数理解析研究所講究録
1479 巻 2006 年 130-141

ON THE FUNCTIONAL DERIVATIVES OF THE GENERATING FUNCTIONAL FOR CORRELATION FUNCTIONS

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ABSTRACT. Let a Lagrangian function be given in $[0,T]\times R^n$. We add a so-called source term $x\cdot J(t)$ to the Lagrangian function, where $J(t)=(J_1(t),\ldots,J_n(t))$ is an R^n-valued continuous function in the interval $[0,T]$. Then the Feynman path integral with this new Lagrangian function can be defined rigorously by the time-slicing method through broken line paths. This path integral is a functional of $J(t)$ into weighted Sobolev spaces and called the generating functional. In the present paper it is rigorously proved that the generating functional is some times continuously differentiable in the Fréchet sense correspondingly to a weighted Sobolev space and its deivatives give correlation funtions. This result has been well known in physics roughly.

* Research partially supported by Grant-in-Aid for Scientific No.16540145, Ministry of Education, Science, and Culture, Japanese Government.

1. INTRODUCTION.

This paper is a continuation of [4, 5, 6]. We consider some charged non-relativistic particles in an electromagnetic field. For the sake of simplicity we suppose the charge and the mass of every particle to be one and $m > 0$, respectively. Let $0 < T < \infty$ be arbitrary. We consider $x \in R^n$ and $t \in [0, T]$. Let $E(t,x) = (E_1, \ldots, E_n) \in R^n$ and $(B_{jk}(t,x))_{1\le j<k\le n} \in R^{n(n-1)/2}$ denote electric strength and magnetic strength tensor, respectively and $(V(t,x), A(t,x)) = (V, A_1, \ldots, A_n) \in R^{n+1}$ an electromagnetic potential, i.e.

$$\begin{aligned} &E = -\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x}, \\ &d(\sum_{j=1}^{n} A_j dx_j) = \sum_{1\le j<k\le n} B_{jk} dx_j \wedge dx_k \quad \text{on } R^n, \end{aligned} \tag{1.1}$$

where $\partial V/\partial x = (\partial V/\partial x_1, \ldots, \partial V/\partial x_n)$. Then the Lagrangian function $\mathcal{L}(t,x,\dot{x})$ ($\dot{x} \in R^n$) is given by

$$\mathcal{L}(t,x,\dot{x}) = \frac{m}{2}|\dot{x}|^2 + \dot{x}\cdot A - V \tag{1.2}$$

and the Hamiltonian operator $H(t)$ is defined by

$$H(t) = \frac{1}{2m}\sum_{j=1}^{n}(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j)^2 + V, \tag{1.3}$$

where we set the Planck constant $\hbar = 1$ in the present paper. Let $U(t,s)f$ be the solution to the Schödinger equation

$$i\frac{\partial}{\partial t}u(t) = H(t)u(t), \quad u(s) = f. \tag{1.4}$$

We define the multiplication operator $\hat{x}_j$ ($j = 1, \ldots, n$) by $(\hat{x}_j f)(x) = x_j f(x)$ and $\hat{x}_j(t)$ by $U(t,0)^{-1}\hat{x}_j U(t,0)$. Let $(R^n)^{[s,t]}$ be the space of all paths $q = (q_1, \ldots, q_n) : [s,t] \ni \theta \to q(\theta) \in R^n$. Then the classical action $S(t,s;q)$ for some $q \in (R^n)^{[s,t]}$ is given by

$$S(t,s;q) = \int_s^t \mathcal{L}(\theta, q(\theta), \dot{q}(\theta))d\theta, \quad \dot{q}(\theta) = \frac{dq}{d\theta}(\theta). \tag{1.5}$$

We denote the Banach space of all continuous paths $q : [s,t] \ni \theta \to q(\theta) \in R^n$ with the norm $\|q\|_\infty := \sum_{j=1}^n \max_{s\le\theta\le t} |q_j(\theta)|$ by $C([s,t];R^n)$. Let $J \in C([s,t];R^n)$. We add a so-called source term $x \cdot J(t)$ to $\mathcal{L}(t,x,\dot{x})$. We write the classical action for this new Lagrangian function

$$S^J(t,s;q) = \int_s^t \Big(\mathcal{L}(\theta, q(\theta), \dot{q}(\theta)) + x \cdot J(\theta)\Big) d\theta. \tag{1.6}$$

For the sake of simplicity let $n = 1$. In physics $|x>$ is an eigenfunction of $\hat{x}$ with an eignvalue x. We set $|x,t> = U(t,0)^{-1}|x>$ and denote its complex conjugate by $<x,t|$. Let $t < t'$ and $t \le \theta_j \le t'$ $(j = 1,\dots,k)$. Then correlation functions $<x',t'|T[\hat{x}(\theta_1)\cdots\hat{x}(\theta_k)]|x,t>$ are formally defined by the integration of the product of $<x',t'|$ and $T[\hat{x}(\theta_1)\cdots\hat{x}(\theta_k)]|x,t>$, where $T[\hat{x}(\theta_1)\cdots\hat{x}(\theta_k)]$ is the product of the operators in decreasing time order, regardless of their order as written (cf. page 223 in [9] or page 182 in [10]). Then we have the path integral formula formally

$$<x',t'|T[\hat{x}(\theta_1)\cdots\hat{x}(\theta_k)]|x,t> = \int e^{iS(t',t;q)} q(\theta_1)\cdots q(\theta_k)\mathcal{D}q, \tag{1.7}$$

where the path integral is taken over the space $\{q \in (R^n)^{[t,t']}; q(t) = x, q(t') = x'\}$ (cf. (6.13) in [9]). The generating functional $\tilde{F}(J)$ of $J \in C([t,t'];R^n)$ is formally defined by the path integral

$$\tilde{F}(J) := \int e^{iS^J(t',t;q)}\mathcal{D}q \tag{1.8}$$

over the space $\{q \in (R^n)^{[t,t']}; q(t) = x, q(t') = x'\}$ (cf. page 223 in [9]). Let's take formally its functional derivative. Then we have

$$\begin{aligned} &<x',t'|T[\hat{x}(\theta_1)\cdots\hat{x}(\theta_k)]|x,t> \\ &= (-i)^k \left.\frac{\delta^k \tilde{F}(J)}{\delta J(\theta_1)\cdots\delta J(\theta_k)}\right|_{J=0} \quad (k = 0,1,\dots) \end{aligned} \tag{1.9}$$

(cf. (6.14) in [9]).

Our aim in the present paper is to show (1.9) rigorously. We note that we write (1.8) a slightly different form to obtain the rigorous result. The author does not know that such a result has been published (cf. [3, 8, 11]). We also note that (1.7) has been already proved rigorously in [6] in a slightly different form.

2. THE MAIN RESULT

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. We denote the space of all linear boounded operators from X into Y by $B(X;Y)$. Let Ω be a subset of X and $G:\Omega\to Y$ Fréchet differentiable on Ω. That is, there exists a Fréchet derivative $D_xG(x_0)\in B(X;Y)$ of G at any $x_0\in\Omega$ such that we have

$$\|G(x_0+h)-G(x_0)-D_xG(x_0)h\|_Y=o(1)\|h\|_X$$

as $\|h\|_X\to 0$. In the same way a k times $(k=0,1,\dots)$ Fréchet derivative $D_x^kG(x_0)\in B(X;B(X;\cdots;B(X;Y))\cdots)$ at $x_0\in\Omega$ can be defined (cf. Chapter I in [12]). Let $G:\Omega\to Y$ be a k times Fréchet differentiable on Ω. We write $D_x^kG(x_0)[h^{(1)},\dots,h^{(k)}]=(\cdots((D_x^kG(x_0)h^{(k)})h^{(k-1)})\cdots)h^{(1)}$ for $h^{(j)}\in X$ $(j=0,\dots,k)$. Then we can identify $D_x^kG(x_0)$ with a k-linear bounded operator from $X\times\cdots\times X$ into Y (cf. Chapter I in [12]). Let $x_0\in\Omega, h\in X$ and $h'\in X$. Then we can easily have

$$\|D_xG(x_0+h')-D_xG(x_0)-D_x^2G(x_0)h'\|_{B(X;Y)}=o(1)\|h'\|_X$$

and so

$$\|D_xG(x_0+h')h-D_xG(x_0)h-(D_x^2G(x_0)h')h\|_Y=o(1)\|h'\|_X.$$

Consequently we have

$$(D_x(D_xG(x)h))h'\Big|_{x=x_0}=(D_x^2G(x_0)h')h=D_x^2G(x_0)[h,h'].$$

In the same way we obtain

$$\begin{aligned} &D_x^k G(x_0)[h^{(1)}, \ldots, h^{(k)}] \\ &= (D_x(D_x(\cdots(D_x(D_xG(x)h^{(1)})h^{(2)})\cdots)h^{(k)}\Big|_{x=x_0}. \end{aligned} \tag{2.1}$$

For an $x = (x_1, \ldots, x_n) \in R^n$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we write $|\alpha| = \sum_{j=1}^n \alpha_j$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. Hereafter we write $X = C([0,T]; R^n)$. Let $\Delta : 0 = \tau_0 < \tau_1 < \ldots < \tau_\nu = T$ be a subdivision of the interval $[0,T]$. We set $|\Delta| = \max_{1 \le j \le \nu}(\tau_j - \tau_{j-1})$. Let $x^{(j)} \in R^n$ $(j = 0, \ldots, \nu-1)$ and x in R^n. We denote by $q_\Delta(\theta; x^{(0)}, \ldots, x^{(\nu-1)}, x) \in (R^n)^{[0,T]}$ the broken line path joining points $x^{(j)}$ $(j = 0, \ldots, \nu, x^{(\nu)} = x)$ at τ_j in order. We set

$$q_{x,y}^{t,s}(\theta) = y + \frac{\theta - s}{t - s}(x - y) \in (R^n)^{[s,t]}. \tag{2.2}$$

For $J \in X$ we write

$$(\mathcal{C}^J(t,s)f)(x) = \begin{cases} \sqrt{m/(2\pi i(t-s))}^{\,n} \int \left(\exp iS^J(t,s; q_{x,y}^{t,s})\right) \\ \qquad \times f(y)dy, & s < t, \\ f(x), & s = t \end{cases} \tag{2.3}$$

for $f \in C_0^\infty(R^n)$, where $C_0^\infty(R^n)$ denotes the space of all infinitely differentiable functions in R^n with compact support. Let $\chi \in C_0^\infty(R^n)$ such that $\chi(0) = 1$ and $\epsilon > 0$. Then we have

$$\begin{aligned} &\left(\prod_{j=1}^{\nu} \sqrt{\frac{m}{2\pi i(\tau_j - \tau_{j-1})}}^{\,n}\right) \int \cdots \int (\exp iS^J(t,s; q_\Delta))\chi(\epsilon x^{(\nu-1)}) \times \\ &\qquad \chi(\epsilon x^{(\nu-2)}) \cdots \chi(\epsilon x^{(0)}) f(q_\Delta(0)) dx^{(0)} \cdots dx^{(\nu-1)} \\ &= \mathcal{C}^J(T, \tau_{\nu-1})\chi(\epsilon\cdot)\mathcal{C}^J(\tau_{\nu-1}, \tau_{\nu-2}) \cdots \chi(\epsilon\cdot)\mathcal{C}^J(\tau_1, \tau_0)\chi(\epsilon\cdot) f \end{aligned} \tag{2.4}$$

for $f \in C_0^\infty(R^n)$.

We have from Lemma 6.1 in [5]

Lemma 2.1. *Let* $\partial_x^\alpha E_j(t,x)$ $(j=1,\dots,n)$, $\partial_x^\alpha B_{jk}(t,x)$ *and* $\partial_t B_{jk}(t,x)$ $(1 \le j < k \le n)$ *be continuous in* $[0,T]\times R^n$ *for all* α. *We assume*

$$|\partial_x^\alpha E_j(t,x)| \le C_\alpha,\ |\alpha| \ge 1,\ j=1,\dots,n \tag{2.5}$$

and

$$|\partial_x^\alpha B_{jk}(t,x)| \le C_\alpha < x >^{-(1+\delta)},\ |\alpha|\ge 1,\ 1\le j<k\le n \tag{2.6}$$

in $[0,T]\times R^n$, *where* $< x >= \sqrt{1+|x|^2}$ *and* $\delta > 0$ *are constants that may depend on* α. *Then there exists a continuous potential* (V,A) *in* $[0,T]\times R^n$ *such that*

$$|\partial_x^\alpha A_j(t,x)| \le C_\alpha,\ |\alpha|\ge 1,\ j=1,\dots,n \tag{2.7}$$

and

$$|\partial_x^\alpha V(t,x)| \le C_\alpha < x >,\ |\alpha| \ge 1 \tag{2.8}$$

in $[0,T]\times R^n$.

We note that two of the Maxwell equations

$$d\Big(\sum_{j=1}^n E_j dx_j\Big) = -\sum_{1\le j<k\le n} \partial_t B_{jk} dx_j \wedge dx_k \quad \text{on } R^n,$$
$$d\left(\sum_{1\le j<k\le n} B_{jk} dx_j \wedge dx_k\right) = 0 \quad \text{on } R^n$$

are used in the proof of Lemma 2.1. The lemma below follows from Lemma 2.1 in [5].

Lemma 2.2. *Suppose the assumptions of Lemma 2.1 and take a potential* (V,A) *satisfying (2.7) and (2.8). Let* $f \in C_0^\infty(R^n)$. *Then* $\partial_x^\alpha(C^J(t,s)f)(x)$ *exist for all* α *and are continuous in* $0\le s\le t\le T$ *and* $x\in R^n$.

Let $L^2 = L^2(R^n)$ be the space of all square integrable functions in R^n with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We introduce weighted Sobolev spaces $B^a := \{f \in L^2;\ \|f\|_{B^a} := \|f\| + \sum_{|\alpha|=a}(\|x^\alpha f\| + \|\partial_x^\alpha f\|) < \infty\}$ $(a = 1, 2, \dots)$. We set $B^0 = L^2$.

Lemma 2.3. *Suppose the assumptions of Lemma 2.1 and take a potential (V, A) satisfying (2.7) and (2.8). Then there exist a $\rho^* > 0$, which is determined from the constants C_α of (2.5)-(2.7), and constants $K_a \geq 0$ $(a = 0, 1, \dots)$ independent of $J \in X$ such that we have*

$$\|\mathcal{C}^J(t, s)f\|_{B^a} \leq e^{K_a(t-s)}\|f\|_{B^a}, \quad 0 \leq t - s \leq \rho^* \tag{2.9}$$

for all $J \in X$.

Proof. Lemma 2.3 follows from the proofs of Theorem 3.3 in [5] and Proposition 3.4 in [6]. Q.E.D.

Let $|\Delta| \leq \rho^*$. Suppose the assumptions of Lemma 2.1 and take a potential (V, A) satisfying (2.7) and (2.8). The multiplication operator $\chi(\epsilon\cdot)$ is a bounded operator from B^a $(a = 0, 1, \dots)$ into B^a. So it follows from Lemma 2.3 that the operator on $C_0^\infty(R^n)$ defined by (2.4) can be extended to a bounded operator from B^a into B^a. Moreover we have for $f \in B^a$

$$\begin{aligned}
&\mathcal{C}^J(T, \tau_{\nu-1})\chi(\epsilon\cdot)\mathcal{C}^J(\tau_{\nu-1}, \tau_{\nu-2})\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)\mathcal{C}^J(\tau_1, 0)\chi(\epsilon\cdot)f \\
&\quad - \mathcal{C}^J(T, \tau_{\nu-1})\mathcal{C}^J(\tau_{\nu-1}, \tau_{\nu-2})\cdots\mathcal{C}^J(\tau_1, 0)f \\
&= \sum_{j=0}^{\nu-1}\mathcal{C}^J(T, \tau_{\nu-1})\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)\mathcal{C}^J(\tau_{j+1}, \tau_j)(\chi(\epsilon\cdot) - 1)\mathcal{C}^J(\tau_j, \tau_{j-1})\cdot \\
&\quad \mathcal{C}^J(\tau_{j-1}, \tau_{j-2})\cdots\mathcal{C}^J(\tau_1, 0)f.
\end{aligned}$$

Consequently we get from the Lebesgue dominated convergence theorem

$$\begin{aligned}
&\lim_{\epsilon\to 0} \mathcal{C}^J(T,\tau_{\nu-1})\chi(\epsilon\cdot)\mathcal{C}^J(\tau_{\nu-1},\tau_{\nu-2})\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)\mathcal{C}^J(\tau_1,0)\chi(\epsilon\cdot)f \\
&= \mathcal{C}^J(T,\tau_{\nu-1})\mathcal{C}^J(\tau_{\nu-1},\tau_{\nu-2})\cdots\mathcal{C}^J(\tau_1,0)f \qquad (2.10)
\end{aligned}$$

in B^a. We write the operator defined by (2.10) as $G_\Delta(J)f$ or $\int \Big(\exp iS^J(T,0;q_\Delta)\Big) f(q_\Delta(0))\mathcal{D}q_\Delta$.

Proposition 2.4. *Let $J \in X$ and $f \in B^a$ $(a = 0,1,\dots)$. Suppose the assumptions of Lemma 2.1 and take a potential (V, A) satisfying (2.7) and (2.8). Then there exists a limit $G(J)f$ of $G_\Delta(J)f$ in B^a as $|\Delta| \to 0$.*

Proof. This proposition follows from Theorem 1 in [6]. We note that $G(J)f$ gives the solution to the Schrödinger equation (1.4) where V is replaced by $V - J(t)\cdot x$. Q.E.D.

Remark 2.1. Suppose the assumptions of Lemma 2.1 and take a potential (V', A') satisfying (2.7) and (2.8). Let (V, A) be an arbitrary potential such that $V, \partial V/\partial x_j, \partial A_j/\partial t$ and $\partial A_j/\partial x_k$ $(j,k = 1,2,\cdots,n)$ are continuous in $[0,T]\times R^n$. Then it follows from the proof of Theorem in [5] that there exists a continuously differentiable function $\psi(t,x)$ in $[0,T]\times R^n$ satisfying

$$(-Vdt + A\cdot dx) - (-V'dt + A'\cdot dx) = d\psi.$$

This gives the gauge invariance of (2.4) and so $G_\Delta(J)f$, i.e.

$$G_\Delta(J)f = e^{i\psi(T,\cdot)}G'_\Delta(J)e^{-i\psi(0,\cdot)}. \qquad (2.11)$$

Hence we get the same assertion as in Proposition 2.4 for $a = 0$.

We sometimes write

$$\int \Big(\exp iS^J(T,0;q)\Big) f(q(0))\mathcal{D}q = G(J)f. \qquad (2.12)$$

Let $0 \le t_1 \le t_2 \le \ldots \le t_k \le T$ $(k = 0, 1, \ldots), \epsilon > 0$ a constant and $|\Delta| \le \rho^*$. We consider for $f \in C_0^\infty(R^n)$

$$\left(\prod_{j=1}^{\nu} \sqrt{\frac{m}{2\pi i(\tau_j - \tau_{j-1})}}^{\,n}\right) \int \cdots \int (\exp i S^J(t, s; q_\Delta)) \chi(\epsilon x^{(\nu-1)}) \chi(\epsilon x^{(\nu-2)}) \cdots$$

$$\chi(\epsilon x^{(0)}) \left(\prod_{j=1}^{k} (q_\Delta)_{l_j}(t_j)\right) f(q_\Delta(0)) dx^{(0)} \cdots dx^{(\nu-1)}. \qquad (2.13)$$

Proposition 2.5. *Suppose the assumptions of Lemma 2.1 and take a potential (V, A) satisfying (2.7) and (2.8). Then the operator (2.13) for $f \in C_0^\infty(R^n)$ can be extended to a bounded operator from B^{a+k} $(a = 0, 1, \ldots)$ into B^a. For $f \in B^{a+k}$ there exists its limit in B^a as $\epsilon \to 0$, which we write $\int \left(\exp i S^J(T, 0; q_\Delta)\right) \left(\prod_{j=1}^{k} (q_\Delta)_{l_j}(t_j)\right) f(q_\Delta(0)) \mathcal{D} q_\Delta$. Moreover, for $f \in B^{a+k}$ there exists its limit $\int \left(\exp i S^J(T, 0; q)\right) \left(\prod_{j=1}^{k} (q)_{l_j}(t_j)\right) f(q(0)) \mathcal{D} q$ in B^a as $|\Delta| \to 0$, which is equal to $U^J(T, t_k) \hat{x}_{l_k} U^J(t_k, t_{k-1}) \cdots \hat{x}_{l_1} U^J(t_1, 0) f$.*

Proof. This proposition follows from Theorem 2 in [6]. Q.E.D.

Let l be an non-negative integer and $f \in B^{a+l}$ $(a = 0, 1, \ldots)$. We can consider

$$G(\cdot) f : X \to B^a \ (\subseteq B^{a+l} \ )$$

from Proposition 2.4. Let $k = 0, 1, \ldots, l, h^{(j)} \in X$ $(j = 1, 2, \ldots, k)$ and $0 \le \theta_1, \ldots, \theta_k \le T$. We know from Proposition 2.5 in the present paper and Theorem 1 in [6] that the path integral $\int \left(\exp i S^J(T, 0; q)\right) \left(\prod_{j=1}^{k} h^{(j)}(\theta_j) \cdot q(\theta_j)\right) \times f(q(0)) \mathcal{D} q$ of variables $\theta_j \in [0, T]$ $(j = 1, 2, \ldots, k)$ is continuous in B^a. Then we have the following as the main theorem in the present paper.

Theorem 2.6. *Suppose the assumptions of Lemma 2.1 and take a potential (V, A) satisfying (2.7) and (2.8). Then for $k = 0, 1, \ldots, l$ the functional $G(\cdot) f$*

from X into B^a has a k-times Fréchet derivative $D_J^k G(J)f[h^{(1)},\dots,h^{(k)}]$, which is equal to

$$i^k \int_0^T d\theta_1 \int_0^T d\theta_2 \cdots \int_0^T d\theta_k \int e^{iS^J(T,0;q)} \left(\prod_{j=1}^k h^{(j)}(\theta_j)\cdot q(\theta_j)\right) f(q(0))\mathcal{D}q. \tag{2.14}$$

Remark 2.2. Suppose the assumptions of Remark 2.1. Then we have the gauge invariance (2.11) of $G_\Delta(J)f$. Hence, taking the appropriate spaces in place of B^a ($a = 0,1,\dots$), the same assertions as in Proposition 2.5 and Theorem 2.6 hold.

We write the integrand of (2.14) with respect to the integration variables θ_j ($j = 1,2,\dots,k$) as

$$\frac{\delta^k G(J)}{\delta J(\theta_1)\cdots\delta J(\theta_k)}[h^{(1)},\dots,h^{(k)}], \tag{2.15}$$

which is called the functional derivative of G with respect to variation of the function $J(t)$ at $\theta_1,\dots,\theta_{k-1}$ and θ_k (cf. §7-2 in [2]). Then we get the result corresponding to (1.9) from Theorem 2.6 in the present paper and Corollary in [6].

Corollary 2.7. *Suppose the assumptions of Lemma 2.1 and take a potential (V,A) satisfying (2.7) and (2.8). Let l be a non-negative integer and $f \in B^{a+l}$ ($a = 0,1,\dots$). Then there exist (2.15) in B^a for $k = 0,1,\dots,l$. Setting $J = 0$, these (2.15) are equal to $i^k \int \left(\exp iS(T,0;q)\right)\left(\prod_{j=1}^k h^{(j)}(\theta_j)\cdot q(\theta_j)\right) \times f(q(0))\mathcal{D}q = U(T,0)\cdot T[h^{(1)}(\theta_1)\cdot\hat{q}(\theta_1),\dots,h^{(k)}(\theta_k)\cdot\hat{q}(\theta_k)]f$.*

Let $\mathcal{C}^J(t,s)$ be the operator defined by (2.3) and $\mathcal{C}(t,s) = \mathcal{C}^0(t,s)$. The following is the key lemma for the proof of Theorem 2.6.

Lemma 2.8. *Suppose the assumptions of Lemma 2.1 and take a potential (V, A) satisfying (2.7) and (2.8). Then we have for $f \in B^{a+2}$ $(a = 0, 1, \dots)$*

$$C^J(t,s)f = C(t,s)f + i\int_s^t d\theta \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n} \int \Big(\exp iS(t,s;q^{t,s}_{x,y})\Big) \times$$
$$\Big(J(\theta)\cdot q^{t,s}_{x,y}(\theta)\Big) f(y)dy + R^J(t,s)f \tag{2.16}$$

and

$$\|R^J(t,s)f\|_{B^a} \le C_a(t-s)^2\|J\|^2_\infty \|f\|_{B^{a+2}}, \tag{2.17}$$

where $C_a \ge 0$ are constants.

Proof. We can easily have

$$e^{i\tau} - 1 - i\tau = -\int_0^1 (1-\theta)e^{i\theta\tau}d\theta\tau^2,$$

where

$$\tau = \int_s^t J(\theta)\cdot q^{t,s}_{x,y}(\theta)d\theta = \int_s^t J(\theta)\cdot\Big(x - \frac{t-\theta}{\sqrt{t-s}}w\Big)d\theta,\ w = \frac{x-y}{\sqrt{t-s}}.$$

We can prove Lemma 2.8 from the above by means of Theorem 4.4 in [5]. Q.E.D.

The detailed proof of Theorem 2.6 will be given in [7].

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