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On the existence of the global attractor for a class of degenerate parabolic equations

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p > \max\{1, 2N/(N+2)\}$ so that the embedding $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ is compact. We shall consider the long time behavior of solutions to the following equation (E):

$$(E) \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u + f(u) = g(x), & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & x \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where Δ_p denotes the p-Laplacian defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

The equation (E) is one of the possible extentions of the semilinear reaction-diffusion equation, which corresponds to the case where $p = 2$. Semiliear reaction-diffusion equations have clear meanings as models for physical phenomena. On the contrary, in general, the equation (E) for the case $p \neq 2$ is a theoretical extention of semilenear reaction-diffusion equations. However, it seems important to study such equations for better understanding of the phenomena described by semilinear reaction-diffusion equations. Although many authors have been studied the equation (E) when $p = 2$ (see [6], [7]), the degenerate case ($p > 2$) and the singular case ($p < 2$) seem to have not been pursued yet. There are, however, some important results closely related to ours. Temam [14] treated the case where $f(u) = -u$. Babin and Vishik [1] considered more general equations than (E). In the book of Cholewa and Dlotko [2] they assumed that f_2 is globally Lipschitz. Takeuchi and Yokota [13] constructed the global attractor and studied its structure in the L^2-setting. L. Dung, in [4] and [5], studied the rate of convergence and the “dissipativity” properties of solutions (E), i.e., to derive the existence of an absorbing set bounded in L^∞ from the existence of L^r-bounded absorbing set for some $r \geq 1$. Resently, Nakao and Aris [9] showed the "dissipativity" of solutions to more general quisilinear equations governed by p-Laplacian-like operators.

In the papers mentioned above, it is more or less assumed that the nonlinearity f is subject to some growth conditions, say, polynomial growth. Roughly speaking, these restrictions mainly arises from the following observation: if one consider the equation (E) in certain phase space $L^q(\Omega)$, then it should be hold that $f(u) \in L^q(\Omega)$. This requirement needs the good regularity properties of solutions and the restrictions for the growth of f due to the Sobolev embedding theorem. On the other hand, if $f(\cdot)$ is bounded on any bounded interval and $L^\infty(\Omega)$ is the phase space, then $f(u)$ lies in the same phase space for $u \in L^\infty(\Omega)$. As motivated by the above simple obsevation, in the present note we are going to work in the L^∞-framework in order to get rid of some restrictions on the growth condition of f. In this connection, see also [7] and [6, Section 5.6].

Assume that $f : \mathbb{R} \to \mathbb{R}$ can be represented as a sum of two functions f_1, f_2 which satisfy the following conditions:

(A1): f_1 is a nondecreasing continuous function with $f_1(0) = 0$.
(A2): f_2 is locally Lipschitz continuous.
(A3): There exist constants $k \in [0,1)$ and $c_0 > 0$ such that for every $u \in \mathbb{R}$ the following holds:

$$|f_2(u)| \le k|f_1(u)| + c_0.$$

(A4): There exists a positive constant K_1 such that

$$\liminf_{|u|\to\infty} \frac{f_1(u)}{u} \ge K_1.$$

Remark 1. *These conditions allow us to take the nonlinearity f as*

$$f(u) = |u|^\alpha u + |u|^\beta u \sin u,$$

where $0 < \beta < \alpha < \beta+1$. On the other hand, the case where $f_1(u) \equiv 0$ and $f_2(u) = -u$ is out of our scope since it violates $(A3)$.

Now we state our main results.

Theorem 1. *Assume $f = f_1+f_2$ satisfies (A1) – (A4). Let $g \in L^\infty(\Omega)$. Then for each $u_0 \in L^\infty(\Omega)$ there exists a unique solution to (E) such that*

$$u \in L^\infty(0,\infty; L^\infty(\Omega)) \cap L^\infty_{loc}(0,\infty; W^{1,p}_0(\Omega)) \cap W^{1,2}_{loc}(0,\infty; L^2(\Omega)).$$

By Theorem 1 we can consider a family of operators $(S(t))_{t\ge0}$ defined by $S(t)u_0 = u(t)$ for $t \ge 0$, where u denotes the solution to (E) with initial value u_0. Then the pair $((S(t))_{t\geq 0}, L^\infty(\Omega))$ becomes a dynamical system associated with (E). It follows from the uniqueness of solutions that $(S(t))_{t\geq 0}$ enjoys the semigroup property. Moreover, for each fixed $t \geq 0$, $S(t)$ is a continuous mapping from $L^\infty(\Omega)$ into itself.

Here we recall the notion of the global attractor. Let $\mathcal{X}$ be a metric space and $(\mathcal{S}(t))_{t\geq 0}$ a semigroup acting on $\mathcal{X}$. A set $\mathcal{A} \subset \mathcal{X}$ is called the global attractor of the dynamical system $((\mathcal{S}(t))_{t\geq 0}, \mathcal{X})$ if it is nonempty, compact and invariant under $(\mathcal{S}(t))_{t\geq 0}$, that is, $\mathcal{S}(t)\mathcal{A} = \mathcal{A}$ for every $t \geq 0$, and it attracts each bounded subset $\mathcal{B}$, that is the following holds:

$$\lim_{t\to\infty} \sup_{b\in\mathcal{B}} \inf_{a\in\mathcal{A}} \mathrm{dist}_{\mathcal{X}}(\mathcal{S}(t)\mathcal{B}, \mathcal{A}) = 0.$$

It is well known that if the mapping $\mathcal{S}(t) : \mathcal{X} \to \mathcal{X}$ is continuous for each fixed $t \geq 0$ and there is a compact absorbing set for $((\mathcal{S}(t))_{t\geq 0}, \mathcal{X})$, then its ω-limit set becomes the global attractor (see [14]).

Due to the regularity results in [3], there is an absorbing set which is bounded in some Hölder space. We have then the following theorem.

Theorem 2. *The dynamical system associated with (E) admits the global attractor.*

2. Outline of Proof of Theorem 1

The proof of the existence result is devided into several steps. Each step is based on the standard arguments. The similar arguments are found in [6, Section 5.6], [10], [11] and [15, Theorem 3.10.1].

Here we give some notation. $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\Omega)$ and $\|\cdot\|_r$ represents the $L^r(\Omega)$-norm.

2.1. Uniqueness. We begin with proving the uniqueness of solutions. Let u, v be two solutions for (E) with $u(0) = u_0$, $v(0) = v_0$ for u_0, $v_0 \in L^\infty(\Omega)$. Then the difference $w := u - v$ satisfies

$$\frac{dw}{dt} - \Delta_p u + \Delta_p v + f_1(u) - f_2(u) + f_2(u) - f_2(v) = 0.$$

For each fixed $r \geq 2$, multiplying this by $|w|^{r-2}w$ and then using the monotonicity of the p-Laplacian, i.e., $(-\Delta_p u - (-\Delta_p v), |w|^{r-2}w)_{L^2} \geq 0$ and by $(A1)$ we get

$$\frac{1}{r}\frac{d}{dt}\|w\|_r^r \leq (|f_2(u) - f_2(v)|, |w|^{r-1})_{L^2}$$

Since $u, v \in L^\infty(0,\infty; L^\infty(\Omega))$ and f_2 is locally Lipschitz by $(A2)$, there is a constant $L > 0$ such that $|f_2(u) - f_2(v)| \leq L|w|$ almost everywhere in $\Omega \times [0, \infty)$. Therefore we arrive at

$$\frac{d}{dt}\|w\|_r \leq L\|w\|_r.$$

Then by Gronwall's lemma

$$\|w(t)\|_r \leq e^{Lt}\|u_0 - v_0\|_r.$$

For each fixed $t \geq 0$, passing to the limit $r \to \infty$, we then have

$$\|u(t) - v(t)\|_\infty \leq e^{Lt}\|u_0 - v_0\|_\infty.$$

Hence the uniqueness and the continuous dependence on initial data of solutions immediately follows.

2.2. **Existence.**

2.2.1. *Step1.* We proceed by the truncation technique. For $\sigma > 0$ and a continuous function h, the cutoff function h^σ is defined by

$$h^\sigma(s) := \begin{cases} h(\sigma), & \textit{if } s \geq \sigma, \\ h(s), & \textit{if } -\sigma < s < \sigma, \\ h(-\sigma), & \textit{if } s \leq -\sigma. \end{cases}$$

It is easy to see that if f_1 and f_2 satisfy the conditions $(A1)$, $(A2)$ and $(A3)$, so do f_1^σ and f_2^σ. Let $f_\sigma := f_1^\sigma + f_2^\sigma$ and $M_\sigma := \max_{|s|\leq\sigma} |f_\sigma(s)|$. For an arbitrary $T > 0$ and $u \in C([0,T]; L^2(\Omega))$, the composite function $f_\sigma(u(\cdot))$ also belongs to $C([0,T]; L^2(\Omega))$ and $\|f_\sigma(u(\cdot))\|_{C([0,T];L^2(\Omega))} \leq M_\sigma|\Omega|^{1/2}$. Then the abstract theory developed in [12] says that for all $\sigma > 0$ the following auxiliary problem

$$(E)^\sigma \begin{cases} \dfrac{\partial u^\sigma}{\partial t} - \Delta_p u^\sigma + f_\sigma(u^\sigma) = g(x), & (x,t) \in \Omega \times (0,\infty), \\ u^\sigma(x,t) = 0, & x \in \partial\Omega \times (0,\infty), \\ u^\sigma(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

permits a solution u^σ which belongs to $C([0,T]; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)) \cap W_{loc}^{1,2}(0,T; L^2(\Omega))$ (see also [15, Theorem 3.10.1]).

The uniqueness follows from much the same argument as above.

2.2.2. *Step2.* Our aim in this step is to show the boundedness of the solution u^σ of $(E)^\sigma$. We employ here the comparison theorem. Observe that $v := u^\sigma - Me^t$, where $M > 0$ is a constant fixed later, satisfies

$$\frac{\partial v}{\partial t} - \Delta_p v + f_\sigma(u^\sigma) = g(x) - Me^t.$$

Multiplying this by $[v]^+(x,t) := \max(v(x,t),0)$ and using the fact that

$$(-\Delta_p v, [v]^+)_{L^2} = \int_\Omega |\nabla [v]^+|^p \geq 0,$$

we have

$$\frac{1}{2}\frac{d}{dt}\|[v]^+\|_2^2 \leq (M_\sigma + \|g\|_\infty - M)\int_\Omega [v]^+(x,t)dx.$$

Choose M large enough so that $M > \max(\|u_0\|_\infty, M_\sigma + \|g\|_\infty)$ holds. Then the function $t \mapsto \|[v]^+(t)\|_2^2$ becomes decreasing and $\|[v]^+(0)\|_2 = 0$. Therefore $u^\sigma(x,t) \leq Me^t$ for almost every (x,t). We get another estimate $u^\sigma \geq -Me^t$ analogously. Thus $u^\sigma \in L^\infty(0,T;L^\infty(\Omega))$ for any $T > 0$.

2.2.3. *Step3.* We next show that if σ is chosen in a suitable way, then u^σ also solves the original equation (E) locally in time.

To this end, first we notice that $f_\sigma(s) \equiv f(s)$ for $|s| \leq \sigma$ and it follows from $(A1)$ and $(A3)$ that

$$|f_2^\sigma(s)||s|^{r-1} \leq kf_1^\sigma(s)|s|^{r-2}s + c_0|s|^{r-1}$$

holds for all $r \geq 2$ and $s \in \mathbb{R}$. Multiply $(E)^\sigma$ by $|u^\sigma|^{r-2}u^\sigma$ and use the above inequality to get

$$\frac{1}{r}\frac{d}{dt}\|u^\sigma(t)\|_r^r + (r-1)\int_\Omega |\nabla u^\sigma|^p|u^\sigma|^{r-2}dx \leq (c_0 + \|g\|_\infty)\|u^\sigma(t)\|_{r-1}^{r-1}.$$

Then we have

$$\frac{d}{dt}\|u^\sigma\|_r \leq (c_0 + \|g\|_\infty)|\Omega|^{1/r}.$$

The integration of this over $(0,t)$ leads us to

$$\|u^\sigma(t)\|_r \leq \|u_0\|_r + (c_0 + \|g\|_\infty)|\Omega|^{1/r}t.$$

Passing to the limit $r \to \infty$, we arrive at

$$\|u^\sigma(t)\|_\infty \leq \|u_0\|_\infty + (c_0 + \|g\|_\infty)t.$$

Therefore, for $\sigma > \|u_0\|_\infty$ there is a $t_\sigma > 0$ satisfying $\|u^\sigma(t)\|_\infty \leq \sigma$ on $[0, t_\sigma]$. It turns out that u^σ is just a solution of (E) on $[0, t_\sigma]$.

2.2.4. *Step4.* The last part of the proof is devoted to the continuation argument. We need an a priori estimate for solutions to (E).

Let $u \in L^\infty(0,T;L^\infty(\Omega))$ be a solution to (E) with $u(0) = u_0$. Multiplying (E) by $|u|^{r-2}u$, we have

$$\frac{1}{r}\frac{d}{dt}\|u\|_r^r + \int |f_1(u)||u|^{r-1} \leq \int |f_2(u)||u|^{r-1} + \|g\|_\infty\|u\|_{r-1}^{r-1}.$$

Since the inequality

$$|f_2(s)||s|^{r-1} \le kf_1(s)|s|^{r-2}s + c_0|s|^{r-1}$$

is valid for $r \ge 2$ and $s \in \mathbb{R}$ by $(A1)$ and $(A3)$, it holds that

$$\frac{1}{r}\frac{d}{dt}\|u\|_r^r + (1-k)\int |f_1(u)||u|^{r-1} \le (c_0 + \|g\|_\infty)\|u\|_{r-1}^{r-1}.$$

Notice that from the condition $(A4)$ there is a constant K_2 such that

$$|f_1(s)| \ge \frac{K_1}{2}|s| - K_2 \quad \textit{for all } s \in \mathbb{R}.$$

Therefore,

$$\frac{d}{dt}\|u\|_r + \frac{(1-k)K_1}{2}\|u\|_r \le \{(1-k)K_2 + c_0 + \|g\|_\infty\}|\Omega|^{1/r}.$$

Then we have

$$\|u(t)\|_\infty \le \|u_0\|_\infty e^{-c_1 t} + c_2, \tag{1}$$

where $c_1 := (1-k)K_1/2$ and $c_2 := (1-k)K_2 + c_0 + \|g\|_\infty$.

Let $\xi := \|u_0\|_\infty + 2c_2 + 1$. Then by Step3, there exists a $t_\xi > 0$ such that there is a unique solution u to (E) on $[0, 2t_\xi]$ satisfying $\|u(t)\|_\infty \le \|u_0\|_\infty + c_1$ a.e. on $(0, 2t_\xi)$. Choose $\tau \in (t_\xi, 2t_\xi)$ so that $\|u(\tau)\|_\infty \le \|u_0\|_\infty + c_1$. Since $\|u(\tau)\|_\infty < \xi$, there exists a solution v to (E) on $[0, 2t_\xi]$ with $v(0) = u(\tau)$. Thus by the uniqueness of solutions of (E), u can be continued to the interval $[2t_\xi, 3t_\xi]$ as a solution to (E). In addition, the estimate (1) holds on $[0, 3t_\xi]$. By repeating this procedure, the solution to (E) can be continued to the interval $[0, \infty)$.

3. Outline of Proof of Theorem 2

Now it is sufficient to prove the existence of a compact absorbing set of $((S(t))_{t\ge 0}, L^\infty(\Omega))$.

Let $R > 0$ and $u_0 \in L^\infty(\Omega)$ satisfy $\|u_0\|_\infty \le R$. Let u be the solution to (E) with $u(0) = u_0$. Multiplying (E) by u, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p + c_1\|u\|_2^2 \le c_2|\Omega|^{1/2}\|u\|_2. \tag{2}$$

Neglect the positive term $\frac{1}{p}\|\nabla u\|_p^p$ on the left-hand side and then integrate the resulting inequality over $(0, t)$ to have

$$\|u(t)\|_2 \le \|u_0\|_2 e^{-c_1 t} + c_2|\Omega|^{1/2} \le (Re^{-c_1 t} + c_2)|\Omega|^{1/2}.$$

Then there exists a $t(R) > 0$ such that $\|u(t)\|_2 \le 2c_2|\Omega|^{1/2}$ for $t \ge t(R)$. On the other hand, for $s \ge t(R)$, integrating (2) over $(s, s+1)$ we have

$$\frac{1}{p}\int_s^{s+1} \|\nabla u(t)\|_p^p dt \le 2c_2|\Omega|^{1/2}(1 + c_2|\Omega|^{1/2}).$$

According to the regularity result of [3, Chapter X, Theorem 1.1], the solution u to (E) belongs to $C^{1/2}(\overline{\Omega})$ on $(0,\infty)$ and there exists a constant $\gamma(p, N, \varepsilon, \int_\varepsilon^T \|\nabla u\|_p^p dt)$ such that

$$\|u(t)\|_{C^{1/2}(\overline{\Omega})} \le \gamma \quad \textit{for}\ t \in [\varepsilon, T].$$

For a nonnegative integer n, let v^n be the solution of (E) with $v^n(0) = u(t(R) + n)$. Then

$$\|v^n(t)\|_{C^{1/2}(\overline{\Omega})} \le \gamma\left(p, N, 1, \int_1^2 \|\nabla v^n\|_p^p dt\right)$$

holds for $t \in [1, 2]$. Since

$$\int_1^2 \|\nabla v^n\|_p^p dt = \int_{t(R)+n+1}^{t(R)+n+2} \|\nabla u\|_p^p dt \le 2pc_2|\Omega|^{1/2}(1 + c_2|\Omega|^{1/2}),$$

we have, for any n,

$$\|u(t)\|_{C^{1/2}(\overline{\Omega})} \le \gamma(p, N, 1, c_3) \quad \textit{for}\ t \in [t(R) + n + 1, t(R) + n + 2],$$

where $c_3 := 2pc_2|\Omega|^{1/2}(1 + c_2|\Omega|^{1/2})$. Therefore it follows that there exists an absorbing set which is bounded in $C^{1/2}(\overline{\Omega})$. Since $C^{1/2}(\overline{\Omega})$ is compactly embedded in $L^\infty(\Omega)$, the proof is completed.

4. Comments

In the last two decades, there has been an open problem: wheather the Hausdorff dimension or the fractal dimension of the global attractor for (E) is finite or infinite. Concerning this question the construction of exponential attractors for (E) is also still open. If f is monotone, the dynamical system associated with (E) admits single point equilibrium which corresponds to the global attractor. In this case, the fractal or the Hausdorff dimension of the global attractor is of course finite. However, the attracting rate is expected as polynomial order like $t^{-1/p}$. By [6, Proposition 7.2], we can construct an exponentially attracting set whose Hausdorff dimension is finite. However we say nothing about the fractal dimension of such a set.

Quite resently, the author heard the news that Professors M. Efendiev and M. Ôtani [8] succeeded to estimate the fractal dimension of the global attractor for (E) with $f(u) = -u$ and $g = 0$. Their showed that the fractal dimension is *infinite*. This situation never occurs for the semilinear case, i.e., when $p = 2$. The author believes that this result brings us a new aspect of the study of the degenerate or singular nonlinear evolution equations.

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