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Bifurcation structure of the stationary solution set to a strongly coupled diffusion system

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Abstract

We study the positive stationary solution set of a strongly coupled diffusion system with the Lotka-Volterra reaction term. We obtain the bifurcation branch of the positive solutions, which extends globally with respect to a bifurcation parameter. Furthermore, by the analysis for the *shadow systems*, we derive the nonlinear diffusion effect of on the positive solution branch.

1 Introduction

Many reaction-diffusion models have been proposed to describe the population dynamics in various ecological situations. In particular, the nonlinear-diffusion systems with the Lotka-Volterra interaction terms have been extensively studied by many mathematicians, since the advocated work by Shigesada-Kawasaki-Teramoto [19].

In this article, we focus on the following strongly coupled diffusion system with the prey-predator interaction terms:

$$\text{(P)}\begin{cases} u_t = \Delta u + u(a-u-cv) & \text{in } \Omega\times(0,\infty),\\ v_t = \Delta\left[\left(\mu+\dfrac{1}{1+\beta u}\right)v\right] + v(b+du-v) & \text{in } \Omega\times(0,\infty),\\ u=v=0 & \text{on } \partial\Omega\times(0,\infty),\\ u(\cdot,t)=u_0\ge 0,\ \ v(\cdot,t)=v_0\ge 0 & \text{on } \Omega, \end{cases}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ with a smooth boundary $\partial\Omega$; $a,b,c,d,$ and μ are all positive constants; β is a nonnegative constant. System (P) is a prey-predator

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model. From the ecological point of view, the unknown functions u and v, respectively, denote the population densities of prey and predator species which are interacting and migrating in the same habitat Ω. In the reaction terms, a and b represent the birth rates of the respective species, c and d denote the prey-predator interactions. In the second equation, the strongly coupled diffusion term $\Delta(\frac{v}{1+\beta u})$ models a situation in which the population pressure of predator species weakens in the high density place of prey species. On the solvability of (P), Le. Dung [8] has recently found the global attractor for a class of the quasilinear parabolic systems involving (P). So it becomes more interesting to study the dynamical structure for the solution set of (P). As the beginning of the study, we have been analyzing for the stationary solution set of (P), since [10]. To my knowledge, there are few works on this fractional type of density dependence diffusions in the field of reaction-diffusion systems. It should be noted that our nonlinear diffusion term is different from the usual cross-diffusion term proposed by [19].

Our purpose is to derive the global bifurcation structure of the stationary solution set. Then we will discuss the associate stationary problem with (P);

$$\text{(SP)}\begin{cases}\Delta u + u(a-u-cv) = 0 & \text{in } \Omega,\\ \Delta\left[\left(\mu + \dfrac{1}{1+\beta u}\right)v\right] + v(b+du-v) = 0 & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega.\end{cases}$$

Among other things, we are interested in the positive solutions of (SP). It is said that (u, v) is a positive solution if both $u > 0$ and $v > 0$ satisfy (SP). From the viewpoint of the prey-predator model, a positive solution (u, v) implies a *coexistence* steady state. Hence, it is an important problem to derive the positive solution set of (SP).

In order to study the positive solution set, we need some notations. Henceforth, we will use $\lambda_1(q)$ to denote the least eigenvalue of the problem

$$-\Delta u + q(x)u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $q(x)$ is a continuous function in $\overline{\Omega}$. We simply write λ_1 instead of $\lambda_1(0)$. It is well known that the following problem

$$\Delta u + u(a-u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.1}$$

has a unique positive solution $u = \theta_a$ if and only if $a > \lambda_1$. Hence, (SP) has a semitrivial solution $(u, v) = (\theta_a, 0)$ if $a > \lambda_1$. Furthermore it is easily verified that (SP) has another semitrivial solution $(u, v) = (0, (\mu+1)\theta_{b/(\mu+1)})$ if $b > (\mu+1)\lambda_1$. Here, $\theta_{b/(\mu+1)}$ represents a positive solution of (1.1) with a replaced by $b/(\mu+1)$. The usual norms of the spaces $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\overline{\Omega})$ are defined by

$$\|u\|_p := \left(\int_\Omega |u(x)|^p\,dx\right)^{1/p} \quad \text{and} \quad \|u\|_\infty := \max_{x\in\overline{\Omega}} |u(x)|.$$

2 Positive Solution Set

2.1 Coexistence Region

In this article, we are restricted on the case $b > (\mu + 1)\lambda_1$. The first theorem gives a sufficient condition of the existence of positive solutions:

Theorem 2.1 ([10]). *Let $a^* = \lambda_1(c(\mu+1)\theta_{b/(\mu+1)})$. If $b > (\mu+1)\lambda_1$, (SP) has a positive solution for $a > a^*$. From the bifurcation structure point of view, the positive solution set of (SP) contains a local bifurcation branch $\Gamma = \{(u(s), v(s), a(s)) \in X \times \mathbf{R} : s \in (0, \delta)\}$, such that $(u(0), v(0), a(0)) = (0, (\mu + 1)\theta_{b/(\mu+1)}, a^*)$. Furthermore, Γ can be extended globally with respect to $a(\to \infty)$ as a positive solution branch of (SP).*

We remark that the above bifurcation point

$$a^* = \lambda_1(c(\mu + 1)\theta_{b/(\mu+1)}) \tag{2.1}$$

depends on (b, c, μ), but is independent of β. Furthermore in [10, Lemma 2.3], we have proved that for any fixed (c, μ), $a^* = a^*(b)$ is a monotone increasing smooth function with respect to $b > (\mu + 1)\lambda_1$ such that $\lim_{b\searrow(\mu+1)\lambda_1} a^*(b) = \lambda_1$ and $\lim_{b\to\infty} a^*(b) = \infty$. Theorem 2.1 enables us to find the *coexistence region* on the (a, b) space. By the monotone property of the curve $a = a^*(b)$ (Theorem 2.1), one can deduce that if (a, b) lies in the region surrounded by $a = a^*(b)$ and $b = (\mu + 1)\lambda_1$, then (SP) has a positive solution (see the region $R_1 \cup R_2$ in Figure 1). This region ,in case $\beta = 0$, corresponds to the exact coexistence region shown by López-Gómez and Pardo [15]. From the viewpoint of the bifurcation theory, positive solutions bifurcate from $(u, v) = (0, (\mu + 1)\theta_{b/(\mu+1)})$ when (a, b) moves across $a = a^*(b)$.

2.2 Asymptotic Behavior of Positive Solutions as $\beta \to \infty$

Next, I will derive the nonlinear effect of large β on the positive solution set. For the sake of the derivation, we will introduce two *shadow systems* as $\beta \to \infty$ in (SP). We assume that $\{\beta_n\}$ is any positive sequence with $\lim_{n\to\infty} \beta_n = \infty$, and that $\{(u_n, v_n)\}$ is any positive solution sequence of (SP) with $\beta = \beta_n$. With some suitable assumptions, we will prove that one of the following two situations necessarily occurs:

(i) There exists a certain positive solution (u, v) of

$$\begin{cases} \Delta u + u(a - u - cv) = 0 & \text{in } \Omega, \\ \mu\Delta v + v(b + du - v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

such that $\lim_{n\to\infty}(u_n, v_n) = (u, v)$ in $C(\overline{\Omega})^2$, passing to a subsequence.

(ii) There exists a certain positive solution (w, v) of

$$\begin{cases} \Delta w + w(a - cv) = 0 & \text{in } \Omega, \\ \Delta\left[\left(\mu + \dfrac{1}{1+w}\right)v\right] + v(b - v) = 0 & \text{in } \Omega, \\ w = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

such that $\lim_{n\to\infty}(\beta_n u_n, v_n) = (w, v)$ in $C(\overline{\Omega})^2$, passing to a subsequence.

Our convergence result (Theorem 4.1) will also ensure that if β is sufficiently large, any positive solution of (SP) can be approximated by a suitable positive solution of either (2.2) or (2.3). Hence it is natural to ask which of (2.2) and (2.3) (or both) can characterize positive solutions of (SP), according to each coefficient value.

The positive solution set of the first shadow system (2.2) has been extensively studied by many mathematicians (e.g., [2], [4], [5], [6], [13], [14], [15], [16], [20]). As a summary of their all results, we know the next result on the positive solution set of (2.2):

Theorem 2.2. *Let $\hat{a} = \lambda_1(c\mu\theta_{b/\mu})$. Then* (2.2) *has a positive solution if and only if $a > \hat{a}$. From the bifurcation structure point of view, the positive solution set of* (2.2) *contains a local bifurcation branch $\Gamma_1 = \{(u(s), v(s), a(s)) \in X \times \boldsymbol{R} : s \in (0, \delta)\}$, such that $(u(0), v(0), a(0)) = (0, \mu\theta_{b/\mu}, \hat{a})$. Furthermore, Γ_1 can be extended in the direction $a > \hat{a}$ as an unbounded positive solution branch of* (2.2).

Here we note that for any fixed (c, μ), $\hat{a} = \lambda_1(c\mu\theta_{b/\mu})$ is a monotone increasing smooth function with respect to $b > \mu\lambda_1$, such that $\lim_{b\searrow\mu\lambda_1} \hat{a}(b) = \lambda_1$ and $\lim_{b\to\infty} \hat{a}(b) = \infty$. Furthermore, it can be verified that $a^*(b) < \hat{a}(b)$ for all $b > (\mu + 1)\lambda_1$ (see Figure 1).

Hence, it becomes a crucial part of this article to study the positive solution set of the second shadow system (2.3). By regarding a as a bifurcation parameter, we will show that the branch of the positive solution set of (2.3) bifurcates from the semitrivial solution with $w \equiv 0$ at $a = a^*$, and moreover that this branch extends globally and blows up with respect to $\|w\|_\infty$ at $a = \hat{a}$:

Theorem 2.3 ([12]). *Suppose that $b > (\mu + 1)\lambda_1$. Positive solutions of* (2.3) *bifurcate from the semitrivial solution curve $\{(0, (\mu + 1)\theta_{b/(\mu+1)}, a) : a \in \boldsymbol{R}_+\}$ if and only if $a = a^*$. To be precise, all positive solutions of* (5.2) *near $(0, (\mu + 1)^2\theta_{b/(\mu+1)}, a^*) \in X \times \boldsymbol{R}_+$ can be parameterized as $\Gamma_2 = \{(w(s), v(s), a(s)) \in X \times \boldsymbol{R}_+ : s \in (0, \delta)\}$, such that $(w(0), v(0), a(0)) = (0, \mu\theta_{b/\mu}, \hat{a})$. Furthermore, $\Gamma_2\,(\subset X \times \boldsymbol{R}_+)$ can be extended as an unbounded positive solution branch (of* (2.3)*), which contains an unbounded smooth curve which is parameterized by a; $\{(w(a), v(a), a) \in X \times [\,\hat{a} - \kappa, \hat{a}\,)\}$ with a certain positive number κ. Here, $(w(a), v(a))$ is a smooth function such that*

$$\lim_{a\nearrow\hat{a}} \|w(a)\|_\infty = \infty, \quad \lim_{a\nearrow\hat{a}} v(a) = \mu\theta_{b/\mu} \ \ \textit{in}\ C^1(\overline{\Omega}).$$

Furthermore, it can be proved that (2.3) has no positive solution if $a \geq \tilde{a} := \lambda_1(c\mu^{-1}(\mu+1)^2\theta_{b/\mu})$. Here, we note that $\tilde{a} = \lambda_1(c\mu^{-1}(\mu+1)^2\theta_{b/\mu})$ is also a monotone increasing smooth function for $b > \mu\lambda_1$, such that $\lim_{b\searrow\mu\lambda_1} \tilde{a}(b) = \lambda_1$ and $\lim_{b\to\infty} \tilde{a}(b) = \infty$. Furthermore, it holds that $a^*(b) < \hat{a}(b) < \tilde{a}(b)$ for all $b > (\mu+1)\lambda_1$ (see Figure 1).

Consequently, it follows that if $a \in (a^*, \hat{a})$, (2.3) has at least one positive solution while (2.2) has no positive solution, and that if $a > \tilde{a}$, (2.3) has no positive solution while (2.2) has at least one positive solution. Owing to such studies on the shadow systems, we will prove the approximate result in large β case:

Theorem 2.4 ([12]). *Suppose that* $\{(u_n, v_n)\}$ *is any positive solution sequence of* (SP) *with* $\beta = \beta_n$ *and* $\lim_{n\to\infty} \beta_n = \infty$. *Let* ε *and* δ *be arbitrary small positive numbers. Then, there exist positive numbers* $\hat{a} > a^*(> \lambda_1)$ *such that if* $a \in (a^*, \hat{a} - \delta\,] \cup [\,\hat{a} + \delta, \infty)$, $b > (\mu+1)\lambda_1$ *and n is sufficiently large, either the following situation* (i) *or* (ii) *necessarily occurs :*

(i) *There exists a certain positive solution* (u, v) *of* (2.2) *such that*

$$\max_{x\in\overline{\Omega}} |u_n(x) - u(x)| + \max_{x\in\overline{\Omega}} |v_n(x) - v(x)| < \varepsilon.$$

(ii) *There exists a certain positive solution* (w, v) *of* (2.3) *such that*

$$\max_{x\in\overline{\Omega}} |\beta_n u_n(x) - w(x)| + \max_{x\in\overline{\Omega}} |v_n(x) - v(x)| < \varepsilon.$$

Furthermore, there exists a number $\tilde{a}(> \hat{a})$ *such that if* $a \in [\,\tilde{a}, \infty)$, *the situation of* (ii) *can not occur, and if* $a \in (a^*, \hat{a} - \delta\,]$, *the situation of* (i) *can not occur.*

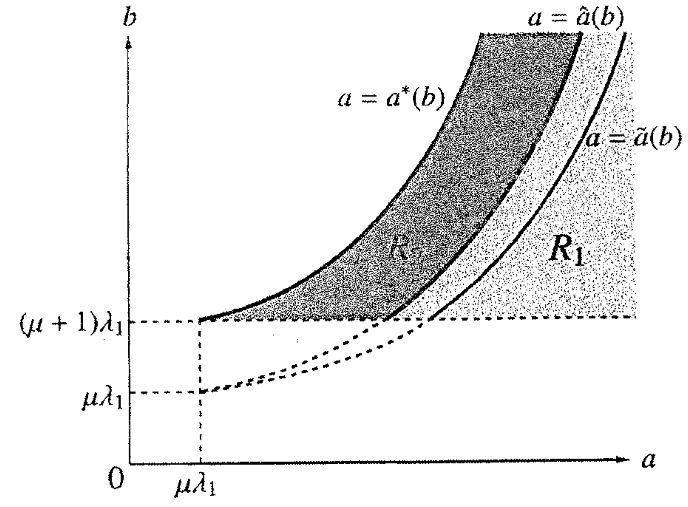

Figure 1: The region R_1 gives the exact coexistence region for (2.2). The region R_2 yields the sufficient condition for existence of positive solutions of (2.3).

3 A Priori Estimates

In this subsection, we introduce a semilinear elliptic system equivalent to (SP), and give some a priori estimates for positive solutions of the semilinear system ([10]). These a priori estimates will play an important role in the succeeding sections. Since we are restricted on nonnegative solutions, it is convenient to introduce the unknown function V by

$$V = \left(\mu + \frac{1}{1+\beta u}\right) v. \tag{3.1}$$

There is a one-to-one correspondence between $(u, v) \geq 0$ and $(u, V) \geq 0$. Then, (SP) is rewritten in the following equivalent form:

$$\text{(EP)} \begin{cases} \Delta u + u(a - u - cv) = 0 & \text{in } \Omega, \\ \Delta V + v(b + du - v) = 0 & \text{in } \Omega, \\ u = V = 0 & \text{on } \partial\Omega, \end{cases}$$

where $v = v(u, V)$ is understood as the function of (u, V) defined by (3.1). It is easy to show that (EP) has two semitrivial solutions

$$(u, V) = (\theta_a, 0) \text{ for } a > \lambda_1 \quad \text{and} \quad (u, V) = (0, (\mu+1)^2\theta_{b/(\mu+1)}) \text{ for } b > (\mu+1)\lambda_1,$$

in addition to the trivial solution $(u, V) = (0, 0)$. We obtain the following a priori estimates for positive solutions of (EP) (or equivalently (SP)):

Lemma 3.1. *Suppose that (u, v) is any positive solution of* (SP). *Let V be the positive function defined by* (3.1). *Then,*

$$0 < u(x) < a, \quad \mu^2\theta_{b/(\mu+1)}(x) < V(x) \leq v(x) < \left(1 + \frac{1}{\mu}\right)(b + ad)$$

for all $x \in \Omega$.

We refer to [10] and [12] for the proof of Lemma 3.1. The next lemma gives a nonexistence region for positive solutions of (EP):

Lemma 3.2. *If $a \leq \lambda_1$ or $(1+\beta a)(b+ad) \leq \lambda_1$,* (EP) (*or equivalently,* (SP)) *has no positive solution.*

Proof. Suppose for contradiction that (u, V) is a positive solution of (EP) with the case $(1+\beta a)(b+ad) \leq \lambda_1$. Since $u < a$ by Lemma 3.1, then

$$-\Delta V = v(b + du - v) = V(1+\beta u)(b + du - v) < V(1+\beta a)(b + ad)$$

in Ω. Then by taking $L^2(\Omega)$ inner product with V, we obtain

$$\|\nabla V\|_2^2 < (1+\beta a)(b+ad)\|V\|_2^2. \tag{3.2}$$

Since $\|\nabla V\|_2^2 \geq \lambda_1\|V\|_2^2$ by Poincaré's inequality, (3.2) obviously yields a contradiction. Observing that $u(a-u-cv) < au$ in Ω, we can derive the assertion in the case $a \leq \lambda_1$ along a similar way.

□

4 Existence of Two Shadow Systems as $\beta \to \infty$

In what follows, we will concentrate ourselves on the special case when β is sufficiently large. Our purpose is to derive the nonlinear effect of large β on the positive solution set of (SP). The next theorem ensures the existence of two shadow systems as $\beta \to \infty$. We refer to [12] for the proof of the theorem.

Theorem 4.1. *Let $\hat{a} := \lambda_1(c\mu\theta_{b/\mu})$ and $b > (\mu+1)\lambda_1$. Suppose that $\{(u_n, v_n)\}$ is any positive solution sequence of* (SP) *with $\beta = \beta_n$ and $\lim_{n\to\infty}\beta_n = \infty$. Then, for any small positive numbers δ and ε, there exists a large integer N (which depends on δ, ε and the coefficients of* (SP)*) such that if*

$$a \in (\lambda_1, \hat{a}-\delta\,] \cup [\,\hat{a}+\delta, \infty)\,(=: I_\delta)$$

and $n \geq N$, either the following property (i) *or* (ii) *holds true* :

(i) *There exist a certain positive solution $(u, v) = (\overline{u}, \overline{v})$ of (2.2) such that*

$$\|u_n - \overline{u}\|_\infty + \|v_n - \overline{v}\|_\infty < \varepsilon.$$

(ii) *There exist a certain positive solution $(w, v) = (\overline{w}, \overline{v})$ of (2.3) such that*

$$\|\beta_n u_n - \overline{w}\|_\infty + \|v_n - \overline{v}\|_\infty < \varepsilon.$$

5 Second Shadow System

5.1 A Priori Estimates

In this section, we will study the second shadow system (2.3). By employing a new unknown function

$$z := \left(\mu + \frac{1}{1+w}\right)v, \tag{5.1}$$

we reduce (2.3) to the following semilinear elliptic system;

$$\begin{cases} \Delta w + w\left\{a - \dfrac{c(1+w)z}{\mu(1+w)+1}\right\} = 0 & \text{in } \Omega, \\ \Delta z + \dfrac{(1+w)z}{\mu(1+w)+1}\left\{b - \dfrac{(1+w)z}{\mu(1+w)+1}\right\} = 0 & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Because of the one-to-one corresponding between $(w, v) \geq 0$ and $(w, z) \geq 0$, we may concentrate ourselves on (5.2). We note that (5.2) has a semitrivial solution $(w, z) = (0, (\mu+1)^2\theta_{b/(\mu+1)})$ if $b > (\mu+1)\lambda_1$. The following lemma gives the a priori bounds for the v (resp. z) component of any positive solution of (2.3) (resp. (5.2)).

Lemma 5.1. *Suppose that $b > (\mu+1)\lambda_1$. Let (w, v) be any positive solution of* (2.3), *and let z be the positive function defined by* (5.1). *Then, it follows that*

$$\frac{\mu^2}{\mu+1}\theta_{b/(\mu+1)} < v < \frac{(\mu+1)^2}{\mu}\theta_{b/\mu} \quad \text{and} \quad \mu^2\theta_{b/(\mu+1)} < z < (\mu+1)^2\theta_{b/\mu} \quad \text{in } \Omega. \tag{5.3}$$

Furthermore, if

$$a \leq \lambda_1\left(\frac{c\mu^2}{\mu+1}\theta_{b/(\mu+1)}\right) \quad \text{or} \quad a \geq \lambda_1\left(\frac{c(\mu+1)^2}{\mu}\theta_{b/\mu}\right),$$

both of (2.3) *and* (5.2) *have no positive solution.*

The above nonexistence region of the positive solutions can be led from (5.3) with the aid of the comparison argument. We refer to [12] for the proof of Lemma 5.1.

5.2 Local Bifurcation Structure of the Positive Solution Set

For the framework of our bifurcation analysis, we prepare two Banach spaces

$$\begin{cases} X := [\,W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,] \times [\,W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,], \\ Y := L^p(\Omega) \times L^p(\Omega) \end{cases}$$

for $p > N$. We note that $X \subset C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ by the Sobolev embedding theorem.

For the positive number $a^* = \lambda_1(c(\mu+1)\theta_{b/(\mu+1)})$ introduced in (2.1), we define the associate positive eigenfunction ϕ^*, which satisfies

$$-\Delta\phi^* + \{c(\mu+1)\theta_{b/(\mu+1)} - a^*\}\phi^* = 0 \quad \text{in } \Omega, \quad \phi^* = 0 \ \text{ on } \partial\Omega, \quad \|\phi^*\|_2 = 1. \tag{5.4}$$

We recall that (5.2) has the semitrivial solution $(w, z) = (0, (\mu+1)^2\theta_{b/(\mu+1)})$. Positive solutions of (5.2) bifurcate from the semitrivial solution curve $\{(0, (\mu+1)^2\theta_{b/(\mu+1)}, a) \in X \times \boldsymbol{R}_+\}$ at the same point $a = a^*$ to the original (EP) case:

Proposition 5.2. *Suppose that $b > (\mu+1)\lambda_1$. Positive solutions of (5.2) bifurcate from the semitrivial solution curve $\{(0,(\mu+1)^2\theta_{b/(\mu+1)},a) : a \in \boldsymbol{R}_+\}$ if and only if $a = a^*$. To be precise, all positive solutions of (5.2) near $(0,(\mu+1)^2\theta_{b/(\mu+1)},a^*) \in X \times \boldsymbol{R}_+$ can be parameterized as*

$$\Gamma_\delta := \{(s(\phi^* + \tilde{W}(s)), (\mu+1)^2\theta_{b/(\mu+1)} + s(\chi + \tilde{z}(s)), a(s)) : 0 < s \le \delta\} \tag{5.5}$$

for some $\delta > 0$ and $\chi \in X$. Here, $(\tilde{W}(s), \tilde{z}(s), a(s))$ is a smooth function with respect to s and satisfies $(\tilde{W}(0), \tilde{z}(0), a(0)) = (0,0,a^*)$ and $\int_\Omega \tilde{W}(s)\phi^* = 0$.

Proof. In view of the nonlinear terms of (5.2), we put

$$\begin{aligned} f(w,z,a) &= w\left\{a - \frac{c(1+w)z}{\mu(1+w)+1}\right\}, \\ g(w,z) &= \frac{(1+w)z}{\mu(1+w)+1}\left\{b - \frac{(1+w)z}{\mu(1+w)+1}\right\}. \end{aligned} \tag{5.6}$$

By Taylor's expansion at the centre of (w^*, z^*), we reduce the differential equations of (5.2) to the form

$$\begin{pmatrix} \Delta w + f(w^*,z^*,a) \\ \Delta z + g(w^*,z^*) \end{pmatrix} + \begin{pmatrix} f_w^* & f_z^* \\ g_w^* & g_z^* \end{pmatrix}\begin{pmatrix} w-w^* \\ z-z^* \end{pmatrix} + \begin{pmatrix} \rho^1(w-w^*,z-z^*,a) \\ \rho^2(w-w^*,z-z^*,a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.7}$$

where $f_w^* := f_w(w^*,z^*,a)$ and the other notations are defined by similar rules. Here, $\rho^i(w-w^*, z-z^*, a)$ $(i = 1, 2)$ are smooth functions such that $\rho^i(0,0,a) = \rho^i_{(w,z)}(0,0,a) = 0$. We note that $f(0,(\mu+1)^2\theta_{b/(\mu+1)},a) = 0$ and

$$g(0,(\mu+1)^2\theta_{b/(\mu+1)}) = (\mu+1)\theta_{b/(\mu+1)}\{b - (\mu+1)\theta_{b/(\mu+1)}\} = -(\mu+1)^2\Delta\theta_{b/(\mu+1)}.$$

By letting $(w^*,z^*) = (0,(\mu+1)^2\theta_{b/(\mu+1)})$ and $\bar{z} := z - (\mu+1)^2\theta_{b/(\mu+1)}$ in (5.9), after some calculations, we obtain

$$\begin{aligned} &\begin{pmatrix} \Delta w \\ \Delta \bar{z} \end{pmatrix} + \begin{pmatrix} a - c(\mu+1)\theta_{b/(\mu+1)} & 0 \\ \theta_{b/(\mu+1)}\{b - 2(\mu+1)\theta_{b/(\mu+1)}\} & \dfrac{b}{\mu+1} - 2\theta_{b/(\mu+1)} \end{pmatrix}\begin{pmatrix} w \\ \bar{z} \end{pmatrix} \\ &\qquad + \begin{pmatrix} \rho^1(w,\bar{z},a) \\ \rho^2(w,\bar{z},a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{5.8}$$

where $\rho^i(w,\bar{z},a)$ $(i = 1, 2)$ are smooth functions satisfying

$$\rho^1_{(w,\bar{z})}(0,0,a) = \rho^2_{(w,\bar{z})}(0,0,a) = 0 \ \text{ for all } \ a > 0. \tag{5.9}$$

We define a mapping $F : X \times \boldsymbol{R}_+ \to Y$ using the left-hand side of (5.10):

$$F(w,\bar{z},a) = \begin{pmatrix} \Delta w + \{a - c(\mu+1)\theta_{b/(\mu+1)}\}w + \rho^1(w,\bar{z},a) \\ \Delta\bar{z} + \theta_{b/(\mu+1)}\{b - 2(\mu+1)\theta_{b/(\mu+1)}\}w + \left(\dfrac{b}{\mu+1} - 2\theta_{b/(\mu+1)}\right)\bar{z} + \rho^2(w,\bar{z},a) \end{pmatrix}. \tag{5.10}$$

Since $(w,z) = (0,(\mu+1)^2\theta_{b/(\mu+1)})$ is a semitrivial solution of (5.2), $F(0,0,a) = 0$ for $a > 0$. It follows from (5.11) and (5.12) that the Fréchet derivative of F at $(w,\bar{z}) = (0,0)$ is given by

$$F_{(w,\bar{z})}(0,0,a)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \Delta h + \{a - c(\mu+1)\theta_{b/(\mu+1)}\}h \\ \Delta k + \theta_{b/(\mu+1)}\{b - 2(\mu+1)\theta_{b/(\mu+1)}\}h + \left(\dfrac{b}{\mu+1} - 2\theta_{b/(\mu+1)}\right)k \end{pmatrix}.$$

From (5.6), we know that $\operatorname{Ker} F_{(w,\bar{z})}(0,0,a)$ is nontrivial for $a = a^*$ and that

$$\operatorname{Ker} F_{(\overline{U},z)}(0,0,a^*) = \operatorname{span}\{\phi^*, \psi\}.$$

Here, ψ is defined by

$$\psi = \left(-\Delta - \frac{b}{\mu+1} + 2\theta_{b/(\mu+1)}\right)^{-1}\left(\theta_{b/(\mu+1)}\{b - 2(\mu+1)\theta_{b/(\mu+1)}\}\phi^*\right),$$

where $\left(-\Delta - \frac{b}{\mu+1} + 2\theta_{b/(\mu+1)}\right)^{-1}$ is the inverse operator of $-\Delta - \frac{b}{\mu+1} + 2\theta_{b/(\mu+1)}$ with the homogeneous Dirichlet boundary condition on $\partial\Omega$. (Recall that $-\Delta - \frac{b}{\mu+1} + 2\theta_{b/(\mu+1)}$ is invertible, see, e.g., [4].) If $(\tilde{h},\tilde{k}) \in \operatorname{Range} F_{(w,\bar{z})}(0,0,a^*)$, then

$$\begin{cases} \Delta h + \{a - c(\mu+1)\theta_{b/(\mu+1)}\}h = \tilde{h} & \text{in } \Omega, \\ \Delta k + \theta_{b/(\mu+1)}\{b - 2(\mu+1)\theta_{b/(\mu+1)}\}h + \left(\dfrac{b}{\mu+1} - 2\theta_{b/(\mu+1)}\right)k = \tilde{k} & \text{in } \Omega, \\ h = k = 0 & \text{on } \partial\Omega \end{cases}$$

for some $(h,k) \in X$. By virtue of the Fredholm alternative theorem, we know that the first equation has a solution h if and only if $\int_\Omega \tilde{h}\phi^* = 0$. For such a solution h, the second equation has a unique solution k because $-\Delta - \frac{b}{\mu+1} + 2\theta_{b/(\mu+1)}$ is invertible. Then, it follows that $\operatorname{codimRange} F_{(w,\bar{z})}(0,0,a^*) = 1$. In order to use the local bifurcation theory of Crandall-Rabinowitz [3] at $(w,\bar{z},a) = (0,0,a^*)$, we need to verify

$$F_{(w,\bar{z}),a}(0,0,a^*)\begin{pmatrix} \phi^* \\ \psi \end{pmatrix} \notin \operatorname{Range} F_{(w,\bar{z})}(0,0,a^*).$$

Since $\rho^i_{(w,\bar{z}),a}(0,0,a^*) = 0$ by (5.11), the differentiation of (5.12) yields

$$F_{(w,\bar{z}),a}(0,0,a^*)\begin{pmatrix} \phi^* \\ \psi \end{pmatrix} = \begin{pmatrix} \phi^* \\ 0 \end{pmatrix}.$$

Suppose for contradiction that there exists a certain function $h \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ such that

$$\Delta h + \{a - c(\mu+1)\theta_{b/(\mu+1)}\}h = \phi^*$$

Multiplying the above equation by ϕ^* and integrating the resulting expression, we have $\|\phi^*\|_2 = 0$, which contradicts the fact that $\|\phi^*\|_2 = 1$. Since $\bar{z} = z - (\mu + 1)^2\theta_{b/(\mu+1)}$, one can obtain expression (5.7) by using the local bifurcation theorem ([3]). We note that the possibility of other bifurcation points except $a = a^*$ is excluded by virtue of the Krein-Rutman theorem. Then we accomplish the proof of Proposition 5.3. □

5.3 Asymptotic Behavior of the Global Bifurcation Branch

In this subsection, we will extend Γ_δ globally as a positive solution branch of (5.2). It will be proved that the global branch is uniformly bounded with respect to (z, a), while $\|w\|_\infty$ blows up along the branch at $a = \hat{a}(= \lambda_1(c\mu\theta_{b/\mu}))$. Before discussing the global extension, we should prove the following inequality.

Lemma 5.3. *Let* $a^* = \lambda_1(c(\mu + 1)\theta_{b/(\mu+1)})$ *and* $\hat{a} = \lambda_1(c\mu\theta_{b/\mu})$. *(These two positive numbers have been introduced in (2.1) and Theorem 4.1, respectively.) If* $b > (\mu + 1)\lambda_1$, $a^* < \hat{a}$.

Lemma 5.4 can be proved by the comparison argument (e.g., [4, Lemma 1]). See [12] for the detail.

Proposition 5.4. *Assume that* $b > (\mu + 1)\lambda_1$. *Let* Γ_δ *be the local bifurcation branch obtained in Proposition 5.3. Then* $\Gamma_\delta (\subset X \times \boldsymbol{R}_+)$ *can be extended as an unbounded positive solution branch* $\hat{\Gamma}$ *of* (5.2). *Furthermore,* $\hat{\Gamma}$ *contains an unbounded smooth curve which is parameterized by* a;

$$\{(w(a), z(a), a) \in X \times [\,\hat{a} - \kappa, \hat{a}\,)\} \tag{5.11}$$

with a certain positive number κ. *Here,* $(w(a), z(a))$ *is a smooth function such that*

$$\lim_{a \nearrow \hat{a}} \|w(a)\|_\infty = \infty, \quad \lim_{a \nearrow \hat{a}} z(a) = \mu^2\theta_{b/\mu} \ \text{ in } \ C^1(\overline{\Omega}). \tag{5.12}$$

Proof. Suppose that $b > (\mu + 1)\lambda_1$. For the local bifurcation branch Γ_δ obtained in Proposition 5.3, let $\hat{\Gamma}$ be a maximum extension of Γ_δ as a solution curve of (5.2). According to the global bifurcation theory (Rabinowitz [18]), one of the following two properties must hold true;

(i) $\hat{\Gamma}$ is unbounded in $X \times \boldsymbol{R}$;

(ii) $\hat{\Gamma}$ meets the trivial or a semitrivial solution curve at a certain point except for $(w, z, a) = (0, (\mu + 1)^2\theta_{b/(\mu+1)}, a^*)$.

We introduce the following positive cone

$$P := \left\{(w, z)\ :\ w > 0,\ z > 0 \text{ in } \Omega, \text{ and } \frac{\partial w}{\partial n} < 0,\ \frac{\partial z}{\partial n} < 0 \text{ on } \partial\Omega\right\},$$

where $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$. Assume that there exists $(\hat{w}, \hat{z}, \hat{a}) \in \hat{\Gamma}$ such that $(\hat{w}, \hat{z}) \in \partial P$. Then it follows from Lemmas 5.1 and 5.2 that

$$\frac{\mu^2}{m+1}\theta_{b/(\mu+1)} \le \hat{z} \le \frac{(\mu+1)^2}{\mu}\theta_{b/\mu} \text{ in } \Omega, \quad \lambda_1\left(\frac{c\mu^2}{m+1}\theta_{b/(\mu+1)}\right) \le \hat{a} \le \lambda_1\left(\frac{c(\mu+1)^2}{\mu}\theta_{b/\mu}\right), \tag{5.13}$$

respectively. Hence $(\hat{w}, \hat{z}) \in \partial P$ implies that $\hat{w} \ge 0$, $\hat{z} \ge 0$ in Ω and

$$\hat{w}(x_0)\hat{z}(x_0) = 0 \text{ at a certain } x_0 \in \Omega \tag{5.14}$$

or

$$\frac{\partial \hat{w}}{\partial \boldsymbol{n}}(x_1)\frac{\partial \hat{z}}{\partial \boldsymbol{n}}(x_1) = 0 \text{ at a certain } x_1 \in \partial\Omega. \tag{5.15}$$

By applying the strong maximum principle to (5.2), it is possible to verify that each of (5.19) and (5.20) leads to $\hat{w} \equiv 0$ or $\hat{z} \equiv 0$. By taking account for (5.18), we must assume that $\hat{w} \equiv 0$ and $\hat{z} > 0$ in Ω. We recall that positive solutions of (5.2) bifurcate from the semitrivial solution curve $\{(0, (\mu+1)^2\theta_{b/(\mu+1)}, a) : a \in \boldsymbol{R}_+\}$ only at $a = a^*$. This fact leads to $(\hat{w}, \hat{z}, \hat{a}) = (0, (\mu+1)^2\theta_{b/(\mu+1)}, a^*)$, which contradicts (ii). Therefore, the situation of (i) necessarily occurs. Together with the a priori estimates of z and a (Lemmas 5.1 and 5.2), we can deduce that $\hat{\Gamma}$ consists of a continuum, which is unbounded with respect to $\|w\|_{W^{1,p}}$. From the continuum, we take any positive solution sequence $\{(w_n, z_n, a_n)\} \subset \hat{\Gamma}$ with $\lim_{n\to\infty}\|w_n\|_{W^{1,p}} = \infty$. In order to prove $\lim_{n\to\infty}\|w_n\|_\infty = \infty$, we use the standard elliptic regularity theory (see e.g., [9]). From the first equation of (5.2), we obtain

$$\|w_n\|_{W^{2,p}} \le C\left(\|w_n\|_p + \left\|w_n\left\{a_n - \frac{c(w_n+1)z_n}{\mu(w_n+1)+1}\right\}\right\|_p\right) \tag{5.16}$$

for a certain positive constant C independent of n. Since z_n and a_n are uniformly bounded with respect to n (see Lemmas 5.1 and 5.2), (5.21) ensures a certain positive constant C' such that $\|w_n\|_{W^{2,p}} \le C'\|w_n\|_\infty$. Hence, it follows that $\lim_{n\to\infty}\|w_n\|_\infty = \infty$. Next we will show $\lim_{n\to\infty} a_n = \hat{a}(= \lambda_1(c\mu\theta_{b/\mu}))$. Since $\{a_n\}$ is a bounded sequence from Lemma 5.2, we can put $a_\infty := \lim_{n\to\infty} a_n$, subject to a subsequence. Furthermore, we put $\tilde{w}_n := w_n/\|w_n\|_\infty$. Therefore, a similar compactness argument to the proof of Theorem 4.1 enables us to find a certain $(\tilde{w}, v_\infty) \in C^1(\overline{\Omega})^2$ such that

$$\lim_{n\to\infty}(\tilde{w}_n, z_n) = (\tilde{w}, \mu v_\infty) \text{ in } C^1(\overline{\Omega})^2, \tag{5.17}$$

and moreover,

$$\begin{cases} \Delta\tilde{w} + \tilde{w}(a_\infty - cv_\infty) = 0 & \text{in } \Omega, \\ \mu\Delta v_\infty + v_\infty(b - v_\infty) = 0 & \text{in } \Omega, \\ \tilde{w} = v_\infty = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.18}$$

passing to a subsequence. Since $v_\infty > 0$ in Ω from (5.22) and Lemma 5.1, the second equation of (5.23) implies $v_\infty = \mu\theta_{b/\mu}$. Therefore, we obtain $a_\infty = \hat{a}$ from the first equation of (5.23). Consequently, we have proved that

$$\lim_{n\to\infty} \|w_n\|_\infty = \infty, \quad \lim_{n\to\infty} z_n = \mu^2\theta_{b/\mu} \text{ in } C^1(\overline{\Omega}), \quad \lim_{n\to\infty} a_n = \hat{a}. \tag{5.19}$$

Next, we will obtain the expression (5.16). Our aim is to prove the non-degeneracy of $\{(w_n, z_n, a_n)\} \subset \hat{\Gamma}$ for sufficiently large $n \in N$, because such a non-degeneracy yields (5.16) by virtue of the implicit function theorem. With respect to (5.2), we define the associate linearized operator at $(w, z) = (w_n, z_n)$ by

$$L_n\begin{pmatrix} h \\ k \end{pmatrix} := -\begin{pmatrix} \Delta h \\ \Delta k \end{pmatrix} - \begin{pmatrix} f_w(w_n, z_n, a_n) & f_z(w_n, z_n, a_n) \\ g_w(w_n, z_n) & g_z(w_n, z_n) \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix},$$

where f and g are nonlinear terms defined by (5.8). By direct computations, we obtain

$$L_n\begin{pmatrix} h \\ k \end{pmatrix} = -\begin{pmatrix} \Delta h \\ \Delta k \end{pmatrix}$$
$$+\begin{pmatrix} \dfrac{c\{\mu(1+w_n)^2 + 2w_n + 1\}z_n}{\{\mu(1+w_n)+1\}^2} - a_n & \dfrac{cw_n(1+w_n)}{\mu(1+w_n)+1} \\ \dfrac{z_n}{\{\mu(1+w_n)+1\}^2}\left\{\dfrac{2(1+w_n)z_n}{\mu(1+w_n)+1} - b\right\} & \dfrac{1+w_n}{\mu(1+w_n)+1}\left\{\dfrac{2(1+w_n)z_n}{\mu(1+w_n)+1} - b\right\} \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}.$$

Henceforth, we write η_n to denote the principal eigenvalue of L_n. Furthermore we put $m_n := \|w_n\|_\infty$ and $\tilde{w}_n := w_n/m_n$. In order to study the behavior of η_n as $n \to \infty$, we modify L_n to the form

$$\tilde{L}_n\begin{pmatrix} h \\ k \end{pmatrix} := -\begin{pmatrix} \Delta h \\ \Delta k \end{pmatrix}$$
$$+\begin{pmatrix} \dfrac{c\{\mu(1+w_n)^2 + 2w_n + 1\}z_n}{\{\mu(1+w_n)+1\}^2} - a_n & \dfrac{cw_n(1+w_n)}{m_n^2\{\mu(1+w_n)+1\}} \\ \dfrac{m_n^2 z_n}{\{\mu(1+w_n)+1\}^2}\left\{\dfrac{2(1+w_n)z_n}{\mu(1+w_n)+1} - b\right\} & \dfrac{1+w_n}{\mu(1+w_n)+1}\left\{\dfrac{2(1+w_n)z_n}{\mu(1+w_n)+1} - b\right\} \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}. \tag{5.20}$$

It is possible to verify that the spectrum set of L_n coincides with that of $\tilde{L}_n$ for any $n \in N$. We recall that

$$\lim_{n\to\infty}(\tilde{w}_n, z_n, a_n) = (\tilde{w}, \mu^2\theta_{b/\mu}, \hat{a}) \text{ in } C^1(\overline{\Omega})^2 \times R, \tag{5.21}$$

where $\tilde{w}$ satisfies the linear elliptic problem

$$-\Delta\tilde{w} + c\mu\theta_{b/\mu}\tilde{w} = \hat{a}\tilde{w} \text{ in } \Omega, \quad \tilde{w}|_{\partial\Omega} = 0, \quad \|\tilde{w}\|_\infty = 1. \tag{5.22}$$

Therefore, letting $n \to \infty$ in (5.25), we know that $\tilde{L}_n$ converges to

$$\tilde{L}_\infty \begin{pmatrix} h \\ k \end{pmatrix} := -\begin{pmatrix} \Delta h \\ \Delta k \end{pmatrix} + \begin{pmatrix} c\mu\theta_{b/\mu} - \hat{a} & 0 \\ \theta_{b/\mu}(2\mu\theta_{b/\mu} - b) & 2\theta_{b/\mu} - \dfrac{b}{\mu} \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}$$

in the sense of the operator norm. (Here we note that the operator norms of the original sequence $\{L_n\}$ are unbounded with respect to n.) Consequently, the associate eigenvalue problem with $\tilde{L}_\infty$ can be expressed as

$$\begin{cases} -\Delta h + (c\mu\theta_{b/\mu} - \hat{a})h = \eta h & \text{in } \Omega, \\ -\Delta k + \theta_{b/\mu}(2\mu\theta_{b/\mu} - b)h + \left(2\theta_{b/\mu} - \dfrac{b}{\mu}\right)k = \eta k & \text{in } \Omega, \\ h = k = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.23}$$

From the first equation of (5.28), we know that all eigenvalues of $\tilde{L}_\infty$ consist of infinitely many real numbers. It follows from (5.27) that $(h, \eta) = (\tilde{w}, 0)$ satisfies the first equation of (5.28). We will show that $\eta = 0$ is the least eigenvalue of $\tilde{L}_\infty$. Since $\lambda_1(q)$ is monotone increase with respect to $q \in C(\overline{\Omega})$, we observe from the second equation of (5.28) that if $h = 0$ and $k \neq 0$,

$$\eta \geq \lambda_1\left(2\theta_{b/\mu} - \frac{b}{\mu}\right) > \lambda_1\left(\theta_{b/\mu} - \frac{b}{\mu}\right) = 0. \tag{5.24}$$

Here, we note that the right equality comes from the definition of $\theta_{b/\mu}$. At once, (5.29) also yields the invertibity of $-\Delta + 2\theta_{b/\mu} - \frac{b}{\mu}$. Therefore, by letting $(h, \eta) = (\tilde{w}, 0)$ in the second equation of (5.28), we obtain

$$k = \left(-\Delta + 2\theta_{b/\mu} - \frac{b}{\mu}\right)^{-1} \left(\theta_{b/\mu}(b - 2\mu\theta_{b/\mu})\tilde{w}\right) (=: k_\infty).$$

Consequently, together with the positivity of $\tilde{w}$, we obtain that $\eta = 0$ is the least eigenvalue of $\tilde{L}_\infty$, and that $(h, k) = (\tilde{w}, k_\infty)$ is the associate eigenfunction. With the aid of the perturbation theory of T.Kato [11], we may assume that η_n are single real eigenvalues for sufficiently large $n \in \boldsymbol{N}$, and that

$$\lim_{n\to\infty}(h_n, k_n, \eta_n) = (\tilde{w}, k_\infty, 0) \text{ in } C^1(\overline{\Omega})^2 \times \boldsymbol{R}. \tag{5.25}$$

Here, (h_n, k_n) denotes the positive eigenfunction of $\tilde{L}_n$ with $\|h_n\|_\infty = 1$. Then, (h_n, k_n)

satisfies

$$\begin{cases} -\Delta h_n + \left[\dfrac{c\{\mu(1+w_n)^2 + 2w_n + 1\}z_n}{\{\mu(1+w_n)+1\}^2} - a_n \right] h_n + \dfrac{cw_n(1+w_n)}{m_n^2\{\mu(1+w_n)+1\}} k_n = \eta_n h_n & \text{in } \Omega, \\ -\Delta k_n + \dfrac{m_n^2 z_n}{\{\mu(1+w_n)+1\}^2} \left\{ \dfrac{2(1+w_n)z_n}{\mu(1+w_n)+1} - b \right\} h_n & \\ \qquad + \dfrac{1+w_n}{\mu(1+w_n)+1} \left\{ \dfrac{2(1+w_n)z_n}{\mu(1+w_n)+1} - b \right\} k_n = \eta_n k_n & \text{in } \Omega, \\ h_n = k_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.26}$$

By multiplying the first equations of (5.2) with $(w, z, a) = (w_n, z_n, a_n)$ by $\tilde{w}$ and integrating the resulting expression, we have

$$\int_\Omega w_n \Delta \tilde{w}\, dx + \int_\Omega \left\{ a_n - \frac{c(1+w_n)z_n}{\mu(1+w_n)+1} \right\} w_n \tilde{w}\, dx. \tag{5.27}$$

By substituting (5.27) for (5.32), we obtain

$$(\hat{a} - a_n) \int_\Omega w_n \tilde{w}\, dx = c \int_\Omega \left\{ \mu\theta_{b/\mu} - \frac{(1+w_n)z_n}{\mu(1+w_n)+1} \right\} w_n \tilde{w}\, dx. \tag{5.28}$$

The same procedure for the first equation of (5.31) leads to

$$\begin{aligned} (\hat{a} - a_n) \int_\Omega h_n \tilde{w}\, dx + c \int_\Omega \left[\frac{\{\mu(1+w_n)^2 + 2w_n + 1\}z_n}{\{\mu(1+w_n)+1\}^2} - \mu\theta_{b/\mu} \right] h_n \tilde{w}\, dx \\ + c \int_\Omega \frac{w_n(1+w_n)}{m_n^2\{\mu(1+w_n)+1\}} k_n \tilde{w}\, dx = \eta_n \int_\Omega h_n \tilde{w}\, dx. \end{aligned} \tag{5.29}$$

Multiplying (5.34) by m_n and letting $n \to \infty$ in the resulting expression, we know along with (5.26) and (5.30) that

$$\begin{aligned} &\|\tilde{w}\|_2^2 \lim_{n\to\infty} m_n \eta_n \\ &= \|\tilde{w}\|_2^2 \lim_{n\to\infty} (\hat{a} - a_n) m_n + c \lim_{n\to\infty} m_n \int_\Omega \left[\frac{\{\mu(1+w_n)^2 + 2w_n + 1\}z_n}{\{\mu(1+w_n)+1\}^2} - \mu\theta_{b/\mu} \right] \tilde{w}^2\, dx. \end{aligned} \tag{5.30}$$

Since $w_n = m_n \tilde{w}_n$, letting $n \to \infty$ in (5.33) yields

$$\|\tilde{w}\|_2^2 \lim_{n\to\infty} (\hat{a} - a_n) m_n = c \lim_{n\to\infty} m_n \int_\Omega \left\{ \mu\theta_{b/\mu} - \frac{(1+w_n)z_n}{\mu(1+w_n)+1} \right\} \tilde{w}^2\, dx. \tag{5.31}$$

Therefore by substituting (5.36) for (5.35), we obtain

$$\begin{aligned} &\|\tilde{w}\|_2^2 \lim_{n\to\infty} m_n \eta_n \\ &= c \lim_{n\to\infty} m_n \int_\Omega \left[\frac{\{\mu(1+w_n)^2 + 2w_n + 1\}z_n}{\{\mu(1+w_n)+1\}^2} - \frac{1+w_n}{\mu(1+w_n)+1} \right] \tilde{w}^2\, dx \\ &= c \lim_{n\to\infty} m_n \int_\Omega \frac{w_n}{\{\mu(1+w_n)+1\}^2} \tilde{w}^2\, dx = \frac{c}{\mu^2} \|\tilde{w}\|_1 > 0. \end{aligned} \tag{5.32}$$

Furthermore, it follows from (5.35) and (5.37) that

$$
\begin{aligned}
&\|\tilde{w}\|_2^2 \lim_{n\to\infty}(\hat{a} - a_n)m_n \\
&= \lim_{n\to\infty} m_n\eta_n + c \lim_{n\to\infty} m_n \int_\Omega \left[\mu\theta_{b/\mu} - \frac{\{\mu(1+w_n)^2 + 2w_n + 1\}z_n}{\{\mu(1+w_n)+1\}^2} \right] \tilde{w}^2\, dx \\
&= \lim_{n\to\infty} m_n\eta_n + \mu(\mu+1)c \lim_{n\to\infty} m_n \int_\Omega \frac{\theta_{b/\mu}}{\{\mu(1+w_n)+1\}^2}\, dx \\
&= \lim_{n\to\infty} m_n\eta_n = \frac{c}{\mu^2}\|\tilde{w}\|_1 > 0.
\end{aligned}
\tag{5.33}
$$

Hence (5.37) and (5.38) imply that $\eta_n > 0$ and $a_n < \hat{a}$ for sufficiently large $n \in N$, respectively. Consequently, we have proved that the linearized operator L_n is non-degenerate if $n \in N$ is large enough. Since L_n is invertible for such $n \in N$, the implicit function theorem gives a positive number κ_n and a neighborhood O_n of $(w_n, z_n) \in X$ such that all positive solutions of (5.2) in $\tilde{O}_n$ can be parameterized as

$$\{(w(a), z(a), a) \;:\; a_n - \kappa_n \le a \le a_n + \kappa_n\},$$

where $\tilde{O}_n := O_n \times (a_n - \kappa_n, a_n + \kappa_n)$ and $(w(a), z(a))$ is a smooth function satisfying $(w(a_n), z(a_n)) = (w_n, z_n)$. By using the *covering* argument (see e.g., Du-Lou [7, Appendix]) for $\{\tilde{O}_n\}$, we can construct the unbounded smooth curve (5.16). Since $a_n < \hat{a}$ for sufficiently large $n \in N$, it follows that $a < \hat{a}$ in (5.16). Hence (5.17) comes from (5.24). Thus we accomplish the proof of Proposition 5.5.

□

By the one-to-one correspondence between $(w, v) > 0$ and $(w, z) > 0$ (see (5.1)), we can give the following result on the positive solution set of (2.3), as a summary of this section:

Theorem 5.5. *If $b > (\mu+1)\lambda_1$, the positive solution set of (2.3) contains a local bifurcation branch $\Gamma_2 = \{(w(s), v(s), a(s)) \in X \times R \;:\; s \in (0, \delta)\}$, such that $(w(0), v(0), a(0)) = (0, (\mu+1)\theta_{b/(\mu+1)}, a^*)$. Furthermore, Γ_2 can be extended as an unbounded positive solution branch $\hat{\Gamma}_2$ of (2.3) with the following properties:*

(i) *Any $(w, v, a) \in \hat{\Gamma}_2$ satisfies*

$$
\begin{aligned}
&\frac{\mu^2}{\mu+1}\theta_{b/(\mu+1)} < v < \frac{(\mu+1)^2}{\mu}\theta_{b/\mu} \quad \text{in } \Omega, \\
&\lambda_1\left(\frac{c\mu^2}{\mu+1}\theta_{b/(\mu+1)}\right) < a < \lambda_1\left(\frac{c(\mu+1)^2}{\mu}\theta_{b/\mu}\right).
\end{aligned}
\tag{5.34}
$$

(ii) $\hat{\Gamma}_2$ *contains an unbounded smooth curve parametrized with respect to* a;

$$\{(w(a), v(a), a) \in X \times [\,\hat{a} - \kappa, \hat{a}\,)\}$$

for a certain positive number κ. *Here* $(w(a), z(a))$ *is a smooth function such that*

$$\lim_{a \nearrow \hat{a}} \|w(a)\|_\infty = \infty, \quad \lim_{a \nearrow \hat{a}} v(a) = \mu\theta_{b/\mu} \quad in \quad C^1(\overline{\Omega}).$$

6 Completion of the Proof of Theorem 2.4

In this section, we will accomplish the proof of Theorem 2.4. Hence Theorem 4.1 yields the convergence properties (i) and (ii) in Theorem 2.4. With respect to the first shadow system, from Theorem 2.2, we know that (2.2) has at least one positive solution if and only if $a > \hat{a}$. On the other hand, form Theorem 5.6, we have proved that the second shadow system (2.3) has at least one positive solution if $a^* < a < \hat{a}$, and no positive solution if $a \geq \tilde{a}$. Here we put $\tilde{a} := \lambda_1(c(\mu + 1)^2\mu^{-1}\theta_{b/\mu})$, which is the number in (5.39). Therefore, by combining Theorem 4.1 with such information on the positive solution sets of two shadow systems, we can deduce that as $\beta \to \infty$, any positive solution of (SP) approaches a certain positive solution of (2.2)(resp. (2.3)) if $a \in (\tilde{a}, \delta^{-1}\,]$ (resp. $a \in (\,a^*, \hat{a} - \delta\,]$). Furthermore, it follows that if β is sufficiently large and $a \in (\,a^*, \hat{a} - \delta\,]$, any positive solution (u, v) of (SP) satisfies $\|u\|_\infty = O(1/\beta)$. Then the proof of Theorem 2.4 is complete.

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