions are imposed on the boundary $\partial\Omega$. Nonnegative initial functions $u_0(x) \geq 0$, $v_0(x) \geq 0$ and $w_0(x) \geq 0$ are given in Ω.

Several authors have already been interested in such a model. Wu [8] studied the stability of travelling wave solutions. Wu and Lin [9] discussed the stability of stationary solutions. Lin and Liu [4] extended this result to a case when the model includes nonlocal effects.

In this paper we intend to construct a global solution to (1.1) for each initial function $U_0 \in K$ and to construct a dynamical system determined from the problem. Furthermore, we are concerned with studying asymptotic behavior of solutions.

We regard and handle the system (1.1) as a degenerate nonlinear diffusion system with respect to (u,v,w). The word "degenerate" here means that the diffusion constants for u and v both vanish. But the general methods for constructing local and global solutions are available if we take an underlying space carefully. In fact, we shall verify that the abstract result obtained in [7, Theorem 3.1] is still applicable for the present problem if X is taken as

$$X = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} ;\ u \in L^\infty(\Omega),\ v \in L^\infty(\Omega) \text{ and } w \in L^2(\Omega) \right\}. \tag{1.2}$$

The space of initial values is taken as

$$K = \left\{ \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} ;\ 0 \leq u_0,\ v_0 \in L^\infty(\Omega) \text{ and } 0 \leq w_0 \in L^2(\Omega) \right\}. \tag{1.3}$$

Nonnegativity of local solutions and a priori estimates for local solutions will be established in ordinary manners.

We have to pay much attention, however, that, owing to the degeneracy of dissipation, we have no longer smoothing effect of solutions. What is even worse, we observe at least numerically (see [6]) that, even if the initial functions (u_0, v_0, w_0) are very smooth, the solution $(u(t), v(t), w(t))$ can tend to a discontinuous stationary solution $(\overline{u}, \overline{v}, \overline{w})$ as $t \to \infty$, $\overline{u}$ and $\overline{v}$ being discontinuous and $\overline{w}$ being continuous in Ω. This suggests furthermore that some trajectories of the dynamical system no longer possess any nonempty ω-limit sets in the usual sense (see [10], [14] and [16]) in the underlying space X given by (1.2). In fact, if a smooth trajectory $(u(t), v(t), w(t))$, $0 \leq t < \infty$, has a cluster point $(\overline{u}, \overline{v}, \overline{w})$ in X, then it is impossible that $\overline{u}$ and $\overline{v}$ are discontinuous in Ω. The dynamical system is neither expected to possess the global attractor in general.

In view of these situations, we are rather led to investigate asymptotic behavior of each trajectory of the dynamical system. We will introduce three kinds of ω-limit sets, namely, $\omega(U_0) \subset L^2\text{-}\omega(U_0) \subset \text{w}^*\text{-}\omega(U_0)$ for $U_0 \in K$. Here, $\omega(U_0)$ is the usual ω-limit set in the topology of X but may be empty for some $U_0 \in K$, $L^2\text{-}\omega(U_0)$ is an ω-limit set with respect to the L^2 topology, and $\text{w}^*\text{-}\omega(U_0)$ is that with respect to the weak* topology of $L^\infty(\Omega)$. Fortunately, we can find a Lyapunov function for our dynamical system. Owing the Lyapunov function, we can obtain many results on these ω-limit sets. Among others, it is proved that $L^2\text{-}\omega(U_0)$ consists of stationary solutions alone. But, for the moment, it is an open problem to prove that $\text{w}^*\text{-}\omega(U_0)$ consists of stationary solutions alone.

As a matter of fact, we can rigorously know existence of discontinuous stationary solutions to the present system (1.1) (see [2]). The interface of a discontinuous stationary solution is then considered as an internal forest boundary or an ecotone of forest which has a significant meaning from the viewpoint of ecology ([3]). In this sense also it is quite natural to choose an underlying space in the form (1.2).

Throughout the paper, Ω is a bounded, convex or $\mathcal{C}^2$ domain in $\mathbb{R}^2$. According to [12], the Poisson problem $-d\Delta w + \beta w = v$ in Ω under the Neumann boundary conditions $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$ enjoys the optimal shift property that $v \in L^2(\Omega)$ always implies that $w \in H^2(\Omega)$. We assume as in [3] that the mortality of young trees is given by a square function of the form

$$\gamma(v) = a(v-b)^2 + c, \tag{1.4}$$

where a, b, $c > 0$ are positive constants. This means that the mortality takes its minimum when the old-age tree density is a specific value b. As mentioned, d, f, h, α, $\beta > 0$ are all positive constants and $0 < \delta \leq 1$.

2. Local solutions and global solutions

In the underlying product space X, we shall formulate the initial boundary value problem (1.1) as the Cauchy problem for an abstract semilinear equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t < \infty, \\ U(0) = U_0. \end{cases}$$

Then we can apply the general results in [7] to construct local solutions.

The linear operator A is defined by

$$A = \begin{pmatrix} f & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \Lambda \end{pmatrix} \quad \text{with} \quad \mathcal{D}(A) = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix}; \ u, v \in L^\infty(\Omega) \ \text{and} \ w \in H^2_N(\Omega) \right\},$$

where Λ is a realization of the operator $-d\Delta + \beta$ in $L^2(\Omega)$ under the homogeneous Neumann boundary condition $\frac{\partial w}{\partial n} = 0$ on the boundary $\partial\Omega$. It is known that Λ is a positive definite self-adjoint operator of $L^2(\Omega)$ with $\mathcal{D}(\Lambda) = H^2_N(\Omega)$ (see [11, 12]), where $H^2_N(\Omega)$ is a closed subspace of $H^2(\Omega)$ consisting of functions w's satisfying the homogeneous Neumann boundary conditions on $\partial\Omega$.

Moreover, for $0 \le \theta \le 1$, $\theta \ne \frac{3}{4}$,

$$A^\theta = \begin{pmatrix} f^\theta & 0 & 0 \\ 0 & h^\theta & 0 \\ 0 & 0 & \Lambda^\theta \end{pmatrix} \quad \text{with} \quad \mathcal{D}(A^\theta) = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} ; \; u, v \in L^\infty(\Omega) \;\text{ and }\; w \in \mathcal{D}(\Lambda^\theta) \right\}.$$

The nonlinear operator F is given by

$$F(U) = \begin{pmatrix} \beta\delta w - \gamma(v)u \\ fu \\ \alpha v \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{D}(A^\eta),$$

where η is an arbitrarily fixed exponent in such a way that $\frac{1}{2} < \eta < 1$. The initial value U_0 is taken from the space K given by (1.3).

It is easy to verify that all assumptions in [7, Theorem 3.1] are satisfied, then we conclude the following result.

Theorem 2.1. *For any $U_0 \in X$, (1.1) possesses a unique local solution in the function space $U \in \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)) \cap \mathcal{C}^1((0, T_{U_0}]; X)$, i.e.,*

$$\begin{cases} u,\ v \in \mathcal{C}([0, T_{U_0}]; L^\infty(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; L^\infty(\Omega)), \\ w \in \mathcal{C}([0, T_{U_0}]; L^2(\Omega)) \cap \mathcal{C}((0, T_{U_0}]; H^2_N(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; L^2(\Omega)). \end{cases} \tag{2.1}$$

Here, $T_{U_0} > 0$ is determined by the norm $\|U_0\|_X = \|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}$ alone. Moreover, the estimate

$$t\|AU(t)\|_X + \|U(t)\|_X \le C_{U_0}, \quad 0 < t \le T_{U_0}$$

holds with some constant C_{U_0} determined by $\|U_0\|_X$ alone.

We next verify that nonnegativity of initial functions implies that of the local solution obtained in Theorem 2.1.

Theorem 2.2. *For any $U_0 \in K$, (1.1) possesses a unique local solution such that*

$$\begin{cases} 0 \le u,\ v \in \mathcal{C}([0, T_{U_0}]; L^\infty(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; L^\infty(\Omega)), \\ 0 \le w \in \mathcal{C}([0, T_{U_0}]; L^2(\Omega)) \cap \mathcal{C}((0, T_{U_0}]; H^2_N(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; L^2(\Omega)). \end{cases}$$

Here, $T_{U_0} > 0$ is determined by the norm $\|U_0\|_X$ alone. Moreover, the estimate

$$t\|AU(t)\|_X + \|U(t)\|_X \le C_{U_0}, \quad 0 < t \le T_{U_0}$$

holds with some constant C_{U_0} determined by $\|U_0\|_X$ alone.

Proof. By Theorem 2.1, (1.1) possesses a unique local solution $U = (u, v, w)$ in function space (2.1) with $T_{U_0} = T_{01}$ determined by the norm $\|U_0\|_X$.

Let us now consider an auxiliary problem

$$
(2.2)\qquad \begin{cases} \dfrac{\partial \widetilde{u}}{\partial t} = \beta\delta\widetilde{w} - \gamma(\widetilde{v})\widetilde{u} - f\widetilde{u} & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial \widetilde{v}}{\partial t} = f\widetilde{u} - h\widetilde{v} & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial \widetilde{w}}{\partial t} = d\Delta\widetilde{w} - \beta\widetilde{w} + \alpha\chi(\mathrm{Re}\,\widetilde{v}) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial \widetilde{w}}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ \widetilde{u}(x,0) = u_0(x),\ \widetilde{v}(x,0) = v_0(x),\ \widetilde{w}(x,0) = w_0(x) & \text{in } \Omega. \end{cases}
$$

Here, $\chi(\widetilde{v})$ is a cutoff function given by

$$
\chi(\widetilde{v}) = \begin{cases} \widetilde{v} & \text{if } \widetilde{v} \geq 0, \\ 0 & \text{if } \widetilde{v} < 0. \end{cases}
$$

By using the same arguments as in the proof of Theorem 2.1, we can deduce that (2.2) possesses a unique local solution $\widetilde{U} = (\widetilde{u}, \widetilde{v}, \widetilde{w})$ in the function space (2.1) with $T_{U_0} = T_{02}$ determined by the norm $\|U_0\|_X$. Our goal is then to show the nonnegativity of $\widetilde{u}, \widetilde{v}$ and $\widetilde{w}$. In this case, $\chi(\widetilde{v}) = \widetilde{v}$ and therefore $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ is also a local solution of (1.1) in $[0, T_{02}]$. Then, by the uniqueness of solutions, we conclude that $(u, v, w) = (\widetilde{u}, \widetilde{v}, \widetilde{w})$ in $[0, T_{U_0}]$ with $T_{U_0} = \min\{T_{01}, T_{02}\}$. That means (1.1) possesses a unique nonnegative local solution in the function space (2.1) with T_{U_0} determined by the norm $\|U_0\|_X$. □

In the next part, we shall establish a *priori* estimates of local solutions, which will then guarantee the existence of global solutions.

Proposition 2.3. *There exist an exponent $\rho > 0$ and a constant $C > 0$ such that the estimates*

$$
(2.3)\qquad \|U(t)\|_X \leq C\{e^{-\rho t}\|U_0\|_X + 1\}, \quad 0 \leq t < T_U
$$

hold for all local solutions U's in the function space (2.1) *on* $[0, T_U]$ *with initial value* $U_0 \in K$.

Proof. Throughout the proof, we shall use notation $C_1, C_2, \dots$ and universal notation C, ρ, ρ' to denote positive constants and positive exponents which are determined by the constants a, b, c, d, f, h, α, β and δ and by Ω. In these, C, ρ and ρ' may be change from occurrence to occurrence.

Step 1. Estimate for $\|U(t)\|_{L^2}$. Multiply the first equation of (1.1) by u and integrate the product in Ω. Then we have

$$
\begin{aligned}
(2.4)\quad \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + f\int_\Omega u^2 dx &= \beta\delta\int_\Omega wu dx - \int_\Omega \gamma(v)u^2 dx \\
&\leq \frac{f}{2}\int_\Omega u^2 dx + C_1\int_\Omega w^2 dx - \int_\Omega \gamma(v)u^2 dx.
\end{aligned}
$$

Multiply the third equation of (1.1) by w and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx+\beta\int_\Omega w^2dx=-d\int_\Omega|\nabla w|^2dx+\alpha\int_\Omega vwdx\le\frac{\beta}{2}\int_\Omega w^2dx+C_2\int_\Omega v^2dx.$$

Let $C_3>0$ be constant such that $C_1C_3\le\frac{\beta}{4}$. Multiply (2.4) by C_3 and add the product to above inequality. Then we obtain that

$$\begin{aligned}(2.5)\quad \frac{C_3}{2}\frac{d}{dt}\int_\Omega u^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx+\frac{C_3f}{2}\int_\Omega u^2dx+\frac{\beta}{4}\int_\Omega w^2dx&\\ \le C_2\int_\Omega v^2dx-C_3\int_\Omega\gamma(v)u^2dx.&\end{aligned}$$

Next, multiply the second equation of (1.1) by v and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2dx+h\int_\Omega v^2dx=f\int_\Omega uvdx.$$

Let $C_4>0$ be constant such that $C_4h\ge 2C_2$. Multiply above equation by C_4 and add the product to the inequality (2.5) to obtain

$$\begin{aligned}\frac{C_3}{2}\frac{d}{dt}\int_\Omega u^2dx+\frac{C_4}{2}\frac{d}{dt}\int_\Omega v^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx+\frac{C_3f}{2}\int_\Omega u^2dx+C_2\int_\Omega v^2dx&\\ +\frac{\beta}{4}\int_\Omega w^2dx\le C_4f\int_\Omega uvdx-C_3\int_\Omega\gamma(v)u^2dx.&\end{aligned}$$

We have notice that

$$\begin{aligned}C_4fuv-C_3\gamma(v)u^2&=-\{C_3a(v-b)^2u^2-C_4f(v-b)u+\frac{C_4^2f^2}{4C_3a}\}\\ &\quad-\{C_3cu^2-C_4fbu+\frac{C_4^2f^2b^2}{4C_3c}\}+\frac{C_4^2f^2}{4C_3}(\frac{1}{a}+\frac{b^2}{c})\le\frac{C_4^2f^2}{4C_3}(\frac{1}{a}+\frac{b^2}{c}).\end{aligned}$$

Therefore,

$$\frac{d}{dt}\int_\Omega(C_3u^2+C_4v^2+w^2)dx+\rho\int_\Omega(C_3u^2+C_4v^2+w^2)dx\le C.$$

Solving this, we conclude that

$$C_3\|u(t)\|_{L^2}^2+C_4\|v(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\le Ce^{-\rho t}(C_3\|u_0\|_{L^2}^2+C_4\|v_0\|_{L^2}^2+\|w_0\|_{L^2}^2)+C.$$

It follows that

$$\begin{aligned}(2.6)\quad \|u(t)\|_{L^2}+\|v(t)\|_{L^2}+\|w(t)\|_{L^2}&\\ \le C[e^{-\rho t}(\|u_0\|_{L^\infty}+\|v_0\|_{L^\infty}+\|w_0\|_{H^{2\mu}})+1],\quad 0\le t<T_U.&\end{aligned}$$

Step 2. Estimate for $\|w(t)\|_{L^\infty}$. Using the representation by the semigroup, we can write $w(t)$ in the form

$$\Lambda^\eta w(t) = \{\Lambda^\eta e^{-\frac{tA}{2}}\}\{e^{-\frac{tA}{2}}w_0\} + \int_0^t \{\Lambda^\eta e^{-\frac{t-\tau}{2}A}\}e^{-\frac{t-\tau}{2}A}\alpha v(\tau)d\tau.$$

Hence,

$$\|w(t)\|_{H^{2\eta}} \le C(1+t^{-\eta})e^{-\frac{\beta t}{2}}\|w_0\|_{L^2} + C\int_0^t \{1+(t-\tau)^{-\eta}\}e^{-\frac{\beta}{2}(t-\tau)}\|v(\tau)\|_{L^2}d\tau,$$

here we used the estimate $\|e^{-tA}\|_{L^2} \le e^{-t\beta}$ for $t \ge 0$. Moreover, by (2.6),

$$\int_0^t \{1+(t-\tau)^{-\eta}\}e^{-\frac{\beta}{2}(t-\tau)}\|v(\tau)\|_{L^2}d\tau \le C\int_0^t \{1+(t-\tau)^{-\eta}\}e^{-\frac{\beta}{2}(t-\tau)}d\tau$$
$$+Ce^{-\rho' t}\int_0^t \{1+(t-\tau)^{-\eta}\}e^{-(\frac{\beta}{2}-\rho')(t-\tau)}e^{-(\rho-\rho')\tau}d\tau\|U_0\|_{L^2} \le C\{e^{-\rho' t}\|U_0\|_{L^2}+1\},$$

where $0<\rho'<\min\{\frac{\beta}{2},\rho\}$. Thus, we have obtained that

$$\|w(t)\|_{L^\infty} \le C\|w(t)\|_{H^{2\eta}} \le C\{(1+t^{-\eta})e^{-\rho t}\|U_0\|_X + 1\}, \quad 0 \le t < T_U.$$

Step 3. Estimate for $\|u(t)\|_{L^\infty}, \|v(t)\|_{L^\infty}$. From the first equation of (1.1), we have

$$u(t) = e^{-\int_0^t \gamma(v(s))+f ds}u_0 + \beta\delta\int_0^t e^{-\int_\tau^t \gamma(v(s))+f ds}w(\tau)d\tau, \quad 0 \le t < T_U.$$

Hence,

$$\|u(t)\|_{L^\infty} \le e^{-ft}\|u_0\|_{L^\infty} + C\int_0^t e^{-f(t-\tau)}\{(1+\tau^{-\eta})e^{-\rho\tau}\|U_0\|_X+1\}d\tau.$$

Therefore, we conclude that

$$\|u(t)\|_{L^\infty} \le C\{e^{-\rho t}\|U_0\|_X + 1\}, \quad 0 \le t < T_U.$$

In a similarly way, by the second equation of (1.1),

$$\|v(t)\|_{L^\infty} \le C\{e^{-\rho t}\|U_0\|_X + 1\}, \quad 0 \le t < T_U.$$

These together with (2.6) finally yield the desired a priori estimates (2.3). □

As an immediate consequence of a priori estimates, we can prove the existence and uniqueness of global solution.

Theorem 2.4. *For any* $U_0 \in K$, *(1.1) possesses a unique global solution such that*

$$\begin{cases} 0 \le u, v \in \mathcal{C}([0,\infty); L^\infty(\Omega)) \cap \mathcal{C}^1((0,\infty); L^\infty(\Omega)), \\ 0 \le w \in \mathcal{C}([0,\infty); L^2(\Omega)) \cap \mathcal{C}((0,\infty); H^2_N(\Omega)) \cap \mathcal{C}^1((0,\infty); L^2(\Omega)). \end{cases}$$

And global solution satisfies the estimates

$$\|U(t)\|_X \le C\{e^{-\rho t}\|U_0\|_X + 1\}, \quad 0 \le t < \infty, \tag{2.7}$$
$$\|w(t)\|_{L^\infty} \le C\{(1+t^{-\eta})e^{-\rho t}\|U_0\|_X + 1\}, \quad 0 \le t < \infty. \tag{2.8}$$

Proof. By Theorem 2.2, there exists a unique local solution U on an interval $[0, T_{U_0}]$. Moreover, by Proposition 2.3, $\|U(T_{U_0})\|_X$ is estimated by $\|U_0\|_X$ alone. This then shows that the solution U can be extended as a local solution on an interval $[0, T_{U_0} + \tau]$, where $\tau > 0$ is determined by $\|U(T_0)\|_X$, and hence depends only on $\|U_0\|_X$. Repeating this procedure, we obtain the result. $\square$

The solution satisfies the following integral equations

$$u(t) = e^{-\int_0^t \{\gamma(v(s))+f\}ds} u_0 + \beta\delta \int_0^t e^{-\int_s^t \{\gamma(v(\tau))+f\}d\tau} w(s)ds, \qquad 0 \le t < \infty, \tag{2.9}$$

$$v(t) = e^{-ht} v_0 + f \int_0^t e^{-(t-s)h} u(s)ds, \qquad 0 \le t < \infty, \tag{2.10}$$

$$w(t) = e^{-tA} w_0 + \alpha \int_0^t e^{-(t-s)A} v(s)ds, \qquad 0 \le t < \infty. \tag{2.11}$$

In addition, we verify the uniform estimates for the derivative and the second order derivative of global solutions.

Proposition 2.5. *Let $U(t) = (u(t), v(t), w(t))$ be the global solution to (1.1) with $U_0 \in K$. Then for $U'(t) = (u'(t), v'(t), w'(t))$,*

$$\|u'(t)\|_{L^\infty} \le (1 + t^{-\eta}) p_1(\|U_0\|_X), \qquad 0 < t < \infty \tag{2.12}$$

$$\|v'(t)\|_{L^\infty} \le p_1(\|U_0\|_X), \qquad 0 < t < \infty \tag{2.13}$$

$$\|w'(t)\|_{L^2} + \|w(t)\|_{H^2} \le (1 + t^{-1}) p_1(\|U_0\|_X), \qquad 0 < t < \infty, \tag{2.14}$$

where $p_1(\cdot)$ is an appropriate continuous increasing function.

Proof. Using (2.7) and (2.8) in the equation on u in (1.1), we immediately observe (2.12). Similarly, from the equation on v in (1.1) we observe (2.13). We know that $v \in \mathcal{C}([0, \infty); L^2(\Omega)) \cap \mathcal{C}^1((0, \infty); L^2(\Omega))$ with the estimate (2.13). Then, (2.14) is deduced by the standard arguments for the linear abstract equation on w in (1.1). $\square$

Proposition 2.6. *Let $U(t) = (u(t), v(t), w(t))$ be the global solution to (1.1) with $U_0 \in K$. Then for the second order derivative $U''(t) = (u''(t), v''(t), w''(t))$,*

$$\|u''(t)\|_{L^\infty} \le (1 + t^{-1-\eta}) p_2(\|U_0\|_X), \qquad 0 < t < \infty, \tag{2.15}$$

$$\|v''(t)\|_{L^\infty} \le (1 + t^{-\eta}) p_2(\|U_0\|_X), \qquad 0 < t < \infty, \tag{2.16}$$

$$\|w''(t)\|_{L^2} + \|w'(t)\|_{H^2} \le (1 + t^{-2}) p_2(\|U_0\|_X), \qquad 0 < t < \infty, \tag{2.17}$$

where $p_2(\cdot)$ is an appropriate continuous increasing function.

Proof. From the second equation in (1.1),

$$v''(t) = fu'(t) - hv'(t), \qquad 0 < t < \infty.$$

Then, $v \in \mathcal{C}^2((0, \infty); L^\infty(\Omega))$ and the estimate (2.16) is seen by (2.12) and (2.13).

With any $\tau > 0$, we consider the Cauchy problem for a linear evolution equation

$$\begin{cases} \frac{dw^1}{dt} + Aw^1 = \alpha v'(t), & \tau < t < \infty, \\ w^1(\tau) = w'(\tau) \end{cases}$$

in $L^2(\Omega)$, where $w^1 = w^1(t)$ is the unknown function. Since v' is in $\mathcal{C}^1([\tau,\infty); L^2(\Omega))$, this problem has a unique solution $w^1 \in \mathcal{C}^1((\tau,\infty); L^2(\Omega))$. By a direct calculation it is verified that $w^1(t) = w'(t)$ for any $t \in [\tau,\infty)$. Therefore,

$$w'(t) = e^{-(t-\tau)A}w'(\tau) + \alpha \int_\tau^t e^{-(t-s)A}v'(s)ds, \qquad \tau \le t < \infty.$$

Taking $\tau = \frac{t}{2}$, we repeat the same argument as for (2.14) to obtain that

$$\|w''(t)\|_{L^2} + \|w'(t)\|_{H^2} \le C(1+t^{-1})\|w'(\tfrac{t}{2})\|_{L^2} + C\{p_2(\|U_0\|_X) + p_1(\|U_0\|_X)\}, \quad 0 < t < \infty.$$

Therefore, (2.17) is obtained in view of (2.14).

As a consequence of (2.14) and (2.17), we have

$$\|w'(t)\|_{L^\infty} \le C\|w'(t)\|_{H^{2\eta}} \le (1+t^{-1-\eta})p(\|U_0\|_X), \qquad 0 < t < \infty.$$

Then, (2.15) is observed directly from

$$u''(t) = \beta\delta w'(t) - \gamma'(v(t))v'(t)u(t) - (\gamma(v(t)) + f)u'(t), \qquad 0 < t < \infty.$$

□

We next verify the Lipschitz continuity of solution in initial data.

Proposition 2.7. *Let U (resp. V) be the solution to* (1.1) *with initial value $U_0 \in \overline{B}^X(0,R)$ (resp. $V_0 \in \overline{B}^X(0,R)$). Then, for each $T > 0$ fixed, there exists some constants $C_{R,T} > 0$ depending on R and T alone such that*

$$t^\eta\|A^\eta\{U(t) - V(t)\}\|_X + \|U(t) - V(t)\|_X \le C_{R,T}\|U_0 - V_0\|_X, \qquad 0 \le t \le T.$$

3. Dynamical system

As shown in preceding section, for each $U_0 \in K$, there exists a unique global solution $U = U(t;U_0)$ to (1.1) and the solution is continuous with respect to the initial value. Therefore, we can define a semigroup $\{S(t)\}_{t\ge0}$ acting on K by $S(t)U_0 = U(t;U_0)$. Such that the mapping $(t,U_0) \mapsto S(t)U_0$ is continuous from $[0,\infty) \times K$ into K, where K is equipped with the distance induced from the universal space X. Hence, we have constructed a dynamical system $(S(t),K,X)$ determined from (1.1).

We now verify that $(S(t),K,X)$ admits a bounded absorbing set. Indeed, let $R > 0$ be any radius and let U_0 be in K with $\|U_0\|_X \le R$. Then, from (2.3) there exists a time t_R such that $\|U(t)\|_X \le \widetilde{C} + 1$ for every $t \ge t_R$, where $\widetilde{C}$ is the constant appearing in (2.3). That is,

$$\sup_{\substack{U_0\in K,\\ \|U_0\|_X\le R}} \ \sup_{t\ge t_R} \|S(t)U_0\|_X \le \widetilde{C} + 1.$$

This then shows that the set

$$\mathcal{B} = \{U \in K; \|U\|_X \le \widetilde{C} + 1\}$$

is a bounded absorbing set of $(S(t), K, X)$.

Since $\mathcal{B}$ itself is absorbed by $\mathcal{B}$, there exists a time $t_{\mathcal{B}} > 0$ such that $S(t)\mathcal{B} \subset \mathcal{B}$ for every $t \geq t_{\mathcal{B}}$. We then consider the set

$$\widetilde{\mathcal{X}} = \bigcup_{0 \leq t < \infty} S(t)\mathcal{B} = \bigcup_{0 \leq t \leq t_{\mathcal{B}}} S(t)\mathcal{B}.$$

It is clean that $\widetilde{\mathcal{X}}$ is an absorbing and invariant bounded set of K. By Theorem 2.2 we then verify that

$$\|AS(t)U_0\| \leq C_{\widetilde{\mathcal{X}}} t^{-1}, \quad 0 < t \leq T_{\widetilde{\mathcal{X}}}, \ U_0 \in \widetilde{\mathcal{X}}$$

with a sufficiently small time $T_{\widetilde{\mathcal{X}}} > 0$ and a constant $C_{\widetilde{\mathcal{X}}} > 0$. In view of such a smoothing effect, we introduce the set

$$\mathcal{X} = S(T_{\widetilde{\mathcal{X}}})\widetilde{\mathcal{X}} \subset \widetilde{\mathcal{X}}.$$

It is easy to see that this set is also an absorbing and invariant set. In addition, $\mathcal{X} \subset \mathcal{D}(A)$ with the estimate

$$\|AU\| = \|AS(T_{\widetilde{\mathcal{X}}})U_0\| \leq C_{\widetilde{\mathcal{X}}} T_{\widetilde{\mathcal{X}}}^{-1}, \quad U = S(T_{\widetilde{\mathcal{X}}})U_0 \in \mathcal{X}, \ U_0 \in \widetilde{\mathcal{X}}.$$

We have thus verified the following result.

Theorem 3.1. *The dynamical system $(S(t), K, X)$ determined from the problem* (1.1) *can be reduced to a dynamical system $(S(t), \mathcal{X}, X)$ in which the phase space is a bounded set of $\mathcal{D}(A)$.*

Since $\mathcal{X}$ is a bounded set of $\mathcal{D}(A)$, it is meaningful to replace the universal space X by $X_\theta = \mathcal{D}(A^\theta)$ with any exponent $0 < \theta < 1$ and consider a dynamical system $(S(t), \mathcal{X}, X_\theta)$, where $\mathcal{X}$ is now a metric space with the distance $d_\theta(U, V) = \|A^\theta(U - V)\|$.

Corollary 3.2. *For each $0 < \theta < 1$, $(S(t), \mathcal{X}, X_\theta)$ is a dynamical system.*

Proof. By the moment inequality (cf. [17]) and the boundedness of $\mathcal{X}$ in $\mathcal{D}(A)$, it follows that

$$\|A^\theta(U - V)\| \leq C\|A(U - V)\|^\theta \|U - V\|^{1-\theta} \leq C_{\mathcal{X}}\|U - V\|^{1-\theta}, \quad U, V \in \mathcal{X}$$

with some constant $C_{\mathcal{X}}$. This shows that the mapping $(t, U) \mapsto S(t)U$ is continuous from $[0, \infty) \times \mathcal{X}$ into X_θ. □

4. Lyapunov function

In this section we shall construct a Lyapunov function $\Psi(U)$ for the dynamical system $(S(t), K, X)$ and shall establish some results concerning the asymptotic behavior of trajectories $S(t)U_0$'s.

Let $U_0 \in K$ and let $S(t)U_0 = U(t) = (u(t), v(t), w(t))$ for $0 \leq t < \infty$. Set $\varphi(t) = fu(t) - hv(t)$, $0 \leq t < \infty$. From the first and second equations of (1.1) it is easily observed that

$$\frac{\partial \varphi}{\partial t} = f\beta\delta w - \{\gamma(v) + f + h\}\varphi - h\{\gamma(v)v + fv\}, \qquad 0 < t < \infty.$$

Multiply this by $\varphi(t) = \frac{\partial v}{\partial t}$ and integrate the product in Ω. Then,

$$(4.1)\quad \frac{1}{2}\frac{d}{dt}\int_\Omega \varphi^2 dx + h\frac{d}{dt}\int_\Omega \Gamma(v)dx - f\beta\delta\int_\Omega \frac{\partial v}{\partial t} w\, dx = -\int_\Omega \{\gamma(v)+f+h\}\left(\frac{\partial v}{\partial t}\right)^2 dx,$$

where $\Gamma(v) = \int_0^v \{\gamma(v)v + fv\}dv$.

While, multiplying the third equation of (1.1) by $\frac{\partial w}{\partial t}$ and integrating the product in Ω, we obtain that

$$(4.2)\qquad \frac{d}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \frac{\beta}{2}\frac{d}{dt}\int_\Omega w^2 dx - \alpha\int_\Omega v\frac{\partial w}{\partial t}\, dx = -\int_\Omega \left(\frac{\partial w}{\partial t}\right)^2 dx.$$

These two energy equalities (4.1) and (4.2) then provide that

$$(4.3)\quad \frac{d}{dt}\int_\Omega \left[\frac{\alpha}{2}\varphi^2 + \frac{df\beta\delta}{2}|\nabla w|^2 + h\alpha\Gamma(v) + \frac{f\beta^2\delta}{2}w^2 - (f\alpha\beta\delta)vw\right]dx$$
$$= -\int_\Omega \left[\alpha\{\gamma(v)+f+h\}\left(\frac{\partial v}{\partial t}\right)^2 + f\beta\delta\left(\frac{\partial w}{\partial t}\right)^2\right]dx \le 0, \qquad 0 < t < \infty.$$

Note that

$$\frac{\alpha}{2}(fu-hv)^2 + \frac{df\beta\delta}{2}|\nabla w|^2 + h\alpha\Gamma(v) + \frac{f\beta^2\delta}{2}w^2 - (f\alpha\beta\delta)vw \ge C$$

with some constant C independent of U. This shows that the functional

$$(4.4)\quad \Psi(U) = \int_\Omega \left[\frac{\alpha}{2}(fu-hv)^2 + \frac{df\beta\delta}{2}|\nabla w|^2 + h\alpha\Gamma(v)\right.$$
$$\left. + \frac{f\beta^2\delta}{2}w^2 - (f\alpha\beta\delta)vw\right]dx, \qquad U \in \mathcal{D}(A^{\frac{1}{2}})$$

is a Lyapunov function for the present dynamical system $(S(t), K, X)$.

From these arguments we obtain the following energy estimates.

Theorem 4.1. *For any trajectory $S(t)U_0 = U(t)$, we have*

$$(4.5)\qquad \int_1^\infty \left\|\frac{dU}{dt}(t)\right\|_{L^2}^2 dt < \infty.$$

Proof. Integrate both the sides of (4.3) in t on an interval $[1,T]$. Then,

$$\int_1^T\int_\Omega \left[\alpha\{\gamma(v)+f+h\}\left(\frac{\partial v}{\partial t}\right)^2 + f\beta\delta\left(\frac{\partial w}{\partial t}\right)^2\right]dxdt$$
$$\le \int_\Omega \left[\frac{\alpha}{2}\varphi(1)^2 + \frac{df\beta\delta}{2}|\nabla w(1)|^2 + h\alpha\Gamma(v(1)) + \frac{f\beta^2\delta}{2}w(1)^2 + f\alpha\beta\delta v(T)w(T)\right]dx.$$

Due to (2.7) and (2.8),

$$(4.6)\qquad \int_1^\infty\int_\Omega \left[\alpha\{\gamma(v)+f+h\}\left(\frac{\partial v}{\partial t}\right)^2 + f\beta\delta\left(\frac{\partial w}{\partial t}\right)^2\right]dxdt < \infty.$$

Differentiating both the sides of the first equations of (1.1), we have

$$\frac{\partial^2 u}{\partial t^2} = \beta\delta\frac{\partial w}{\partial t} - (\gamma(v) + f)\frac{\partial u}{\partial t} - 2au(v-b)\frac{\partial v}{\partial t}, \qquad 0 < t < \infty.$$

Multiply this by $\frac{\partial u}{\partial t}$ and integrate the product in Ω. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx &= \int_\Omega \left(\beta\delta\frac{\partial w}{\partial t} - 2au(v-b)\frac{\partial v}{\partial t}\right)\frac{\partial u}{\partial t}dx - \int_\Omega (\gamma(v)+f)\left(\frac{\partial u}{\partial t}\right)^2 dx\\ &\leq Cp(\|U_0\|_X)\int_\Omega \left\{\left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right\}dx - \frac{f}{2}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx.\end{aligned}$$

Integrating both the sides in t, we obtain that

$$f\int_1^T \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dxdt \leq \int_\Omega \left(\frac{\partial u}{\partial t}(1)\right)^2 dx + Cp(\|U_0\|_X)\int_1^T \int_\Omega \left\{\left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right\}dxdt.$$

Therefore, in view of (4.6), we conclude that

$$\int_1^\infty \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dxdt < \infty.$$

This together with (4.6) then yields the desired estimate (4.5). □

Theorem 4.2. *For any trajectory $S(t)U_0 = U(t)$, as $t \to \infty$, the derivative $\frac{dU}{dt}(t)$ tends to 0 in the L^2 topology.*

Proof. We prove the assertion of theorem by contradiction. Suppose that $\frac{dU}{dt}(t)$ might not converge to 0 in $L^2(\Omega)$ as $t \to \infty$. Then there would exist a number $\varepsilon > 0$ and a time sequence $\{t_n\}$ tending to ∞ such that

$$\left\|\frac{dU}{dt}(t_n)\right\|_{L^2}^2 \geq \varepsilon, \qquad n = 1, 2, 3, \ldots.$$

In the meantime, by Propositions 2.5 and 2.6, we have

$$\left|\frac{d}{dt}\left\|\frac{dU}{dt}(t)\right\|_{L^2}^2\right| = 2\left|\left(\frac{d^2U}{dt^2}(t), \frac{dU}{dt}(t)\right)_{L^2}\right| \leq M, \qquad 1 \leq t < \infty$$

with some constant M. Consequently, by the mean-value theorem,

$$\left\|\frac{dU}{dt}(t)\right\|_{L^2}^2 \geq \begin{cases} M(t - t_n + \frac{\varepsilon}{M}), & t_n - \frac{\varepsilon}{M} \leq t \leq t_n,\\ -M(t - t_n - \frac{\varepsilon}{M}), & t_n \leq t \leq t_n + \frac{\varepsilon}{M}.\end{cases}$$

This is a contradiction to the fact that $\|\frac{dU}{dt}(t)\|_{L^2}^2$ is integrable in $(1, \infty)$, i.e., (4.5). □

5. ω-limit sets

In this section, we shall introduce three types of ω-limit sets, namely, $\omega(U_0)$, L^2-$\omega(U_0)$ and w*-$\omega(U_0)$, and shall investigate their relations.

As well known, the (usual) ω-limit set of $S(t)U_0$, $U_0 \in K$, is defined by

$$\omega(U_0) = \bigcap_{t\geq 0} \overline{\{S(\tau)U_0;\ t \leq \tau < \infty\}} \qquad \text{(closure in the topology of } X\text{)},$$

namely, $\overline{U} \in \omega(U_0)$ if and only if there exists a time sequence $\{t_n\}$ tending to ∞ such that $S(t_n)U_0 \to \overline{U}$ in the topology of X. There is some numerical simulation (see [6]) suggests that there exists a trajectory which starts from a continuous initial functions $U_0 = (u_0(x), v_0(x), w_0(x)) \in K$ but, as $t \to \infty$, converges to a discontinuous stationary solution $\overline{U} = (\overline{u}(x), \overline{v}(x), \overline{w}(x))$. If this phenomenon is true, then any sequence $S(t_n)U_0$ cannot converge to $\overline{U}$ in the topology of X, namely, it is possible that $\omega(U_0) = \emptyset$.

We define the L^2 topology of X as follows. A sequence $\{(u_n, v_n, w_n)\}$ in X is said to be L^2 convergent to $(u_0, v_0, w_0) \in X$ as $n \to \infty$, if

$$\begin{cases} u_n \to u_0 & \text{strongly in } L^2(\Omega), \\ v_n \to v_0 & \text{strongly in } L^2(\Omega), \\ w_n \to w_0 & \text{strongly in } L^2(\Omega). \end{cases}$$

Then, using this topology we define the L^2-ω-limit set of $S(t)U_0$, $U_0 \in K$, by

$$L^2\text{-}\omega(U_0) = \bigcap_{t\geq 0} \overline{\{S(\tau)U_0;\ t \leq \tau < \infty\}} \qquad \text{(closure in the } L^2 \text{ topology of } X\text{)}.$$

In addition, we may equip X with the weak* topology. A sequence $\{(u_n, v_n, w_n)\}$ in X is said to be weak* convergent to $(u_0, v_0, w_0) \in X$ as $n \to \infty$, if

$$\begin{cases} u_n \to u_0 & \text{weak* in } L^\infty(\Omega), \\ v_n \to v_0 & \text{weak* in } L^\infty(\Omega), \\ w_n \to w_0 & \text{strongly in } L^2(\Omega). \end{cases}$$

Using this topology, we define the w*-ω-limit set of $S(t)U_0$, $U_0 \in K$, by

$$\text{w}^*\text{-}\omega(U_0) = \bigcap_{t\geq 0} \overline{\{S(\tau)U_0;\ t \leq \tau < \infty\}} \quad \text{(closure in the weak* topology of } X\text{)}.$$

Theorem 5.1. *For each $U_0 \in K$, w*-$\omega(U_0)$ is a nonempty set.*

Proof. Let $U_0 \in K$ and $U(t) = S(t)U_0$. Since $\mathcal{B}$ is an absorbing set of $(S(t), K, X)$, it follows that there exists a sequence of time $t_n \to \infty$ such that $S(t_n)U_0 \in \mathcal{B}$. Therefore, $\{u(t_n)\}$ is a bounded sequence in $L^\infty(\Omega)$. By Banach-Alaoglu's theorem, we can take a subsequence $\{u(t_{n'})\}$ of $\{u(t_n)\}$ such that $u(t_{n'}) \to \overline{u}$ weak* in $L^\infty(\Omega)$. Similarly, from the bounded sequence $\{v(t_{n'})\}$, we have a subsequence $\{v(t_{n''})\}$ such that $v(t_{n''}) \to \overline{v}$ weak* in $L^\infty(\Omega)$. Finally, by the boundedness of sequence $\{w(t_{n''})\}$ in $H^{2\eta}(\Omega)$, there exists a subsequence $\{w(t_{n'''})\}$ such that $w(t_{n'''}) \to \overline{w}$ strongly in $L^2(\Omega)$. Then, by the definition, we deduce that $(\overline{u}, \overline{v}, \overline{w})$ belongs to w*-$\omega(U_0)$. $\square$

In general we observe the following relations.

Theorem 5.2. *For each $U_0 \in K$, $\omega(U_0) \subset L^2\text{-}\omega(U_0) \subset \mathrm{w}^*\text{-}\omega(U_0)$.*

Proof. The first relation $\omega(U_0) \subset L^2\text{-}\omega(U_0)$ is obvious by the definition.

Let $\overline{U} = (\overline{u}, \overline{v}, \overline{w}) \in L^2\text{-}\omega(U_0)$. Then, there exists a sequence $\{t_n\}$ tending to ∞ such that $S(t_n)U_0 = (u(t_n), v(t_n), w(t_n)) \to \overline{U}$ in the L^2 topology of X. Let $\varphi \in L^1(\Omega)$. For any $f \in L^2(\Omega)$,

$$\left| \int_\Omega \varphi\{u(t_n) - \overline{u}\} dx \right| \le \|\varphi - f\|_{L^1} \|u(t_n) - \overline{u}\|_{L^\infty} + \left| \int_\Omega f\{u(t_n) - \overline{u}\} dx \right|.$$

Since $L^2(\Omega)$ is dense in $L^1(\Omega)$ and since (2.7) is valid, we verify that, as $t_n \to \infty$,

$$\left| \int_\Omega \varphi\{u(t_n) - \overline{u}\} dx \right| \to 0.$$

Hence, $u(t_n) \to \overline{u}$ in the weak* topology of $L^\infty(\Omega)$. Due to (2.7), it is the same for the weak* convergence of $v(t_n)$ to $\overline{v}$. Thus we have $\overline{U} \in \mathrm{w}^*\text{-}\omega(U_0)$. □

We do not know whether the converse relation $\mathrm{w}^*\text{-}\omega(U_0) \subset L^2\text{-}\omega(U_0)$ is true in general or not. We can however prove some weak result.

Theorem 5.3. *For $U_0 \in K$, let there exist a sequence $\{t_n\}$ tending to ∞ such that $S(t_n)U_0 = (u(t_n), v(t_n), w(t_n))$ converges to a triplet of functions $\overline{U} = (\overline{u}, \overline{v}, \overline{w}) \in X$ almost everywhere in Ω. Then, $\overline{U} \in L^2\text{-}\omega(U_0)$.*

Proof. By virtue of (2.7) and (2.8), the almost everywhere convergence implies L^2 convergence for each sequence of $u(t_n)$, $v(t_n)$ and $w(t_n)$. Hence, $\overline{U} \in L^2\text{-}\omega(U_0)$. □

The rest of this section is devoted to proving some structural results for the ω-limit sets under specific conditions assumed to hold for the coefficients of equations in (1.1).

Theorem 5.4. *Assume that $h > \frac{f\alpha\delta}{c+f}$. Then, $\omega(U_0) = L^2\text{-}\omega(U_0) = \mathrm{w}^*\text{-}\omega(U_0) = \{(0,0,0)\}$ for every $U_0 \in K$.*

Proof. Let $U_0 = (u_0, v_0, w_0) \in K$ and let $S(t)U_0 = (u(t), v(t), w(t))$ be the global solution. Multiply the first equation of (1.1) by $2(c+f)u$ and integrate the product in Ω. Then,

$$\begin{aligned}(5.1)\quad (c+f)\frac{d}{dt}\int_\Omega u^2 dx + 2(c+f)^2 \int_\Omega u^2 dx - 2(c+f)\beta\delta \int_\Omega wu dx \\ = -2a(c+f)\int_\Omega (v-b)^2 u^2 dx \le 0, \qquad 0 < t < \infty.\end{aligned}$$

Similarly, multiply the second equation of (1.1) by $\frac{2(c+f)\alpha\delta}{f} v$ and integrate the product in Ω. Then,

$$\begin{aligned}(5.2)\quad \frac{(c+f)\alpha\delta}{f}\frac{d}{dt}\int_\Omega v^2 dx + 2(\alpha\delta)^2 \int_\Omega v^2 dx - 2(c+f)\alpha\delta \int_\Omega uv dx \\ + \frac{2(c+f)\alpha\delta}{f}\left(h - \frac{f\alpha\delta}{c+f}\right)\int_\Omega v^2 dx = 0, \qquad 0 < t < \infty.\end{aligned}$$

Multiply the third equation of (1.1) by $2\beta\delta^2 w$ and integrate the product in Ω. Then,

$$\begin{aligned}(5.3)\quad \beta\delta^2\frac{d}{dt}\int_\Omega w^2dx + 2(\beta\delta)^2\int_\Omega w^2dx - 2\alpha\beta\delta^2\int_\Omega vwdx \\ = -2d\beta\delta^2\int_\Omega |\nabla w|^2dx \le 0, \qquad 0<t<\infty.\end{aligned}$$

Summing up (5.1), (5.2) and (5.3), we obtain that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\Big((c+f)u^2 + \frac{(c+f)\alpha\delta}{f}v^2 + \beta\delta^2 w^2\Big)dx + 2\int_\Omega\{((c+f)u)^2 + (\alpha\delta v)^2 + (\beta\delta w)^2\}dx \\ &\qquad - 2\int_\Omega\{(c+f)u\alpha\delta v + \alpha\delta v\beta\delta w + \beta\delta w(c+f)u\}dx + 3\int_\Omega \varepsilon v^2dx \le 0,\end{aligned}$$

where $\varepsilon = \frac{2(c+f)\alpha\delta}{3f}\Big(h - \frac{f\alpha\delta}{c+f}\Big) > 0$. We here notice that

$$\begin{aligned}&2\{((c+f)u)^2 + (\alpha\delta v)^2 + (\beta\delta w)^2 - (c+f)u\alpha\delta v - \alpha\delta v\beta\delta w - \beta\delta w(c+f)u\} + 3\varepsilon v^2 \\ &\quad = \Big\{\frac{\big((c+f)\alpha\delta\big)^2}{\alpha^2\delta^2+\varepsilon}u^2 - 2(c+f)u\alpha\delta v + (\alpha^2\delta^2+\varepsilon)v^2\Big\} \\ &\quad + \Big\{(\alpha^2\delta^2+\varepsilon)v^2 - 2\alpha\delta v\beta\delta w + \frac{(\alpha\delta)^2(\beta\delta)^2}{\alpha^2\delta^2+\varepsilon}w^2\Big\} + \{\beta\delta w - (c+f)u\}^2 \\ &\quad + \varepsilon\Big\{\frac{(c+f)^2}{\alpha^2\delta^2+\varepsilon}u^2 + v^2 + \frac{(\beta\delta)^2}{\alpha^2\delta^2+\varepsilon}w^2\Big\}.\end{aligned}$$

Therefore, with an appropriate exponent $\rho > 0$ and appropriate constants $C_i > 0$, $i =$ 1, 2, 3,

$$\frac{d}{dt}\int_\Omega(C_1u^2 + C_2v^2 + C_3w^2)dx + \rho\int_\Omega(C_1u^2 + C_2v^2 + C_3w^2)dx \le 0.$$

We thus conclude that

$$\begin{aligned}&C_1\|u(t)\|^2_{L^2} + C_2\|v(t)\|^2_{L^2} + C_3\|w(t)\|^2_{L^2} \\ &\qquad \le e^{-\rho t}(C_1\|u_0\|^2_{L^2} + C_2\|v_0\|^2_{L^2} + C_3\|w_0\|^2_{L^2}), \qquad 0<t<\infty.\end{aligned}$$

As a result, as $t\to\infty$, $S(t)U_0$ converges to $(0,0,0)$ in the L^2 topology. More strongly, since $\|w(t)\|_{L^\infty} \le C_\varepsilon\|w(t)\|_{H^{1+\varepsilon}} \le C_\varepsilon\|w(t)\|^{(1-\varepsilon)/2}_{L^2}\|w(t)\|^{(1+\varepsilon)/2}_{H^2}$, we deduce from the L^2 convergence of $w(t)$ that in the L^∞ topology (due to (2.14)). Furthermore, from the formula (2.9) and (2.10), this implies convergence of $u(t)$ and $v(t)$ to 0 in the L^∞ topology. In this way, we ultimately conclude that, as $t\to\infty$, $S(t)U_0$ converges to $(0,0,0)$ in the L^∞ topology. From this the assertion of theorem follows immediately. □

Theorem 5.5. *Assume that $ab^2 < 3(c+f)$. Then, L^2-$\omega(U_0)$ = w*-$\omega(U_0)$ for every $U_0\in K$.*

Proof. Let $S(t)U_0 = U(t) = (u(t), v(t), w(t))$. Consider any time sequence $\{t_n\}$ which tends to ∞ as $n\to\infty$. By (2.7), $\|w(t_n)\|_{H^2}$ is a bounded sequence; so, we can choose a subsequence $\{t_{n'}\}$ for which $\{w(t_{n'})\}$ is convergent to $\overline{w}$ in $H^{1+\varepsilon}(\Omega)$ and hence in $L^\infty(\Omega)$. From the first and second equations of (1.1) it is easily observed that

$$\Big(\gamma(v(t_{n'})) + f\Big)v(t_{n'}) = \frac{f}{h}\left\{\beta\delta w(t_{n'}) - \frac{du}{dt}(t_{n'}) - \frac{\gamma(v(t_{n'})) + f}{f}\frac{dv}{dt}(t_{n'})\right\}.$$

Here, we introduce the cubic function

$$P(v) \equiv (\gamma(v) + f)v = av^3 - 2abv^2 + (ab^2 + c + f)v, \qquad -\infty < v < \infty.$$

It is easy to see the following property.

Lemma 5.6. *When $ab^2 < 3(c+f)$, $w = P(v)$ is a monotone increasing function for $v \in (-\infty, \infty)$. Its inverse function $P^{-1}(w)$ is a single-valued smooth function for w with uniformly bounded derivative in the whole real axis $w \in (-\infty, \infty)$.*

Proof of lemma. Obviously we have

$$P'(v) = 3av^2 - 4abv + (ab^2 + c + f) = 3a\left(v - \frac{2b}{3}\right)^2 - \frac{ab^2 - 3(c+f)}{3} > 0.$$

Therefore, the assertion of lemma is clear. □

Using $P^{-1}(w)$, we can write

$$v(t_{n'}) = P^{-1}\left(\frac{f}{h}\left\{\beta\delta w(t_{n'}) - \frac{du}{dt}(t_{n'}) - \frac{\gamma(v(t_{n'})) + f}{f}\frac{dv}{dt}(t_{n'})\right\}\right).$$

Since $w(t_{n'}) \to \overline{w}$ in $L^\infty(\Omega)$ and since Theorem 4.2 is true, we conclude that $v(t_{n'})$ converges to $\overline{v} = P^{-1}(\frac{f\beta\delta}{h}\overline{w})$ in $L^2(\Omega)$. Since Theorem 4.2 provides in particular that, as $t \to \infty$, $fu(t) - hv(t) \to 0$ in $L^2(\Omega)$, we conclude also that $u(t_{n'})$ converges to $\frac{h}{f}\overline{v}$ in $L^2(\Omega)$. Thus we have shown that $(u(t_{n'}), v(t_{n'}), w(t_{n'})) \to (\overline{u}, \overline{v}, \overline{w})$ in $L^2(\Omega)$.

We now know that any sequence $(u(t_n), v(t_n), w(t_n))$ has a subsequence which converges to some vector of X in the L^2 topology. Hence, the relation w*-$\omega(U_0) \subset L^2$-$\omega(U_0)$ is proved, cf., Proof of Theorem 5.2. □

6. Constituents of L^2 ω-limit sets

In this the section, we shall show that every L^2 ω-limit set consists of stationary solutions of (2.1). For this end, we begin with verifying the following Proposition.

Proposition 6.1. *For each $U_0 \in K$, L^2-$\omega(U_0)$ is an invariant set of $S(t)$, i.e.,*

$$S(t)\Big(L^2\text{-}\omega(U_0)\Big) \subset L^2\text{-}\omega(U_0), \qquad t > 0.$$

Proof. In the proof of this proposition, it is essential to show that $S(t)$ is continuous from K into itself in the L^2 topology.

To see this, consider two initial values $U_{01} = (u_{01}, v_{01}, w_{01})$ and $U_{02} = (u_{02}, v_{02}, w_{02})$ in K, and let $(u_1(t), v_1(t), w_1(t))$ and $(u_2(t), v_2(t), w_2(t))$ be the solutions to (2.1) with the initial value U_{01} and U_{02}, respectively. Let $T > 0$ be arbitrarily fixed time, and let t varies in the bounded interval $[0, T]$.

Then, from (2.9),

$$u_i(t) = e^{-\int_0^t\{\gamma(v_i)+f\}ds}u_{0i} + \beta\delta \int_0^t e^{-\int_\tau^t\{\gamma(v_i)+f\}ds}w_i(\tau)d\tau, \qquad i = 1,\, 2.$$

Consequently,

$$\begin{aligned} u_2(t) - u_1(t) &= e^{-\int_0^t\{\gamma(v_1)+f\}ds}\big(e^{-\int_0^t\{\gamma(v_2)-\gamma(v_1)\}ds} - 1\big)u_{01} \\ &\quad + e^{-\int_0^t\{\gamma(v_2)+f\}ds}(u_{02} - u_{01}) + \beta\delta \int_0^t e^{-\int_\tau^t\{\gamma(v_2)+f\}ds}(w_2(\tau) - w_1(\tau))d\tau \\ &\quad + \beta\delta \int_0^t e^{-\int_\tau^t\{\gamma(v_1)+f\}ds}\big(e^{-\int_\tau^t\{\gamma(v_2)-\gamma(v_1)\}ds} - 1\big)w_1(\tau)d\tau. \end{aligned}$$

In view of (2.7) and (2.8), we obtain that

$$\begin{aligned} &\|u_2(t) - u_1(t)\|_{L^2} \le \|u_{02} - u_{01}\|_{L^2} \\ &\quad + Cp(\|U_{01}\|_X + \|U_{02}\|_X)\Big\{ \Big\|e^{-\int_0^t\{\gamma(v_2)-\gamma(v_1)\}ds} - 1\Big\|_{L^2} + \int_0^t \|w_2(\tau) - w_1(\tau)\|_{L^2}d\tau \\ &\quad + \int_0^t \Big\|e^{-\int_\tau^t\{\gamma(v_2)-\gamma(v_1)\}ds} - 1\Big\|_{L^2}(1 + \tau^{-\eta})d\tau\Big\}, \qquad 0 \le t \le T. \end{aligned}$$

For any $R > 0$, there exists a constant $C_R > 0$ such that $|e^\xi - 1| \le C_R|\xi|$ holds for all $|\xi| \le R$. Using this estimate, we verify that

$$\Big\|e^{-\int_0^t\{\gamma(v_2)-\gamma(v_1)\}ds} - 1\Big\|_{L^2} \le Cp(\|U_{01}\|_X + \|U_{02}\|_X) \int_0^t \|v_2(\tau) - v_1(\tau)\|_{L^2}d\tau.$$

Similarly,

$$\begin{aligned} &\int_0^t \Big\|e^{-\int_\tau^t\{\gamma(v_2)-\gamma(v_1)\}ds} - 1\Big\|_{L^2} \tau^{-(1+\varepsilon)/2}d\tau \\ &\quad \le Cp(\|U_{01}\|_X + \|U_{02}\|_X) \int_0^t \int_\tau^t \|v_2(s) - v_1(s)\|_{L^2}\tau^{-(1+\varepsilon)/2}dsd\tau \\ &\quad \le Cp(\|U_{01}\|_X + \|U_{02}\|_X) \int_0^t \|v_2(s) - v_1(s)\|_{L^2}ds. \end{aligned}$$

Hence,

$$\begin{aligned} (6.1)\quad &\|u_2(t) - u_1(t)\|_{L^2} \le \|u_{02} - u_{01}\|_{L^2} \\ &+ Cp(\|U_{01}\|_X + \|U_{02}\|_X) \int_0^t \{\|v_2(\tau) - v_1(\tau)\|_{L^2} + \|w_2(\tau) - w_1(\tau)\|_{L^2}\}d\tau, \qquad 0 \le t \le T. \end{aligned}$$

In a similar way, from (2.10) it follows that

$$\|v_2(t) - v_1(t)\|_{L^2} \le \|v_{02} - v_{01}\|_{L^2} + C\int_0^t \|u_2(\tau) - u_1(\tau)\|_{L^2} d\tau, \qquad 0 \le t \le T. \tag{6.2}$$

Finally, from (2.11) we have

$$w_2(t) - w_1(t) = e^{-tA}(w_{02} - w_{01}) + \alpha\int_0^t e^{-(t-\tau)A}\{v_2(\tau) - v_1(\tau)\}d\tau.$$

Therefore,

$$\|w_2(t) - w_1(t)\|_{L^2} \le \|w_{02} - w_{01}\|_{L^2} + \alpha\int_0^t \|v_2(\tau) - v_1(\tau)\|_{L^2} d\tau, \qquad 0 \le t \le T. \tag{6.3}$$

Summing up (6.1), (6.2) and (6.3) and using Gronwall's inequality, we conclude that

$$\begin{aligned}\|u_2(t) - u_1(t)\|_{L^2} &+ \|v_2(t) - v_1(t)\|_{L^2} + \|w_2(t) - w_1(t)\|_{L^2} \\ &\le \|U_{02} - U_{01}\|_{L^2} e^{C_p(\|U_{01}\|_X + \|U_{02}\|_X)t}, \qquad 0 \le t \le T.\end{aligned}$$

This shows that, for $0 \le t \le T$, the semigroup $S(t)$ is continuous in the L^2 topology. But, as $T > 0$ is arbitrary, it is the same for any $0 \le t < \infty$.

It is now immediate to prove the assertion of theorem. Let $\overline{U} \in L^2\text{-}\omega(U_0)$. By definition there exists a sequence t_n tending to ∞ such that $S(t_n)U_0 \to \overline{U}$ in the L^2 topology. By the L^2 continuity proved above, we have $S(t_n + t)U_0 = S(t)S(t_n)U_0 \to S(t)\overline{U}$ in L^2. Therefore, $S(t)\overline{U} \in L^2\text{-}\omega(U_0)$. □

Theorem 6.2. *For any $U_0 \in K$, $L^2\text{-}\omega(U_0)$ consists of equilibria of the dynamical system.*

Proof. Let $\overline{U} = (\overline{u}, \overline{v}, \overline{w}) \in L^2\text{-}\omega(U_0)$. There exists a sequence $t_n \to \infty$ such that $S(t_n)U_0 = U(t_n) \to \overline{U}$ in the L^2 topology. Since $w(t_n)$ is a bounded sequence in $H^2(\Omega)$, we can take a subsequence $\{w(t_{n'})\}$ of $\{w(t_n)\}$ such that $w(t_{n'}) \to \overline{w}'$ strongly in $H^1(\Omega)$. It is then easy to see that $\overline{w} = \overline{w}'$. Meanwhile, in view of (2.7), $u(t_n) \to \overline{u}$ and $v(t_n) \to \overline{v}$ in any L^p topology with finite p such that $2 \le p < \infty$.

By these facts we conclude that the Lyapunov function $\Psi(U(t_{n'}))$ given by (4.4) is convergent to $\Psi(\overline{U})$ as $t_{n'} \to \infty$. That is,

$$\Psi(\overline{U}) = \lim_{n' \to \infty} \Psi(U(t_{n'})) = \inf_{0 \le t < \infty} \Psi(S(t)U_0) \equiv \Psi_\infty.$$

This means that $\Psi(\overline{U}) \equiv \Psi_\infty$ for all $\overline{U}$'s of vectors in $L^2\text{-}\omega(U_0)$. By Proposition 6.1, $S(t)\overline{U} \in L^2\text{-}\omega(U_0)$ for every $t > 0$. Hence,

$$\Psi(S(t)\overline{U}) \equiv \Psi_\infty, \qquad 0 < t < \infty,\ \overline{U} \in L^2\text{-}\omega(U_0).$$

Furthermore, let $S(t)\overline{U} = \overline{U}(t) = (\overline{u}(t), \overline{v}(t), \overline{w}(t))$; then, by (4.3), we have

$$\frac{d}{dt}\Psi(\overline{U}(t)) = -\int_\Omega \left[\alpha\{\gamma(\overline{v}) + f + h\}\left(\frac{\partial \overline{v}}{\partial t}\right)^2 + f\beta\delta\left(\frac{\partial \overline{w}}{\partial t}\right)^2\right] dx \equiv 0, \quad 0 < t < \infty.$$

Hence, $\frac{\partial \overline{v}}{\partial t}(t) \equiv \frac{\partial \overline{w}}{\partial t}(t) \equiv 0$ for $0 < t < \infty$. In addition, from the second equation of (2.1), it follows that $f\overline{u}(t) \equiv h\overline{v}(t)$; hence, $\frac{\partial \overline{u}}{\partial t}(t) \equiv 0$ for $0 < t < \infty$. Thus it has been shown that $S(t)\overline{U} \equiv \overline{U}$ for every $0 < t < \infty$, namely, $\overline{U}$ must be an equilibrium. □

References

[1] L. H. Chuan and A. Yagi, *Asymptotic behavior of solutions for forest kinetic equations*, Preprint.

[2] L. H. Chuan and A. Yagi, *Stationary solutions to forest equations*, Preprint.

[3] Yu A. Kuznetsov, M. Ya. Antonovsky, V. N. Biktashev and A. Aponina, *A cross-diffusion model of forest boundary dynamics*, J. Math. Biol. **32** (1994), 219-232.

[4] Y. Lin and Y. Liu, *Qualitative analysis for a model of forest with diffusion and nonlocal effects*, Nonlinear Analysis **39** (2000), 217-229.

[5] E. Nakaguchi and A. Yagi, *Fully discrete approximation by Galerkin Runge-Kutta methods for quasilinear parabolic systems*, Hokkaido Math. J. **31** (2002), 385–429.

[6] H. Nakata, *Numerical simulations for forest boundary dynamics model*, Master's thesis, Osaka University (2004).

[7] K. Osaki and A. Yagi, *Global existence for a chemotaxis-growth system in* $\mathbb{R}^2$, Adv. Math. Sci. Appl. **12** (2002), 587-606.

[8] Y. Wu, *Stability of travelling waves for a cross-diffusion model*, J. Math. Anal. Appli. **215** (1997), 388-414.

[9] Y. Wu and Y. Lin, *The stability of steady states for a model with diffusion and spatial average*, J. Math. Anal. Appli. **232** (1999), 259-271.

[10] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.

[11] R. Dautray and J. L. Lions, *Mathematical analysis and numerical methods for science and technology*, Vol. 2, Springer-Verlag, Berlin, 1988.

[12] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, London, 1985.

[13] J. D. Murray, *Mathematical Biology, II: Spacial Models and Biomedical Applications*, Springer, New York, 2003.

[14] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge Univ. Press, Cambridge, 2001.

[15] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations*, Springer, New York, 2002.

[16] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed., Springer-Verlag, Berlin, 1997.

[17] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, 1978.