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Root of unity symmetries in the 8 and 6 vertex models

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Abstract

We review the recently discovered symmetries of the 8 and 6 vertex models which exist at roots of unity and present their relation with representation theory of affine Lie algebras, Drinfeld polynomials and Bethe vectors.

Keywords: lattice models, Bethe equations, quantum groups, loop algebras, Drinfeld polynomial

I. INTRODUCTION

The 8 vertex model is a lattice model in statistical mechanics whose transfer matrix is given in ref.[1] by

$$\mathbf{T}_8(v)|_{\mu,\nu} = \mathrm{Tr}W(\mu_1,\nu_1)W(\mu_2,\nu_2)\cdots W(\mu_N,\nu_N) \tag{1.1}$$

where $\mu_j,\nu_j=\pm 1$ and $W(\mu,\nu)$ is a 2×2 matrix whose nonvanishing elements are given as

$$\begin{aligned}
W(1,1)|_{1,1} = W(-1,-1)|_{-1,-1} &= \rho\Theta(2\eta)\Theta(v-\eta)H(v+\eta) = a(v)\\
W(-1,-1)|_{1,1} = W(1,1)|_{-1,-1} &= \rho\Theta(2\eta)H(v-\eta)\Theta(v+\eta) = b(v)\\
W(-1,1)|_{1,-1} = W(1,-1)|_{-1,1} &= \rho H(2\eta)\Theta(v-\eta)\Theta(v+\eta) = c(v)\\
W(1,-1)|_{1,-1} = W(-1,1)|_{-1,1} &= \rho H(2\eta)H(v-\eta)H(v+\eta) = d(v).
\end{aligned} \tag{1.2}$$

The definition and some useful properties of $H(v)$ and $\Theta(v)$ are summarized in appendix A. This model is characterized by the important property that for all fixed η all elliptic nomes p and all chain lengths N it satisfies the commutation relation[1]

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$$[\mathbf{T}(v), \mathbf{T}(v')] = 0. \tag{1.3}$$

The 8 vertex model also has the important property that there are many cases in which the eigenvalues and eigenvectors of the transfer matrix may be computed, but, in contrast with the commutation relation (1.3), qualifying statements must be made on the allowed values of η and N. Some of these qualifying statements are present in the original studies of the eigenvalues[1,2] and eigenvectors[3-5] by Baxter and others have been recently observed by the present authors[6-8]

Conditions for eigenvalues

1) N unrestricted with $\eta = mK/L$ with L even or L and m odd (ref.[1])
2) N even and η unrestricted (ref.[3,2])
3) $\eta = mK/L$ with L odd, m even and N even with $N \leq L-1$ (ref.[6]).

Conditions for some eigenvectors

1) N even and $\eta = mK/L$ (ref.[3-5])
2) N odd and $\eta = mK/L$ with L odd, m even and $N = 2n_B + n_L L$ with n_B and n_L integers (ref.[3-5]).

This array of qualifying conditions is in contrast with the special case of the 6 vertex model where the nome p vanishes, the Boltzmann weight $d(v)$ vanishes and the remaining nonvanishing weights are

$$\begin{aligned} W(1,1)|_{1,1} = W(-1,-1)|_{-1,-1} &= \rho' \sin(v+\eta) = a(v) \\ W(-1,-1)|_{1,1} = W(1,1)|_{-1,-1} &= \rho' \sin(v-\eta) = b(v) \\ W(-1,1)|_{1,-1} = W(1,-1)|_{-1,1} &= \rho' \sin(2\eta) = c(v) \end{aligned} \tag{1.4}$$

where the Bethe form of the eigenvectors is known to hold[9] for all eigenvectors and all eigenvalues are computed for all η and N. On the other hand it was recently discovered[10-13] that if the root of unity condition

$$\gamma = 2\eta = m\pi/L \tag{1.5}$$

holds then the 6 vertex model has an sl_2 loop algebra symmetry group. In this article we review this infinite dimensional symmetry algebra of the 6 vertex model at roots of unity and discuss its relation to the qualifying restrictions given above for the solution of the 8 vertex model at (elliptic) roots of unity

$$\eta = mK/L. \tag{1.6}$$

II. LOOP ALGEBRA SYMMETRY OF THE 6 VERTEX MODEL

For generic (irrational) values of γ/π the spectrum of eigenvalues of the 6 vertex transfer matrix is nondegenerate. However, when the root of unity condition (1.5) holds degenerate multiplets occur if $N > L$. These multiplets may be described in terms of the operator

$$S^z = \frac{1}{2}\sum_{k=1}^{N} \sigma_k^z \tag{2.1}$$

which commutes with the transfer matrix of the 6 vertex model. Call S^z_{max} the maximum value of S^z in the multiplet. Then in the sector

$$S^z \equiv 0 \ (\mathrm{mod} L) \tag{2.2}$$

the number of degenerate states in the multiplet with the value of S^z given by

$$S^z = S^z_{\mathrm{max}} - lL \quad \text{with } 0 \le l \le 2S^z_{\mathrm{max}}/L \tag{2.3}$$

is

$$\binom{2S^z_{\mathrm{max}}/L}{l} \tag{2.4}$$

and thus the total number of states in the degenerate multiplet is $2^{2S^z_{\mathrm{max}}/L}$. When

$$S^z \equiv n \ne 0 (\mathrm{mod} L) \tag{2.5}$$

there are three types of multiplets with degeneracies

$$\binom{[2S^z_{\mathrm{max}}/L] + (-1,0,1)}{l} \quad \text{with } 0 \le l \le [2S^z_{\mathrm{max}}/L] + (-1,0,1) \tag{2.6}$$

where $[x]$ is the greatest integer contained in x.

These degenerate multiplets signal the existence of a symmetry of the system which is not present in the finite system for γ/π irrational. This symmetry algebra was discovered in ref.[10] where it was shown that the operators

$$\begin{aligned} S^{\pm(L)} = \textstyle\sum_{1\le j_1<\cdots<j_L\le N} & q^{L\sigma^z/2} \otimes \cdots \otimes q^{L\sigma^z/2}\sigma^{\pm}_{j_1} \otimes q^{(L-2)\sigma^z/2} \otimes \cdots \otimes q^{(L-2)\sigma^z/2} \\ & \otimes \sigma^{\pm}_{j_2} \otimes q^{(L-4)\sigma^z/2} \otimes \cdots \otimes \sigma^{\pm}_{j_L} \otimes q^{-L\sigma^z/2} \otimes \cdots \otimes q^{-L\sigma^z/2} \end{aligned} \tag{2.7}$$

$$\begin{aligned} T^{\pm(L)} = \textstyle\sum_{1\le j_1<\cdots<j_L\le N} & q^{-L\sigma^z/2} \otimes \cdots \otimes q^{-L\sigma^z/2}\sigma^{\pm}_{j_1} \otimes q^{-(L-2)\sigma^z/2} \otimes \cdots \otimes q^{-(L-2)\sigma^z/2} \\ & \otimes \sigma^{\pm}_{j_2} \otimes q^{-(L-4)\sigma^z/2} \otimes \cdots \otimes \sigma^{\pm}_{j_L} \otimes q^{L\sigma^z/2} \otimes \cdots \otimes q^{L\sigma^z/2} \end{aligned} \tag{2.8}$$

with $q = -e^{i\gamma}$ satisfy in the sector (2.2) the commutation relations with the 6 vertex transfer matrix $\mathbf{T}_6(v)$ [illegible]

$$[S^{\pm(L)}, \mathbf{T}_6(v)e^{-iP}] = [T^{\pm(L)}, \mathbf{T}_6(v)e^{-iP}] = 0 \tag{2.9}$$

where e^{-iP} is the lattice translation operator. We note that $S^{\pm(L)} = T^{\pm(L)*}$. The operators $S^{\pm(L)}$, $T^{\pm(L)}$, and S^z satisfy the defining relations of the Chevalley generators of the loop algebra of sl_2

$$[S^{+(L)}, T^{+(L)}] = [S^{-(L)}, T^{-(L)}] = 0 \tag{2.10}$$

$$[S^{\pm(L)}, S^z] = \pm L S^{\pm(L)}, \quad [T^{\pm(L)}, S^z] = \pm L T^{\pm(L)} \tag{2.11}$$

$$[S^{+(L)}, S^{-(L)}] = [T^{+(L)}, T^{-(L)}] = -(-q)^L \tfrac{2}{L} S^z \tag{2.12}$$

$$S^{+(L)3}T^{-(L)} - 3S^{+(L)2}T^{-(L)}S^{+(L)} + 3S^{+(L)}T^{-(L)}S^{+(L)2} - T^{-(L)}S^{+(L)3} = 0 \tag{2.13}$$

$$S^{-(L)3}T^{+(L)} - 3S^{-(L)2}T^{+(L)}S^{-(L)} + 3S^{-(L)}T^{+(L)}S^{-(L)2} - T^{+(L)}S^{-(L)3} = 0 \tag{2.14}$$

$$T^{+(L)3}S^{-(L)} - 3S^{+(L)2}S^{-(L)}T^{+(L)} + 3T^{+(L)}S^{-(L)}T^{+(L)2} - S^{-(L)}T^{+(L)3} = 0 \tag{2.15}$$

$$T^{-(L)3}S^{+(L)} - 3T^{-(L)2}S^{+(L)}T^{-(L)} + 3T^{-(L)}S^{+(L)}T^{-(L)2} - S^{+(L)}T^{-(L)3} = 0 \tag{2.16}$$

In the sector (2.5) we further define the operators

$$S^{\pm}=\sum_{j=1}^{N}q^{\sigma^z/2}\otimes\cdots q^{\sigma^z/2}\otimes\sigma_j^{\pm}\otimes q^{-\sigma^z/2}\otimes\cdots\otimes q^{-\sigma^z/2} \tag{2.17}$$

$$T^{\pm}=\sum_{j=1}^{N}q^{-\sigma^z/2}\otimes\cdots q^{-\sigma^z/2}\otimes\sigma_j^{\pm}\otimes q^{\sigma^z/2}\otimes\cdots\otimes q^{\sigma^z/2} \tag{2.18}$$

$$\tag{2.19}$$

and we found in ref.[10] that for even N the following eight operators commute with $\mathbf{T}_6(v)e^{-iP}$

$$\begin{aligned}
&(T^+)^n(S^-)^nS^{-(L)}, \quad S^{-(L)}(S^-)^{L-n}(T^+)^{L-n},\\
&S^{+(L)}(S^+)^n(T^-)^n, \quad (T^-)^{L-n}(S^+)^{L-n}S^{+(L)},\\
&T^{+(L)}(T^+)^n(S^-)^n, \quad (S^-)^{L-n}(T^+)^{L-n}T^{+(L)},\\
&(S^+)^n(T^-)^nT^{-(L)}, \quad T^{-(L)}(T^-)^{L-n}(S^+)^{L-n}.
\end{aligned} \tag{2.20}$$

The operators

$$(T^+)^n(S^-)^n,\ (S^+)^n(T^-)^n,\ (T^-)^{L-n}(S^+)^{L-n},\ (S^-)^{L-n}(T^+)^{L-n} \tag{2.21}$$

which appear in (2.20) each have a large null space but they are not in themselves projection operators. For $L=2$ and numerically on the computer for $L=3$ we have constructed the projection operators onto the eigenspace of the nonzero eigenvalues of the operators (2.21) and using (2.20) have constructed the corresponding projections of $S^{\pm(L)}$ and $T^{\pm(L)}$ and have verified that for the projected operators in the sector (2.5) that (2.10),(2.11) and (2.13)-(2.16) hold without modification but that in (2.12) the constant $2/L$ is replaced by an expression which depends on both L and n. We thus conclude that for even N the sl_2 loop algebra is a symmetry algebra of all sectors of the 6 vertex model.

III. EVALUATION REPRESENTATIONS, DRINFELD POLYNOMIALS AND BETHE VECTORS

The degenerate multiplets of the 6 vertex model are an example of a "highest weight" phenomenon where all eigenvectors of the multiplet may be obtained by letting the generators of the symmetry algebra operate on the "highest weight vector" $|\Omega>$ of the multiplet. In the sector $S^z\equiv 0$ (mod$L$) the highest weight vector $|\Omega>$ is defined[14],[15] in terms of the Chevalley generators $S^{\pm(L)}$ and $T^{\pm(L)}$ of the previous section as

$$S^{+(L)}|\Omega>=T^{+(L)}|\Omega>=0 \tag{3.1}$$

$$\frac{T^{+(L)r}}{r!}\frac{S^{-(L)r}}{r!}|\Omega>=\mu_r|\Omega>,\ \frac{S^{+(L)r}}{r!}\frac{T^{-(L)r}}{r!}|\Omega>=\mu_r|\Omega>, \tag{3.2}$$

$$S^z|\Omega>=S^z_{\rm max}|\Omega>. \tag{3.3}$$

To further study these finite dimensional representation we need to recall a fundamental property of all affine Lie algebras that, besides the Chevalley basis, they are also characterized by a "mode" basis. In this basis the elements of the sl_2 loop algebra are $e(n),\ f(n)$ and $h(n)$, where n is an integer, which satisfy the commutation relations

$$\begin{aligned}
[e(m),f(n)]&=h(m+n)\\
[e(m),h(n)]&=-2e(m+n)\\
[f(m),h(n)]&=2f(m+n).
\end{aligned} \tag{3.4}$$

The relation to the Chevalley basis of the previous section is

$$e(0) = T^{-(L)}, \quad e(-1) = S^{-(L)} \quad f(0) = T^{+(L)}, \quad f(1) = S^{+(L)}. \tag{3.5}$$

In terms of this mode basis the evaluation representations are specified by vectors $|a_j, m_j>$ where for all integer n (positive, negative or zero)

$$\begin{aligned} e(n)|a_j, m_j> &= a_j^n e_{m_j}|a_j, m_j> \\ f(n)|a_j, m_j> &= a_j^n f_{m_j}|a_j, m_j> \\ h(n)|a_j, m_j> &= a_j^n h_{m_j}|a_j, m_j> \end{aligned} \tag{3.6}$$

where a_j are called evaluation parameters and e_{m_j}, f_{m_j} and h_{m_j} are a spin $m_j/2$ representation of sl_2. An important theorem[14,15] is that the evaluation parameters a_j are the roots with multiplicities m_j of what is called the (classical) Drinfeld polynomial $P_\Omega(z)$

$$P_\Omega(z) = \sum_{r \geq 0} \mu_r(-z)^r \tag{3.7}$$

where μ_r are the eigenvalues defined in (3.2).

To apply this notion of highest weight vector to the present case we need to find the relation between $|\Omega>$ and the Bethe form of the eigenvectors. In the region $S^z \geq 0$ these Bethe vectors are specified by the coordinate x_k of $n = \frac{N}{2} - S^z$ "down" spins which satisfy $1 \leq x_1 < x_2 < \cdots < x_n \leq N$ and the form of the wave function is

$$|x_1, x_2, \cdots x_n> = \sum_P A_P e^{i(k_{P1}x_1 + k_{P2}x_2 + \cdots + k_{Pn}x_n)} \tag{3.8}$$

where the sum is over all $n!$ permutations P of $1, \cdots n$, The A_P are specified functions of the k_{Pj} and the k_j are given in terms of v_j by $e^{ik} = (e^{i\gamma} - e^{iv})/(e^{i(v+\gamma)} - 1)$ where the v_j satisfy

$$\frac{\sin(v_j + \gamma/2)}{\sin(v_j - \gamma/2)} = \prod_{l=1, \neq j}^{N/2 - S^z} \frac{\sin(v_j - v_l + \gamma)}{\sin(v_j - v_l - \gamma)}. \tag{3.9}$$

These equations uniquely specify the eigenvectors as long as the root of unity condition (1.5) does not hold. In terms of the "algebraic Bethe Ansatz" presented in ref.[16,17] the states (3.8) are given as $\prod_j B(v_j)|0>$ where the v_j are determined from (3.9),$|0>$ is the state with no down spins and the operator $B(v)$ is the upper right hand element of the 2×2 monodromy matrix $\mathbf{M}(v)$ given as

$$\begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix} = \mathbf{M}(v) = W(\mu_1, \nu_1) W(\mu_2, \nu_2) \cdots W(\mu_N, \nu_N) \tag{3.10}$$

When the root of unity condition (1.5) does hold there is ambiguity in the solution of (3.9) because L of the v_j may be of the form

$$v_{j;k} = v_j^c + km\pi/L, \quad k = 0, 1, \cdots, L-1. \tag{3.11}$$

These sets of roots are called "complete strings" and give factors of 0/0 which cancel out of (3.9). It is indirectly shown in ref.[13] that the highest weight vectors $|\Omega>$ are all Bethe vectors where the "Bethe roots" v_j do not contain any complete L strings. The remaining members of the multiplet, while they are still of the Bethe form (3.8), will contain complete L strings. For γ at roots of unity the algebraic Bethe Ansatz does not give a proper construction of eigenstates belonging to degenerate multiplets as the operator $\prod_{k=0}^{L-1} B(v-2ik\gamma)$ which formally creates a L-string vanishes[18]. We have shown in[13] that the creation operator of the L-string part of a state vector is

$$B^{(L)}(v) =$$
$$\sum_{k=0}^{L-1}\left(\prod_{l=0}^{k-1} B(v-2il\gamma)\right)\left(B_\gamma(v-2ik\gamma)+\frac{X(v-2ik\gamma)}{Y(v)}B_v(v-2ik\gamma)\right)\times\left(\prod_{l=k+1}^{L-1} B(v-2il\gamma)\right) \quad (3.12)$$

where $B_\gamma(v)$ and $B_v(v)$ specify derivatives of $B(v)$ with respect to γ and v respectively and where

$$X(v) = 2i\sum_{l=0}^{L-1}\frac{l\sinh^N\frac{1}{2}(v-(2l+1)i\gamma)}{\prod_{k=1}^{n}\sinh\frac{1}{2}(v-v_k-2il\gamma)\sinh\frac{1}{2}(v-v_k-2i(l+1)\gamma)} \quad (3.13)$$

and

$$Y(v) = \sum_{l=0}^{L-1}\frac{\sinh^N\frac{1}{2}(v-(2l+1)i\gamma)}{\prod_{k=1}^{n}\sinh\frac{1}{2}(v-v_k-2il\gamma)\sinh\frac{1}{2}(v-v_k-2i(l+1)\gamma)} \quad (3.14)$$

and v_k with $k=1,\cdots,n$ are the ordinary Bethe roots. We show in[13] that $Y(v)$ satisfies the periodicity condition

$$Y(v+m\pi/L) = Y(v) \quad (3.15)$$

and thus is a Laurent polynomial in $z = e^{2iLv}$. Furthermore we define the degrees $d_\pm$ by

$$Y(v) = C_\pm e^{\pm 2iLd_\pm v} \quad \text{as } v \to \pm i\infty. \quad (3.16)$$

The Drinfeld polynomial $P_\Omega(z)$ is then given as[13]

$$P_\Omega(z) = e^{d_- 2iLv}Y(v) \quad (3.17)$$

which is a polynomial in z of degree $d = d_+ + d_-$.

If the zeros of $Y(v)$ all have multiplicity one then the number of eigenvalues in the multiplet specified by the evaluation parameters of the highest weight vector $|\Omega>$ is 2^d. There exists no analytic proof that the roots of $Y(v)$ are all simple but this has been verified in all numerical evaluations which have been made.

IV. THE 8 VERTEX MODEL AT ROOTS OF UNITY

The eigenvalues of the 8 vertex model develop degenerate multiplets when η satisfies (1.6) in a manner almost identical with the degeneracies of the 6 vertex model at (1.5). The only difference is that some of the 6 vertex multiplets split into two multiplets for the 8 vertex model when the elliptic nome $p \neq 0$. Thus the numerical evidence for the existence of a symmetry algebra for the 8 vertex model at the roots of unity points (1.6) is just as compelling as for the 6 vertex model.

However, despite the great similarity in the numerical evidence the treatment of the degeneracies of the 8 vertex model is very different from what was done for the 6 vertex model. In particular we do not have expressions for operators like $S^{\pm(L)}$ and $T^{\pm(L)}$ which commute with $\mathbf{T}_8 e^{-iP}$ and we do not know what is the symmetry algebra which produces the degeneracy. Accordingly we cannot give a group theoretic explanation in terms of highest weight states and evaluation parameters,

Instead of using group theory we study the multiplets of the 8 vertex model at roots of unity (1.6) by use of the matrix first introduced by Baxter[1] in 1972 in his original solution of the 8 vertex model. This matrix (which we call $\mathbf{Q}_{72}(v)$) satisfies the functional equation

$$\mathbf{T}_8(v)\mathbf{Q}(v) = [\rho\Theta(0)h(v-\eta)]^N\mathbf{Q}(v+2\eta) + [\rho\Theta(0)h(v+\eta)]^N\mathbf{Q}(v-2\eta) \tag{4.1}$$

where $h(v) = \Theta(v)H(v)$ and the commutation relations

$$[\mathbf{T}_8(v), \mathbf{Q}(v')] = [\mathbf{Q}(v), \mathbf{Q}(v')] = 0. \tag{4.2}$$

In Baxter's paper[1] the following explicit form is given for $\mathbf{Q}_{72}(v)$

$$\mathbf{Q}_{72}(v) = \mathbf{Q}_R(v)\mathbf{Q}_R^{-1}(v_0) \tag{4.3}$$

where v_0 is an arbitrary normalization point at which $\mathbf{Q}_R(v)$ is nonsingular. The matrix $\mathbf{Q}_R(v)$ is defined as

$$\mathbf{Q}_R(v)|_{\alpha|\beta} = \mathrm{Tr}\mathbf{S}(\alpha_1,\beta_1)\mathbf{S}(\alpha_2,\beta_2)\cdots\mathbf{S}(\alpha_N,\beta_N) \tag{4.4}$$

where α_j and $\beta_j = \pm 1$ and $S(\alpha,\beta)$ is an $L\times L$ matrix given as (C16) of ref.[1]

$$\mathbf{S}(\alpha,\beta) = \begin{pmatrix} z_0 & z_{-1} & 0 & 0 & \cdot & 0 \\ z_1 & 0 & z_{-2} & 0 & \cdot & 0 \\ 0 & z_2 & 0 & z_{-3} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & z_{1-L} \\ 0 & 0 & 0 & \cdot & z_{L-1} & z_L \end{pmatrix} \tag{4.5}$$

with $z_m = q(\alpha,\beta,m|v)$ and

$$\begin{aligned} q(+,\beta,m|v) &= H(v+K+2m\eta)\tau_{\beta,m}, \\ q(-,\beta,m|v) &= \Theta(v+K+2m\eta)\tau_{\beta,m} \end{aligned} \tag{4.6}$$

The $\tau_{\beta,m}$ are generically arbitrary but we note that if they are all set equal to unity then $\mathbf{Q}_R(v)$ is so singular that its rank becomes 1. On the other hand as long as the $\tau_{\beta,m}$ are chosen so that there is a v_0 such that $\mathbf{Q}_R(v_0)$ is not singular then $\mathbf{Q}_{72}(v)$ is independent of $\tau_{\beta,m}$.

For many years it seems to have been assumed that for all L and all N there did exist a v_0 for which $\mathbf{Q}_R(v)$ was nonsingular but in ref.[6] we made a computer study of $\mathbf{Q}_{72}(v)$ which demonstrated that such a general statement of nonsingularity does not hold. Instead we found that while $\mathbf{Q}_R(v)$ is nonsingular if L is even or when L and m are odd that if L is odd and m is even then $\mathbf{Q}_R(v)$ is singular for all v when N is odd or when N is even and $L \geq N-1$. When N is even the eigenvalues of $\mathbf{T}_8(v)$ may still be studied by means of the TQ equation (4.1) even though $\mathbf{Q}_R(v)$ is singular for $L \geq N-1$ by use of the symmetry

$$\mathbf{T}_8(v+K; K-\eta) = \mathbf{S}\mathbf{T}_8(v;\eta) \tag{4.7}$$

where

$$\mathbf{S} = \prod_{j=1}^{N} \sigma_j^z \tag{4.8}$$

and we note that

$$[\mathbf{S}, \mathbf{T}_8(v)] = 0, \quad [\mathbf{S}, \mathbf{Q}_{72}(v)] = 0. \tag{4.9}$$

However for odd N with L odd and m even no such symmetry exists.

Even though $\mathbf{T}_8(v)$ has many degenerate eigenvalues when η satisfies the root of unity condition (1.6) the matrix $\mathbf{Q}_{72}(v)$ has the remarkable property (discovered numerically but never proven analytically) that it has no degenerate eigenvalues. Therefore if we can find a criterion to determine the class of eigenvalues of $\mathbf{Q}_{72}(v)$ which have the same eigenvectors of the degenerate eigenvalues of $\mathbf{T}_8(v)$ we may determine the degeneracy of an eigenvalue of $\mathbf{T}_8(v)$ by counting the corresponding eigenvalues of $\mathbf{Q}_{72}(v)$. To determine this relation between eigenvalues of $\mathbf{T}_8(v)$ and $\mathbf{Q}_{72}(v)$ we note, as demonstrated in ref.[6], that $\mathbf{Q}_{72}(v)$ obeys the following quasiperiodicity conditions

$$\mathbf{Q}_{72}(v+2K) = \mathbf{S}\mathbf{Q}_{72}(v) \tag{4.10}$$
$$\mathbf{Q}_{72}(v+2iK') = p^{-N}\exp(-iN\pi v/K)\mathbf{Q}_{72}(v). \tag{4.11}$$

It follows from (4.9) that $\mathbf{S}$ and $\mathbf{Q}_{72}(v)$ may be simultaneously diagonalized and thus in the basis where $\mathbf{S}$ is diagonal with eigenvalues $(-1)^{\nu'}$ with $\nu' = 0, 1$ we obtain from (4.10),(4.11) the quasiperiodicity conditions for the eigenvalues

$$Q_{72}(v+2K) = (-1)^{\nu'} Q_{72}(v) \tag{4.12}$$
$$Q_{72}(v+2iK') = p^{-N}\exp(-iN\pi v/K) Q_{72}(v). \tag{4.13}$$

and thus the eigenvalues $Q_{72}(v)$ of the matrix $\mathbf{Q}_{72}(v)$ may be expressed in a factored form as

$$Q_{72}(v) = \mathcal{K}(p; v_k)\exp(-i\nu\pi v/2K)\prod_{j=1}^{N} H(v - v_j) \tag{4.14}$$

where we have the sum rules

$$\nu = \sum_{j=1}^{N} \mathrm{Im} v_j/K' = \text{even integer } -\nu' = N \tag{4.15}$$

$$N + \sum_{j=1}^{N} \mathrm{Re} v_j/K = \text{even integer}. \tag{4.16}$$

From the commutation relations (4.2) we see that all matrices in the TQ equation (4.1) may be simultaneously diagonalized and making this diagonalization we obtain an equation for eigenvalues $T_8(v)$ and $Q(v)$

$$T_8(v)Q(v) = [\rho\Theta(0)h(v-\eta)]^N Q(v+2\eta) + [\rho\Theta(0)h(v+\eta)Q(v-2\eta) \tag{4.17}$$

Then using the factored form (4.14) of the eigenvalues $Q_{72}(v)$ in (4.17) and setting $v = v_j$ we obtain the equation for the zeros v_j of $Q_{72}(v)$

$$\left(\frac{h(v_l - mK/L)}{h(v_l + mK/L)}\right)^N = e^{2\pi i \nu m/L} \prod_{j=1,\neq l}^{N} \frac{H(v_l - v_j - 2mK/L)}{H(v_l - v_j + 2mK/L)}. \tag{4.18}$$

If in the complete set of N roots v_j there are sets of L roots

$$v_{j;k} = v_j^c + 2kmK/L \tag{4.19}$$

we see from the eigenvalue expression (4.17) that all terms with v_j^c cancel and thus eigenvalues of $Q_{72}(v)$ which differ only in the location of the string centers v_j^c have the same degenerate eigenvalues $T_8(v)$. Thus we can count the degeneracy of an eigenvalue of $\mathbf{T}_8(v)$ by determining all those eigenvalues $Q_{72}(v)$ which differ only by their L strings. These L string solutions are the analogue for the 8 vertex model of the L string solutions (3.11) of the 6 vertex model and like the L strings of the 6 vertex model the string centers cannot be determined from the equation (4.22). We note that the strings (4.19) which contain L roots are not the same as the strings of ref.[5],[19] which contain $2L$ roots and are invariant under translation by iK'.

Unlike the 6 vertex equations (3.9) which have $N/2 - S^z$ Bethe roots v_j the 8 vertex equations (4.18) have N roots for all eigenstates of $\mathbf{T}_8(v)$. Moreover there are distinct features in the solutions v_j of (4.18) which depend on N which do not occur for (3.9). To see these features it is (at present) necessary to do a numerical study of the zeroes of the eigenvalues of $\mathbf{Q}_{72}(v)$. We have done this for N even in ref.[6] and for N odd in ref.[8] and have found the following results for the roots of unity condition (1.6).

Even N with m odd and L even or odd

There are n_B pairs of roots

$$v_j^N, \quad v_j^B + iK' \tag{4.20}$$

which we call Bethe roots and n_L complete L strings of the form (4.19) where

$$2n_B + Ln_L = N \tag{4.21}$$

When this form is used in the TQ equation (4.22) we find that the n_B Bethe roots satisfy

$$\left(\frac{h(v_l^B - mK/L)}{h(v_l^B + mK/L)}\right)^N = e^{2\pi i(\nu - n_B)m/L} \prod_{j=1,\neq l}^{n_B} \frac{h(v_l^B - v_j - 2mK/L)}{h(v_l^B - v_j + 2mK/L)}. \tag{4.22}$$

The L string roots are not determined from this equation and we proved for $L = 2$ in ref.[7] and conjectured for $L \geq 3$ in ref.[6] that they are determined in terms of the n_B Bethe roots by the functional equation

$$\mathbf{A}'e^{-N\pi iv/2K}\mathbf{Q}_{72}(v-iK')=\sum_{l=0}^{L-1}\frac{h^N(v-(2l+1)\eta)\mathbf{Q}_{72}(v)}{\mathbf{Q}_{72}(v-2l\eta)\mathbf{Q}_{72}(v-2(l+1)\eta)} \tag{4.23}$$

where $\mathbf{A}'$ is a matrix which commutes with $\mathbf{Q}_{72}(v)$, is independent of v and depends on the normalization in the construction of $\mathbf{Q}_{72}(v)$. The left hand side of (4.23) is an entire function and thus the apparent poles on the right hand side at the zeroes of $\mathbf{Q}_{72}(v)$ must cancel. This cancellation leads to the equation for the Bethe roots v_j^B of (4.22). The remaining zeroes of the right hand side give the string solutions $v_{j;k}^L$ and have the property that if $v_{j;k}^L=v_j^c+2kmK/L$ is a solution then $v_{j;k}^L+iK'$ is also a solution. Therefore the number of eigenvalues of $\mathbf{Q}_{72}(v)$ which correspond to a degenerate eigenvalue of $\mathbf{T}_8(v)$ is 2^{n_L}. Thus for even N we have explained the degeneracies of the 8 vertex model transfer matrix without an appeal to group theory.

Odd N with m odd and L even or odd

There are no paired roots and no L strings. All the N roots v_j are determined from (4.18). For every set of roots v_j which solves (4.18) there is a second solution v_j+iK' which also solves (4.18) and thus all eigenvalues of $\mathbf{T}_8(v)$ are doubly degenerate. This is to be expected from the fact that the transfer matrix is invariant under spin reversal and that all states have half integer total spin. Because there are never any L strings there are no further degeneracies in the eigenvalue spectrum of $\mathbf{T}_8(v)$.

There remains the case L odd and m even where $\mathbf{Q}_{72}$ does not exist. For N even this is related to the case m odd by use of the symmetry (4.7). For N odd, there is no such symmetry. However, it was demonstrated in ref.[3]-[5] that if there exist integer n_B and n_L such that (4.21) holds that the eigenvalues of the transfer matrix $\mathbf{T}_8(v)$ may be computed. Because N and L are odd this requires that n_L be odd (and in particular non zero) for the method to apply.

There is no proof that for N odd, L odd and m even that there is any matrix $\mathbf{Q}(v)$ which satisfies the TQ equation (4.1) and the commutation relations (4.2). Nevertheless we conjecture that such a matrix $\mathbf{Q}(v)$ does exist which satisfies the quasiperiodicity conditions (4.12) and (4.13). We have studied this numerically in ref.[8] and found that for N odd, with L odd and m even there are two types of eigenvalues $Q(v)$ which occur:

Type I.

There are n_B pairs of Bethe roots (4.20) and n_L complete L strings of the form (4.19) where (4.21) holds with $n_L\neq 0$. The Bethe roots are determined from (4.22) and the method of ref.[3]-[5] applies. As with the case of even N we find that for each complete L string $v_{j;k}^L$ there is a companion string $v_{j;k}^L+iK'$. Therefore the degeneracy of the transfer matrix eigenvalue is 2^{n_L}.

Type II.

There are no pairs of Bethe roots and no L strings. To every set of roots v_j there is a companion set of roots v_j+iK' and thus all eigenvalue of the transfer matrix $\mathbf{T}_8(v)$ are doubly degenerate. Because there are no paired roots and no complete L string the condition (4.21) does not hold and the method of computing eigenvectors of ref.[3]-[5] does not apply to these states. On the other hand we have verified that the function $Q(v)$ constructed from v_k according to (4.14) satisfies the equation (4.23) with $A'=0$ from which (4.18) follows. This case for which the methods of ref.[3-5] fail is particularly interesting. A special case was first studied in ref.[20] and the 6 vertex limit has been studied extensively by several authors[21-24].

V. OUTLOOK

The results and computations presented above provide a detailed explanation of the degeneracies in the eigenvalue spectrum of the 6 and 8 vertex models at roots of unity. However it is clear that there are many open questions which need to be resolved before the problem of the degeneracies can be considered to be solved.

For the 6 vertex model with $S^z \equiv 0$ (mod$L$) where the generators of the symmetry algebra are explicitly known the highest weight property (3.1)-(3.3) of the Bethe vectors which contain no strings needs to be directly proven[15]. For $S^z \neq 0$ (mod$L$) the projection operators needed for the several different sectors need to be studied in such a form that the symmetry algebra in the sectors may be analytically established. For all sectors a direct proof of the expression for the Drinfeld polynomial (3.17) in terms of the Bethe roots needs to be given and a proof that the roots of the Drinfeld polynomial are all simple needs to be found. In addition it would be desirable to find a physical interpretation for the basis of degenerate eigenvectors which is specified by the evaluation parameters.

For the 8 vertex model much more needs to be done because here the symmetry algebra is not known even though such an algebra must exist. This algebra for the 8 vertex model must contain information about the sectoring of the 6 vertex model for $S^z \neq 0$ (mod$L$). There should presumably be some analogue for this symmetry algebra of the highest weight phenomenon and the righthand side of (4.23) should be some sort of elliptic generalization of a Drinfeld polynomial with the zeroes providing a generalization of the evaluation parameters.

Moreover, while for the 6 vertex model the eigenvectors and eigenvalues of the transfer matrix $T_6(v)$ are known for all γ and all N to follow from the Bethe form of the eigenvectors the same is not true for the 8 vertex model. For generic η and even N a matrix $\mathbf{Q}_{73}(v)$ is known[3-5,2] which satisfies the TQ equation (4.1) and the commutation relations (4.2) but this matrix does not specialize to $\mathbf{Q}_{72}(v)$ when η is root of unity. This demonstrates that the matrix $\mathbf{Q}(v)$ which satisfies (4.1) and (4.2) is not unique and it is of interest to find how many arbitrary parameters can be contained in the solutions. This would generalize the studies of the $\mathbf{Q}(v)$ matrices made for the 6 vertex model[25-28]. Moreover for odd N in the case L odd with m even or for generic η no $\mathbf{Q}(v)$ matrix is proven to exist even though we have numerically seen that the TQ equation can be satisfied. The case N odd L odd and m even needs to be investigated in the detail for which the 6 vertex limit has been studied in ref.[21-24]. The recent work of Bazhanov and Mangazeev[29] for $\eta = 2K/3$ is an important development in this study.

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APPENDIX A: PROPERTIES OF THETA FUNCTIONS

The definition of Jacobi theta functions of nome p is

$$H(v) = 2\sum_{n=1}^{\infty}(-1)^{n-1}p^{(n-\frac{1}{2})^2}\sin[(2n-1)\pi v/2K] \tag{A1}$$

$$\Theta(v) = 1 + 2\sum_{n=1}^{\infty}(-1)^n p^{n^2}\cos(nv\pi/K) \tag{A2}$$

where K and K' are the standard elliptic integrals of the first kind and

$$p = e^{-\pi K'/K} \tag{A3}$$

These theta functions satisfy the quasiperiodicity relations

$$\begin{aligned} H(v+2K) &= -H(v) && \text{(A4)}\\ H(v+2iK') &= -p^{-1}e^{-\pi iv/K}H(v) && \text{(A5)}\\ \Theta(v+2K) &= \Theta(v) && \text{(A6)}\\ \Theta(v+2iK') &= -p^{-1}e^{-\pi iv/K}\Theta(v) && \text{(A7)}\end{aligned}$$

From (A1) and (A2) we see that $\Theta(v)$ and $H(v)$ are not independent but satisfy

$$\begin{aligned} \Theta(v \pm iK') &= \pm ip^{-1/4}e^{\mp\frac{\pi iv}{2K}}H(v)\\ H(v \pm iK') &= \pm ip^{-1/4}e^{\mp\frac{\pi iv}{2K}}\Theta(v) \end{aligned} \tag{A8}$$