数理解析研究所講究録
1480 巻 2006 年 79-93

Representation Theory and Baxter's TQ equation for the six-vertex model. A pedagogical overview.

Christian Korff

School of Mathematics, University of Edinburgh
Mayfield Road, Edinburgh EH9 3JZ, UK

November 24, 2004

1 Introduction

Recent years have seen renewed interest in the six and eight-vertex model at rational coupling values, that is when the crossing parameter is evaluated at roots of unity. Deguchi, Fabricius and McCoy pointed out that the extra degeneracies in the spectrum of the six-vertex transfer matrix can be understood in terms of an affine symmetry algebra in certain commensurate spin-sectors [1]. This raised the question of how the representation theory of this algebra manifests itself in the Bethe ansatz and determines the level of the degeneracies [1, 2, 3]. The analogous investigation for the eight-vertex model [4, 5, 6, 7] has led to new developments and a better understanding of Baxter's celebrated TQ equation [8, 9, 10]. Here T denotes the transfer matrix and Q stands for the auxiliary matrix. In the trigonometric limit some of the zeroes of the auxiliary matrix correspond to the solutions of the six-vertex Bethe ansatz equations first derived in [11, 12].

The significance of a better understanding of the six as well as the eight-vertex model at roots of unity has been further highlighted by results when the order of the root of unity is three. Then Baxter's TQ equation can be explicitly solved for the groundstate eigenvalue provided the square lattice has an odd number of columns; see [13, 14] for the six and [6, 15] for the eight-vertex case. One finds two linearly independent solutions. These arise because the TQ equation is a second order difference equation, see the discussion in [16] and [17]. However, both linearly independent solutions do not always exist at roots of unity. An additional aspect which motivated further investigation is the connection with combinatorial aspects such as the enumeration of alternating-sign matrices or plane partitions, see e.g. [18, 19, 20].

1.1 Auxiliary matrices for the six-vertex model

The described developments prompted the series of papers [21, 22, 23, 24, 25] on a representation theoretic construction of Q-operators for the six-vertex model at roots of unity. See also [26] for an earlier, but different construction and [27, 28] for a discussion away from roots of unity. The idea to use representation theory of quantum groups to solve Baxter's TQ equation (or generalizations thereof) on the level of operators parallels the approach [29] put forward in the context of the Liouville model. However, there are subtle differences between the continuum theory and the model on the finite lattice and the root of unity case is only marginally discussed in [29].

In particular, at roots of unity the auxiliary space used in the construction of Q can be kept finite-dimensional (this will be explained in more detail below) which simplifies calculations drastically and prevents any problems with convergence; see the discussions in [28] and [23]. For numerical calculations this is of great practical importance.

But why using Baxter's TQ equation for the six-vertex model at all instead of the Bethe ansatz? From the brief outline given above it should be clear that we are interested in an approach which applies to both the six and the eight-vertex model. This rules out the Bethe ansatz which rests on the conservation of the total spin or the existence of a pseudovacuum. But there are additional equally significant reasons:

- To see the full symmetry present at roots of unity one should not choose a basis of eigenvectors which diagonalizes the total spin-operator *even for the six-vertex model.* By dropping this requirement a much wider set of solutions to a generalization of Baxter's TQ equation can be obtained which do not preserve the total spin and reveal a rich geometric structure. This provides a new, different perspective on the symmetries at roots of unity; see [21] and the discussion in Section 4.2 of this article.

- The other aspect is the existence of a second linearly independent solution to the TQ equation [16, 17]. One solution gives the Bethe roots above the equator, the second solution, which is related to the first by spin-reversal, yields the Bethe roots below the equator. The explicit construction of Q-operators provides the platform to rigorously prove existence of these solutions and discuss the question of completeness; see [25]. This discussion is closely related to the aforementioned explicit ground state solutions [14, 6, 15] and the connection between the TQ equation in conformal field theory and ordinary differential equations [30].

1.2 The construction procedure

Having made the case of using the concept of auxiliary matrices for the six-vertex model let us briefly comment on the choice of the construction procedure. There are a number of technical difficulties with the construction described in [10] which can be overcome by the use of representation theory of quantum groups. The advantages are:

1. A simple algebraic form of the auxiliary matrix which drastically simplifies calculations and rests on the Yang-Baxter equation with its underlying algebraic framework. It applies to lattices with an even as well as an odd number of columns.

2. A representation theoretic derivation of functional equations. In particular, there is no need for an "a priori" knowledge of the form of the TQ equation.

3. It enables a generalization to a wider class of models associated with quantum groups of higher rank and is independent of the choice of the representation chosen for the quantum space.

1.3 Outline of the article

As the title already indicates this article is intended to provide an easy-to-digest overview of the construction of auxiliary matrices and their relation to the special symmetries at roots of unity. We will therefore omit any technical computations and proofs referring the interested reader to the original manuscripts [21, 22, 23, 24, 25]. Instead we will emphasize that solving the TQ equation is simple step-by-step procedure which we illustrate with numerous diagrams.

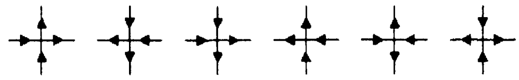
Section 2 briefly reviews the definition of the six-vertex model and its underlying quantum group structure. Section 3 explains in general and simple terms how the Q-operator is constructed and the TQ equation is derived for "generic" q. Section 4 makes contact with the affine symmetry algebra at roots of unity and a geometric picture of the auxiliary matrices. For a particular subvariety of solutions to the TQ equation the spectrum is analyzed. Section 5 states some concluding remarks.

2 Preliminaries on the six-vertex model

In order to keep this article self-contained we repeat some well-known facts from the definition of the six-vertex model and its associated quantum group structure.

2.1 Definition

Consider an $M \times M'$ square lattice where one assigns to each vertex one of the six configurations depicted in Figure 1. Each configuration occurs with a probability determined by the Boltzmann weights a, b, c, c'.



$a \quad a \quad b \quad b \quad c' \quad c$

Figure 1. The six allowed vertex configurations and their Boltzmann weights.

The partition function $Z = \mathrm{Tr}_{\mathcal{H}} T^{M'}$ of the associated statistical six-vertex model can be written in terms of the transfer matrix,

$$T = \operatorname*{Tr}_{V_0} R_{0M} R_{0M-1} \cdots R_{01} \in \mathrm{End}\,\mathcal{H}, \quad \mathcal{H} = (\mathbb{C}^2)^{\otimes M} \,. \tag{1}$$

Here $\mathcal{H}$ is known as "quantum space", $V_0 \cong \mathbb{C}^2$ is called "auxiliary space". The matrix R is defined over $\mathbb{C}^2 \otimes \mathbb{C}^2$ and contains the Boltzmann weights associated with the different vertex configurations,

$$R = \tfrac{a+b}{2} 1 \otimes 1 + \tfrac{a-b}{2} \sigma^z \otimes \sigma^z + c\sigma^+ \otimes \sigma^- + c'\sigma^- \otimes \sigma^+ = \begin{pmatrix} a & & & \\ & b & c & \\ & c' & b & \\ & & & a \end{pmatrix} . \tag{2}$$

The symbols $\{\sigma^x, \sigma^y, \sigma^z\}$ denote the Pauli matrices with $\sigma^{\pm} = (\sigma^x \pm i\sigma^y)/2$ and the lower indices in (1) indicate on which pair of spaces the R-matrix acts in the $(M+1)$-fold tensor product of $\mathbb{C}^2$. Two transfer matrices commute with each other provided their Boltzmann weights leave the quantity $\Delta = \left(a^2 + b^2 - cc'\right)/2ab$ invariant. The well-known symmetries of the model are expressed in terms of the following commutators

$$[T(z), T(w)] = [T(z), S^z] = [T(z), \mathfrak{R}] = [T(z), \mathfrak{S}] = 0 \,, \tag{3}$$

where the respective operators are defined as

$$S^z = \frac{1}{2} \sum_{m=1}^{M} \sigma^z_m, \quad \mathfrak{R} = \sigma^x \otimes \cdots \otimes \sigma^x = \prod_{m=1}^{M} \sigma^x_m, \quad \mathfrak{S} = \sigma^z \otimes \cdots \otimes \sigma^z = \prod_{m=1}^{M} \sigma^z_m \,. \tag{4}$$

These symmetries hold for spin-chains of even as well as odd length. We now review the connection of this model with the affine quantum group $U_q(\widehat{sl}_2)$. This will set the stage for the representation theoretic construction of the auxiliary matrices.

2.2 The quantum group $U_q(\widehat{sl}_2)$

There is a quasi-triangular Hopf algebra $U_q(\widehat{sl}_2)$ which is generated by the six elements $\{e_i, f_i, q^{\pm h_i}\}_{i=0,1}$ obeying the relations

$$q^{h_i} q^{h_j} = q^{h_j} q^{h_i}, \quad q^{h_i} q^{-h_i} = q^{-h_i} q^{h_i} = 1, \tag{5}$$

$$q^{h_i} e_j q^{-h_i} = q^{\mathcal{A}_{ij}} e_j, \quad q^{h_i} f_j q^{-h_i} = q^{-\mathcal{A}_{ij}} f_j, \tag{6}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad i, j = 0, 1, \quad \mathcal{A}_{ij} = (-1)^{i+j} 2 \,. \tag{7}$$

In addition, for $i \neq j$ the q-deformed Chevalley-Serre relations hold,

$$x_i^3 x_j - [3]_q x_i^2 x_j x_i + [3]_q x_i x_j x_i^2 - x_j x_i^3 = 0, \quad x = e, f \,. \tag{8}$$

Further defining relations can be found in e.g. [31]. For our present purposes only one more structure is important, the coproduct of $U_q(\widehat{sl}_2)$ which is given by[1]

$$\Delta(e_i) = 1 \otimes e_i + q^{h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i, \quad \Delta(q^{h_i}) = q^{h_i} \otimes q^{h_i} . \tag{9}$$

There is an opposite coproduct $\Delta^{\rm op}$ which is obtained by permuting the two factors. Moreover, there exists an abstract universal R-matrix intertwining these two coproduct structures

$$\mathbf{R}\,\Delta(x) = \Delta^{\rm op}(x)\,\mathbf{R}, \quad x \in U_q(\widehat{sl}_2), \quad \mathbf{R} \in U_q(b_+) \otimes U_q(b_-) . \tag{10}$$

Here $U_q(b_\pm)$ denote the upper and lower Borel subalgebra generated by $\{e_i, q^{h_i}, q^{-h_i}\}$ and $\{f_i, q^{h_i}, q^{-h_i}\}$, respectively. In a particular representation of the quantum group the intertwiner can now be identified with the building blocks of the transfer matrix, i.e. the R-matrix (2).

2.3 Evaluation representation of spin $\frac{1}{2}$

Let $z \in \mathbb{C}$ be nonzero then the mapping

$$e_0 \to z\, f,\ f_0 \to z^{-1} e,\ q^{h_0} \to q^{-h},\ e_1 \to e,\ f_1 \to f,\ q^{h_1} \to q^h \quad . \tag{11}$$

defines an algebra homomorphism $\mathrm{ev}_z : U_q(\widehat{sl}_2) \to U_q(sl_2)$. Here $U_q(sl_2)$ is isomorphic to the Hopf algebra generated by either $\{e_1, f_1, q^{h_1}\}$ or $\{e_0, f_0, q^{h_0}\}$. Denote by $\pi_z = \pi \circ ev_z$: $U_q(\widetilde{sl}_2) \to \mathrm{End}\,\mathbb{C}^2$ the spin 1/2 evaluation representation of the quantum group defined by the relations

$$\pi(e) = \sigma^+, \quad \pi(f) = \sigma^-, \quad \pi(q^h) = q^{\sigma^z} \quad . \tag{12}$$

With respect to this representation one can now consider the intertwiner of the tensor product $\pi_z \otimes \pi_1$, i.e.

$$R(z)\,(\pi_z \otimes \pi_1)\,\Delta(x) = [(\pi_z \otimes \pi_1)\,\Delta^{\rm op}(x)]\,R(z), \quad x \in U_q(\widehat{sl}_2) . \tag{13}$$

The matrix elements of the intertwiner can be explicitly computed using and one finds up to an arbitrary overall normalization factor that it coincides with the six-vertex R-matrix with the following parametrization of the Boltzmann weights,

$$a = zq^{\frac{3}{2}} - q^{-\frac{1}{2}}, \quad b = zq^{\frac{1}{2}} - q^{\frac{1}{2}}, \quad c = (q - q^{-1})q^{\frac{1}{2}}, \quad c' = (q - q^{-1})zq^{\frac{1}{2}}, \quad \Delta = \frac{q + q^{-1}}{2} .$$

We could have chosen a different parametrization but this one ensures that the eigenvalues of T are polynomials in z and that the functional equations we are going to derive look more symmetric.

3 The construction of Q for "generic" q

We will now solve the eigenvalue problem of the six-vertex transfer matrix by introducing an additional matrix Q^p which commutes with T and satisfies a functional equation of the following type

$$T(z)Q^p(z) = Q^p(z)T(z) = \phi_1^M(z)Q^{p'}(zq^2) + \phi_2^M(z)Q^{p''}(zq^{-2}) . \tag{14}$$

Here ϕ_1, ϕ_2 are scalar functions depending on z, q and the upper index of the auxiliary matrix Q^p denotes the possible dependence on additional free parameters which we collectively denote by p. Note that these parameters are allowed to shift in the functional equation. This is one of the main differences with Baxter's procedure [10]. Although it appears marginal at first sight this modification is crucial to maintain a simple algebraic form of the auxiliary matrix. The free parameters are also needed to break the symmetries of the transfer matrix such as spin-reversal symmetry, spin-conservation and, when q is a root of unity, the affine symmetry [1]; see [21] for details.

[1] This choice of Hopf algebra is different from the one which often is used in the physics literature involving a symmetric coproduct. See the comments in [21], Section 2.1, pages 5237-8. We comment on this further in Section 4.1 of this article.

3.1 The Yang-Baxter equation

In contrast to Baxter's approach described in Chapter 9 of his book [10] we now first ensure that the auxiliary matrix commutes with the transfer matrix and then afterwards derive the functional equation. Also our algebraic form of the auxiliary matrix differs from the one in [10]. Namely, to ensure the greatest possible compatibility with the definition of the transfer matrix we choose the auxiliary matrix to be of the form

$$Q^p(w) = \operatorname*{Tr}_{V_0=\pi^p_w} L^p_{0M}(w) L^p_{0(M-1)}(w) \cdots L^p_{01}(w) \tag{15}$$

with L^p being a solution of the Yang-Baxter equation,

$$L^p_{12}(w/z) L^p_{13}(w) R_{23}(z) = R_{23}(z) L^p_{13}(w) L^p_{12}(w/z) \,. \tag{16}$$

Obviously, this is sufficient to guarantee that T and Q commute. Contrary, to Baxter [10] we do not at the moment require the commutation relation $[Q^p(w), Q^p(z)] = 0$. We will come back to this point later.

The major achievement of Drinfel'd [32] and Jimbo [33] was to show that the solutions of (16) naturally arise from quantum groups. Instead of solving the Yang-Baxter equation one solves the simpler intertwining relation

$$L^p(w)(\pi^p_w \otimes \pi_1)\Delta(x) = [(\pi^p_w \otimes \pi_1)\,\Delta^{\rm op}(x)]\, L^p(w), \quad x \in U_q(b_+) \,. \tag{17}$$

Here π^p_w denotes a suitably chosen representation. Its precise form is for the moment not important, it will be specified explicitly below (Section 4.2) when q is a primitive root of unity. What matters at the moment is that it depends on additional free parameters and that it is similar to an evaluation representation. Both properties hold true for "generic" q, the difference lies in the fact that π^p is finite-dimensional when q is a root of unity but infinite-dimensional otherwise. In both cases the intertwiner L^p has been explicitly constructed, see [28] and [21]. However, when $q^N \neq 1$ then one has to specify how a meaningful definition of the trace over an infinite-dimensional space in (15) can be given. This can be achieved by introducing quasi-periodic boundary conditions or for $|q|^{\pm 1} < 1$ through the restriction to positive (negative) spin-sectors; see [23] for details.

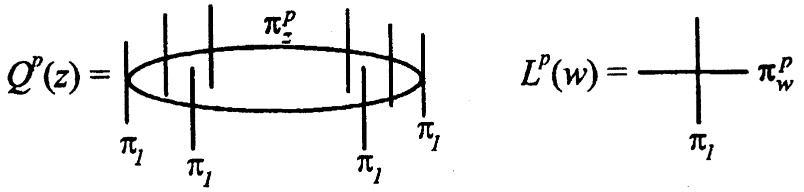


Figure 2. Graphical depiction of the auxiliary matrix and the intertwiner L.

3.2 The derivation of the TQ equation

Having fixed our object Q^p we now need to verify whether it actually satisfies a functional equation with the transfer matrix which enables us to express the eigenvalues of T in terms of Q^p. Since by construction T and Q are built out of intertwiners the same applies to their operator product, namely we have

$$Q^p(w)T(z) = \operatorname*{Tr}_{V_0\otimes V_{0'}=\pi^p_w\otimes\pi_z} L^p_{0M}(w) R_{0'M}(z) \cdots L^p_{01}(w) R_{0'1}(z) \tag{18}$$

with $L^p_{0m}(w)R_{0'm}(z)$ being the intertwiner with respect to the three-fold tensor product $[\pi^p_w \otimes \pi_z] \otimes \pi_1$. In Figure 3 below the product of Q^p and T is diagrammatically presented by the object to the utmost left. There are two horizontal lines one representing the auxiliary space $V_0 = \pi^p_w$ of the Q-operator and one $V_{0'} = \pi_z$ the auxiliary space of the transfer matrix. This interpretation now motivates to investigate the properties of the tensor product representation $\pi^p_w \otimes \pi_z$.

So far the parameter w has been assumed to be free, we now fix it to a value (depending on z and the parameters p) such that the tensor product $\pi^p_w \otimes \pi_z$ becomes a reducible representation. That is, it contains a proper subrepresentation of the quantum group which turns out be similar to π^p_w up to a possible shift in the parameters, $p \to p'$ and $w \to w'$. We explain momentarily how the value w and the subrepresentation $\pi^{p'}_{w'}$ can be explicitly computed. Since we are dealing with a non semi-simple algebra we can not expect that its reducible representations decompose always into a direct sum of irreducible representations. Instead we have to take the quotient space $\pi^p_w \otimes \pi_z / \pi^{p'}_{w'}$ which once more turns out to be another representation $\pi^{p''}_{w''}$. This decomposition of the joint auxiliary space of Q^p and T is conveniently summarized in the following non-split exact sequence,

$$0 \to \pi^{p'}_{w'} \xhookrightarrow{\imath} \pi^p_w \otimes \pi_z \xrightarrow{\tau} \pi^{p''}_{w''} \to 0 \,. \tag{19}$$

Here $\imath$ denote the inclusion map which identifies $\pi^{p'}_{w'}$ and τ the projection map which sends $\pi^{p'}_{w'}$ to zero and determines $\pi^{p''}_{w''}$. Both maps need to be explicitly constructed. Once this is done, the TQ equation depicted in Figure 3 can be derived.

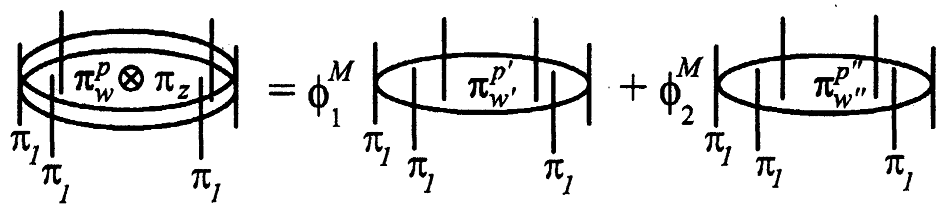


Figure 3. Graphical depiction of the TQ equation.

Before we describe this derivation and the computation of the scalar coefficients $\phi_{1,2}$ let us point out how one finds the value of w and the subrepresentation $\pi^{p'}_{w'}$ which provide the starting point. This information is extracted from the intertwiner of the tensor product $\pi^p_w \otimes \pi_z$ which is simply $L^p(w/z)$, i.e. our building block of the auxiliary matrix. The tensor product $\pi^p_w \otimes \pi_z$ is reducible if and only if $L^p(w/z)$ has a non-trivial kernel which coincides with the image of $\pi^{p'}_{w'}$ under the inclusion map, i.e. we must have

$$\det L^p(w/z) = 0 \qquad \text{and} \qquad \ker L^p(w/z) = \imath(\pi^{p'}_{w'}) \tag{20}$$

under an appropriate choice of w. This also determines the representation space $\pi^{p''}_{w''}$ by taking the quotient $\pi^p_w \otimes \pi_z / \pi^{p'}_{w'}$ as pointed out before. In order to fully specify the inclusion and projection map we now need to pick a suitable basis in the respective representation spaces.

3.2.1 The inclusion

The inclusion $\imath : \pi^{p'}_{w'} \hookrightarrow \pi^p_w \otimes \pi_z$ has so far only been characterized by identifying its image. For the actual calculations and the derivation of the coefficient ϕ_1 in the TQ equation one needs a concrete identification on the level of basis vectors. This is particularly simple if $\pi^p_w \otimes \pi_z, \pi^{p'}_{w'}$ are highest or lowest weight representations. For instance, let us assume that we have lowest weight representations and denote the respective vectors by $|0\rangle$ and $|0\rangle'$ then we must have $\pi^p_w(e_1)\,|0\rangle = \imath[\pi^{p'}_{w'}(e_1)\,|0\rangle'] = 0$. The representation $\pi^{p'}_{w'}$ can then be generated by the quantum group action via the identification

$$\imath[\pi^{p'}_{w'}(x)\,|0\rangle'] = (\pi^p_w \otimes \pi_z)\Delta(x)\;|0\rangle\,, \qquad x \in U_q(\widehat{sl}_2)\,. \tag{21}$$

This relation is also used to determine the parameters p', w'. As the quantum group action also fixes the intertwiner via (10) up to a possible normalization constant, the intertwiner of the three-fold tensor product $\pi^p_w \otimes \pi_z \otimes \pi_1$ must coincide on the subspace $\imath(\pi^{p'}_{w'}) \otimes \pi_1$ with the intertwiner of $\pi^{p'}_{w'} \otimes \pi_1$ up to a scalar multiple. That is, we have the identity

$$L^p_{13}(w)R_{23}(z)(\imath \otimes 1) = \phi_1\,(\imath \otimes 1)L^{p'}(w'), \tag{22}$$

which is graphically depicted in Figure 4. The right-pointing fork represents the inclusion map and each intersection on the various lines stands for the respective intertwiners. (Note the similarity with the bootstrap equation in factorizable scattering theories of QFT.)

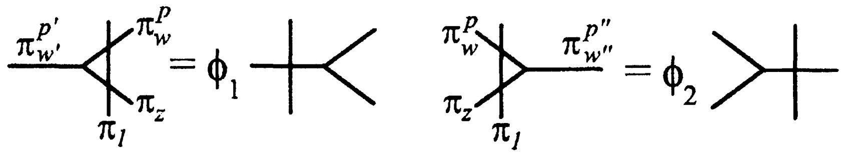


Figure 4. Graphical depiction of the equations (22) and (24).

The scalar multiple occurring fixes the coefficient of the first term in the TQ equation and must be explicitly calculated.

3.2.2 The projection

In the case of the projection map $\tau : \pi^p_w \otimes \pi_z \to \pi^{p''}_{w''}$ we proceed analogously. The only minor complication is that we have now to identify all the vectors in the image of the inclusion map with the null vector in $\pi^{p''}_{w''}$, i.e. the composition $\imath \circ \tau$ vanishes. If τ is presented as a left-pointing fork then its composition with $\imath$ yields the bubble diagram shown in Figure 5.

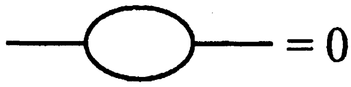


Figure 5. The combination of inclusion and projection vanishes.

Again, the construction of the map τ is simplified when we are dealing with highest or lowest weight representations and the counterpart of equation (21) is then

$$\tau[(\pi^p_w \otimes \pi_z)\Delta(x)\ |0\rangle] = \pi^{p''}_{w''}(x)\,|0\rangle'' , \qquad x \in U_q(\widehat{sl}_2) \,. \tag{23}$$

By the same argument as before we must have the following identity for the respective intertwiners

$$(\tau \otimes 1)L^p_{13}(w)R_{23}(z) = \phi_2\, L^{p''}(w'')(\tau \otimes 1) \,. \tag{24}$$

The diagram corresponding to this equation is shown in Figure 4. Note that the orientation matters as the coefficients ϕ_1, ϕ_2 are in general different.

Our findings can now be briefly summarized as follows. The intertwiner of the three-fold tensor product $[\pi^p_w \otimes \pi_z] \otimes \pi_1$ appearing under the trace in (18) can be written (in a simplified notation) as the upper-triangular matrix

$$L^p_{13}R_{23} = \begin{pmatrix} \phi_1\, L^{p'} & * \\ 0 & \phi_2\, L^{p''} \end{pmatrix}$$

from which the TQ equation (14) is now easily deduced. To ensure that the TQ equation also holds on the level of eigenvalues we need to show that all operators in (14) simultaneously commute. This has been done in [21] for the root-of-unity case employing the intertwiners constructed in [26]. For $q^N \neq 1$ this has been indirectly shown using the algebraic Bethe ansatz [23]. The stronger condition $[Q^p(z), Q^{p'}(w)] = 0$ which would correspond to Baxter's axiom (v) in Chapter 9.5 of [10] is at the moment only rigorously proved for some cases at roots of unity; see (36) below.

4 Roots of unity

We now make contact with the special symmetries mentioned in the introduction which are present when we set the deformation parameter q of the quantum group to be a primitive root of unity, $q^N = 1$. We set $N' = N/2$ when N is even and $N' = N$ when the order is odd. Crucial for the following discussion is to appreciate that the full enriched algebraic structure available can only be seen if one is directly at a root of unity and not by simply taking the root-of-unity limit in the formulae for "generic" q. This is reflected on the level of the underlying algebra by the fact that there exist now *two* alternative, fundamentally different versions of the quantum group:

(1) The unrestricted quantum group $U_q(\widehat{sl}_2)$ [34]. This version has an enlarged centre compared to the case of "generic" q. The additional central elements are generated by $\{e_i^{N'}, f_i^{N'}, q^{N'h_i}\}$ which can take non-zero values in the representation theory at roots of unity. We will exploit this fact for the construction of the auxiliary matrices.

(2) The restricted quantum group $U_q^{\text{res}}(\widehat{sl}_2)$ [35]. The second version of the quantum group does not have an enlarged centre and can be thought of as an algebra of derivations acting on $U_q(\widehat{sl}_2)$. For instance, one identifies the quantum group generators at generic q with the derivation

$$e_i^{(n)} \equiv e_i^n/[n]_q \to [e_i^{(n)}, \cdot\,], \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n \in \mathbb{N},$$

which can be uniquely extended to the case $q^N = 1$. In particular, this still holds true when $n = 0 \bmod N'$ and the subalgebra of the restricted quantum group generated by these elements can be identified with the non-deformed, classical affine algebra $U(\widehat{sl}_2)$ via the quantum Frobenius homomorphism,

$$F : U_q^{\text{res}}(\widehat{sl}_2) \to U(\widehat{sl}_2), \quad e_i^{(n)} \to \begin{cases} \bar{e}_i^{n/N'}, & n = 0 \bmod N' \\ 0, & \text{otherwise} \end{cases}.$$

This is the symmetry algebra of the six-vertex model at roots of unity discovered by Deguchi, Fabricius and McCoy in [1].

We now discuss the various aspects of these two versions and their relation to the six-vertex model at roots of unity separately.

4.1 The affine symmetry algebra: the restricted quantum group

Let us briefly review the affine symmetry algebra of the six-vertex transfer matrix at roots of unity discovered in [1]. Therein the symmetry was established based on the Temperly-Lieb algebra, however, we recall here the simplified proof given in [37] based on the intertwining property of the six-vertex monodromy matrix

$$\mathbf{T} = R_{0M} \cdots R_{01} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{25}$$

Namely, because of our earlier choice of the coproduct (9) we now have

$$\mathbf{T}\;(\pi_z \otimes \pi_{\mathcal{H}})\,\Delta(x) = [(\pi_z \otimes \pi_{\mathcal{H}})\,\Delta^{\text{op}}(x)]\,\mathbf{T} \quad \text{with} \quad \pi_{\mathcal{H}} = \pi_1^{\otimes M}, \quad x \in U_q(\widehat{sl}_2). \tag{26}$$

From this relation it is straightforward to derive the commutation relations between the quantum group generators and the matrix elements of $\mathbf{T}$. For instance, we find for the transfer matrix $T = A + D$ and $x = e_1$,

$$\pi_{\mathcal{H}}(e_1^n)T = (q^n A + q^{-n} D)\,\pi_{\mathcal{H}}(e_1^n) + [n]_q(1 - q^{2S^z})C\,\pi_{\mathcal{H}}(e_1^{n-1}). \tag{27}$$

Analogous relations hold for the remaining quantum group generators; see [37]. As (26), (27) hold true for "generic" q we infer for roots of unity $q^N = 1$ and the restricted quantum group generators with $n = N'$ that

$$\pi_{\mathcal{H}}(e_1^{(N')})T = q^{N'}T\,\pi_{\mathcal{H}}(e_1^{(N')}) \quad \text{iff} \quad 2S^z = 0 \bmod N. \tag{28}$$

The above condition defines the commensurate spin-sectors of the affine symmetry algebra, only there the second term on the right hand side of (27) vanishes. This does not mean that there are no degeneracies outside these spin-sectors. In fact, it has been shown [1] for the free fermion case, $q^4 = 1$, and by a numerical construction for $N = 3$ that certain projection operators have to be introduced to obtain the symmetry algebra in all sectors. The general case is still open. Notice, however, that for lattices with M odd and even roots of unity there are no commensurate sectors and, indeed there are no extra degeneracies in the spectrum of the transfer matrix besides spin-reversal. We will come back to this point below.

In [37] it has been established that for a special choice of quasi-periodic boundary condition the affine symmetry (28) extends to all spin-sectors without the need of projection operators, although it is broken down to $U_q(b_\pm)$ outside the commensurate sectors.

4.2 Auxiliary matrices: the non-restricted quantum group

The construction of the auxiliary matrices Q^p relies on an appropriate choice of the representation π^p_w. We now specify this representation as an irreducible representation of the non-restricted quantum group which has an enlarged centre. We make the (slightly restrictive) assumption that π^p_w is an evaluation representation, i.e. similarly as in the case of the transfer matrix we set $\pi^p_w = \pi^p \circ \mathrm{ev}_w$ with ev being Jimbo's evaluation homomorphism (11) and π^p a representation of the finite quantum group $U_q(sl_2)$. This allows us to make contact with the discussion in [34].

The finite-dimensional irreducible representations of $U_q(sl_2)$ at a primitive root of unity $q^N = 1$ are determined (up to isomorphism) by the values of the following central elements,

$$\mathbf{x} = \left[(q-q^{-1})e\right]^{N'}, \quad \mathbf{y} = \left[(q-q^{-1})f\right]^{N'}, \quad \mathbf{z} = (q^h)^{N'} \tag{29}$$

and the Casimir operator $\mathbf{c} = q^{h+1} + q^{-h-1} + (q - q^{-1})^2 ef$. Each representation now assigns to these central elements four complex numbers which comprise our parameters p, i.e. $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c}) \to p = (p_1, p_2, p_3, p_4) \in \mathbb{C}^4$. An example for such a representation π^p over the vector space $\mathbb{C}^{N'}$ is given by [36, 34],

$$\begin{aligned} \pi^p(q^h)\,|n\rangle &= \lambda q^{-2n}\,|n\rangle\,, \quad \pi^p(f)\,|n\rangle = |n+1\rangle\,, \quad \pi^p(f)\,|N'-1\rangle = \zeta\,|0\rangle\,, \\ \pi^p(e)\,|n\rangle &= \xi_n\,|n-1\rangle\,, \quad \pi^p(e)\,|0\rangle = \xi\,|N'-1\rangle \end{aligned} \tag{30}$$

where $n = 0, 1, 2, ..., N' - 1$ and

$$\xi_n := \frac{\lambda q + \lambda^{-1}q^{-1} - \lambda q^{-2n+1} - \lambda^{-1}q^{2n-1}}{(q-q^{-1})^2} + \xi\zeta, \quad n > 0\,. \tag{31}$$

The values $p = p(\xi, \zeta, \lambda)$ of the central elements (29) are given by

$$\pi^p(\mathbf{x}) = (q-q^{-1})^{N'}\xi \prod_{n=1}^{N'-1} \xi_n, \quad \pi^p(\mathbf{y}) = \zeta(q-q^{-1})^{N'}, \; \pi^p(\mathbf{z}) = \lambda^{N'}$$

and for the Casimir element one finds $\pi^p(\mathbf{c}) = q\lambda + q^{-1}\lambda^{-1} + (q - q^{-1})^2\xi\zeta$. Note that the representation (30) is cyclic when $\xi, \zeta \neq 0$, i.e. there are no highest or lowest weight vectors.

Having specified the representation π^p, and therefore also the evaluation representation π^p_w of the affine quantum group, one can construct the intertwiner L^p via (17), which forms the building block of the auxiliary matrix (15). However, due to the enlarged centre of the quantum group this intertwiner might only exist for special values of the central elements, in other words the concept of the universal R-matrix in (10) breaks down; see the discussion in [21] and references therein. For instance, we find that for even roots of unity we must set $\xi = \zeta = 0$ in order to find a solution of (17).

For odd roots of unity, however, there are no restrictions on ξ, ζ or λ and for $\xi, \zeta \neq 0$ the resulting Q-operators do not preserve the spin. We comment on this case in more detail, as it provides us with an interesting geometric picture of the solutions to the TQ equation (14).

4.2.1 A geometric picture for N odd

As the reader might have already noticed from the representation (30) the central elements (29) and the Casimir operator are not algebraically independent. In fact, their values ought to lie on a three-dimensional hypersurface Spec Z specified by the following identity,

$$\operatorname{Spec} Z: \quad \mathbf{xy}+\mathbf{z}+\mathbf{z}^{-1}=\prod_{\ell=0}^{N-1}\left(\mathbf{c}+q^{\ell}+q^{-\ell}\right)-2\,. \tag{32}$$

Here we now interpret $(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{c})$ as $\mathbb{C}$-numbers which have to solve (32). To each solution, i.e. a point on the hypersurface, we can then associate a representation π^{p} respectively π_w^{p} giving rise to a solution Q^{p} of the operator functional equation (14). So far we have not specified the points p',p'' appearing in the TQ equation. To this end recall from [34] that Spec Z has locally the structure of an N-fold covering space over the base manifold Spec $Z_0=\{\mathbf{x},\mathbf{y},\mathbf{z}\}=\mathbb{C}^3$. The three points appearing in (14) lie in the same fibre and decomposing $\mathbf{c}=\mu+\mu^{-1}$ are explicitly given by

$$p=(\mathbf{x},\mathbf{y},\mathbf{z},\mu+\mu^{-1}),\quad p'=(\mathbf{x},\mathbf{y},\mathbf{z},\mu q+\mu^{-1}q^{-1}),\quad p''=(\mathbf{x},\mathbf{y},\mathbf{z},\mu q^{-1}+\mu^{-1}q)\,. \tag{33}$$

See Figure 6 for a simplified graphical depiction. The values for the evaluation parameters and coefficients $\phi_{1,2}$ are [21]

$$w=z/\mu,\quad w'=zq/\mu,\quad w''=zq^{-1}/\mu,\quad \phi_1(z)=\phi_2(zq^{-2})=z-1\,. \tag{34}$$

The motivation for making this connection lies in the rich structure of the hypersurface (32) described in [34]. Spec Z is endowed with an infinite-dimensional group action G, called the quantum coadjoint action, which induces holomorphic transformations in the coordinates $(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{c})$ and acts transitively on the hypersurface. This action can be carried over to the auxiliary matrices (15). However, what is missing at the moment is an implementation on the lattice, i.e. a map D from the group G of transformations into the matrices acting on the quantum space $\pi_{\mathcal{H}}$ such that

$$D: G\to \operatorname{End}\pi_{\mathcal{H}},\quad D(g)Q^{p}D(g^{-1})=Q^{g\cdot p}, \tag{35}$$

where $g\cdot p$ is the point on the hypersurface obtained under the coordinate transformation $g\in G$.

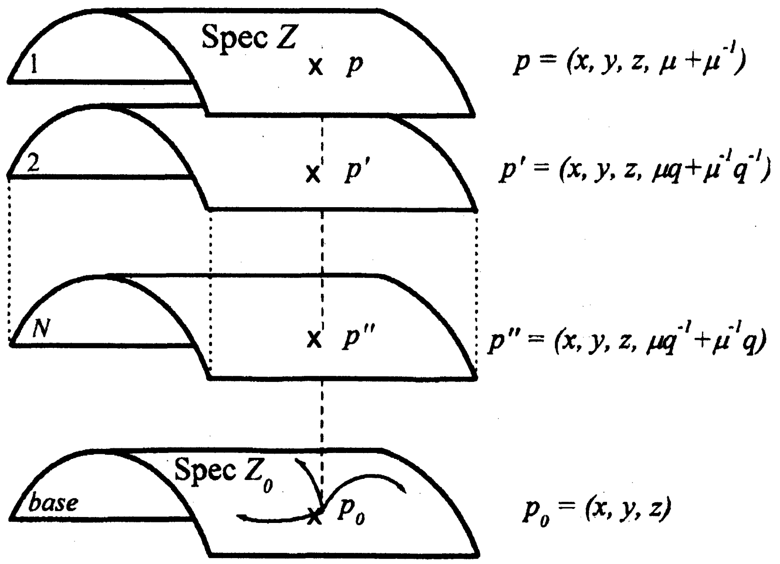


Figure 6. A simplified picture of the hypersurface (32).

The map (35) might provide the key for the construction of the symmetry algebra (28) outside the commensurate sectors. It would also simplify the calculation of the spectrum of the general set of auxiliary matrices Q^{p}. As of yet the spectrum has only been calculated for representations with $\mathbf{x}=\mathbf{y}=0$, which are called nilpotent and which we discuss next.

4.3 The spectrum for nilpotent representations

We now lift the temporary restriction to odd roots of unity but, henceforth, shall only consider representations $\pi^{\mu} \equiv \pi^{p_\mu}$ in the set $\{p_\mu = (\mathbf{x}=0, \mathbf{y}=0, \mathbf{z}=\mu^{-N'}, \mu+\mu^{-1}) : \mu \in \mathbb{C}\}$. Note that the relations (33) and (34) remain true in the limit $\mathbf{x}, \mathbf{y} \to 0$. Denoting the associated auxiliary matrices by $Q_\mu \equiv Q^{p_\mu}$ it has been proved in [22] for $N=3$ and in [25] for $N=4,6$ that the following commutation relation holds,

$$[Q_\mu(z), Q_\nu(w)] = 0, \quad \forall z, w, \mu, \nu \in \mathbb{C} \quad . \tag{36}$$

The proofs given rely on the explicit construction of the quantum group intertwiners with respect to the tensor product $\pi^{\mu}_{w} \otimes \pi^{\nu}_{1}$. Although the construction for the general case is an open problem the necessary condition for the intertwiner to exist are satisfied and numerical computations for $N=5,7,8$ up to $M=11$ confirm that it is correct also for higher roots of unity. From (36) one can deduce two cardinal facts [22, 24]:

1. The auxiliary matrix Q_μ is normal and hence diagonalizable.
2. The eigenvalues of $Q_\mu(z)$ are polynomials in z which are at most of degree M.

The additional information required to determine the characteristics of the eigenvalues is yet again derived from another operator functional equation [24],

$$Q_\mu(z\mu^2q^2)Q_\nu(z) = (zq^2-1)^M Q_{\mu\nu q}(z\mu^2q^2) + Q_{\mu\nu q^{N'+1}}(z\mu^2q^2)T^{(N'-1)}(zq^2), \tag{37}$$

which similar as in the case of the TQ equation is deduced from the decomposition of a tensor product [24],

$$0 \longrightarrow \pi^{\mu'}_{w'} \stackrel{\imath}{\hookrightarrow} \pi^{\mu}_{w} \otimes \pi^{\nu}_{1} \stackrel{\tau}{\twoheadrightarrow} \pi^{\mu''}_{w''} \otimes \pi^{(N'-2)}_{z'} \longrightarrow 0 \; . \tag{38}$$

Here $T^{(N'-1)}$ denotes the fusion matrix of degree $N'-1$, i.e. the analogue of the transfer matrix for the six-vertex model with spin $(N'-2)/2$. The various parameters appearing in the representations are not all independent but satisfy the relations

$$w = \mu\nu q^2, \quad \mu' = \mu\nu q, \quad w' = \mu q, \quad \mu'' = \mu\nu q^{-N'+1}, \quad w'' = \mu q^{N'+1}, \quad z' = \nu q^{N'+1} \; . \tag{39}$$

We do not want to go into the details of the derivation as it follows the analogous steps as detailed in Section 3. However, it provides a more complicated example and underlines the general nature of the approach which not only applies to the TQ equation. See Figure 7 for a graphical depiction of the equation.

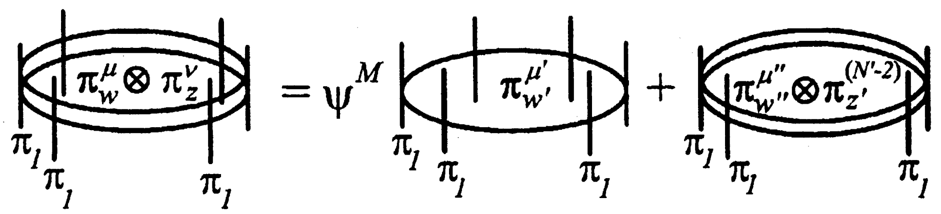


Figure 7. Graphical depiction of the functional equation (37).

The result on the structure of the eigenvalues, which we denote by the same symbol as the corresponding operator, can be summarized as follows. It factorizes into two polynomials,

$$Q_\mu(z) = Q^+(z)Q^-_\mu(z), \tag{40}$$

one of which, Q^+, does not depend on the free parameter μ. The other factor Q^-_μ can be expressed through the first one in the special limit $\mu \to q^{-N'}$,

$$Q^-(z) := \lim_{\mu \to q^{-N'}} Q^-_\mu(z) = q^{(N'+1)s} Q^+(z) \sum_{\ell=1}^{N'} \frac{q^{-2\ell s}(zq^{2\ell}-1)^M}{Q^+(zq^{2\ell})Q^+(zq^{2\ell-2})} \; . \tag{41}$$

It is not immediately apparent that the right hand side of this equation defines a polynomial. This follows from the TQ equation (14) which implies that the zeroes of Q^+ satisfy the six-vertex Bethe ansatz equations. The parameter s in (41) can be identified with

$$s = 2n_0 + S^z \bmod N' , \tag{42}$$

where n_0 denotes the number of Bethe roots which vanish in the root of unity limit $q^N \to 1$. That is, in general we have less Bethe roots than in the case when $q^N \neq 1$, $\deg Q^+ \leq M/2 - S^z$ instead of $\deg Q^+ = M/2 - S^z$

4.3.1 Case-by-case discussion

In order to further characterize the spectrum and explain how the free parameter μ enters we have now to distinguish various cases. As we already saw earlier the loop symmetry (28) is absent for lattices with an number of columns and even roots of unity. This will be reflected in the spectrum of the auxiliary matrices. It needs to be emphasized that the results of [24] presented here are in accordance with the findings of Fabricius and McCoy in the eight-vertex case, see [6] and the corresponding article in this proceedings volume [7]. We will comment further on this in the conclusion.

M even, N arbitrary. In this case the second factor takes the following form

$$Q^-_\mu(z) = \mathcal{N}_\mu\, z^{n_\infty} Q^+(z\mu^{-2})\, P_S(z^{N'}), \quad P_S(z^{N'}) = \prod_{i=1}^{n_S}(1 - z^{N'} a_i)\,. \tag{43}$$

Here $\mathcal{N}_\mu$ is a normalization constant which only depends on μ and q, the power of the polynomial depends on the number of Bethe roots which have either vanished or gone off to infinity in the root of unity limit. The last factor P_S, which drops out of the TQ equation (14), determines the degeneracy of the corresponding eigenvalue of the transfer matrix T. Denote by V_T the associate degenerate eigenspace of the transfer matrix then we have

$$\dim V_T = 2^{\deg P_S}\ . \tag{44}$$

This result follows from the fact that the auxiliary matrix Q_μ lifts the degeneracy of the transfer matrix. Inside the degenerate eigenspace V_T the set of eigenvalues of Q_μ varies only through the dependence of each zero a_i of P_S on the parameter μ. There are only two choices: either a_i depends on μ through a simple multiplicative factor $\mu^{2N'}$ or it does not depend on it at all. Hence, the maximal number of eigenvectors of the auxiliary matrix Q_μ in a degenerate eigenspace V_T of the transfer matrix is given by (44).

In addition, it has been shown for several examples with $N = 3$ in [22] that the zeroes $\{a_i\}$ coincide with the evaluation parameters of the loop algebra, i.e. P_S has been identified with the Drinfel'd polynomial. To prove this assertion for general N a deeper understanding of the highest weight vectors of the symmetry algebra (28) is desirable; see [1, 2, 3].

M odd, N odd. For odd chains the just presented picture remains true with the possible exception that some eigenvalues of the auxiliary matrix may now vanish, i.e. we have the two possible cases

$$\mathcal{N} = 0 \quad \text{or} \quad Q^-_\mu(z) = \mathcal{N}_\mu\, z^{n_\infty} Q^+(z\mu^{-2})\, P_S(z^{N'}) \quad \text{with} \quad \mathcal{N} \neq 0\,.$$

The vanishing of some eigenvalues seems at first to be a serious drawback. However, the vanishing of the eigenvectors occurs only for singlet states, i.e. non-degenerate eigenvectors of the transfer matrix. The number of such vectors rapidly decreases as M starts to exceed the order N [24]. More importantly, since the relation (41) remains true and Q^+ is non-vanishing one can derive a set of difference equations which yield constraints on the Bethe roots, i.e. the zeroes of Q^+ [24]. These constraints are polynomial equations of order $N - 2$ in contrast to the Bethe ansatz equation which are of order M. For $N = 3$ these can be explicitly solved and one obtains Stroganov's result [14]; see [24] and references therein for further details.

M **odd,** N **even.** For the last case we discuss, the spectrum of the six-vertex transfer matrix shows no degeneracies except for spin-reversal symmetry. Now Q_μ^- does not factorize as in the previous cases and we have $Q_\mu^-(z) = Q^-(z\mu^{-2})$ up to some normalization factor which is not important. The second factor Q^- now constitutes a second linearly independent solution to Baxter's TQ equation on the level of eigenvalues and its zeroes are the solutions to the Bethe ansatz equations below the equator. This is an explicit construction of the analogous scenario discussed in [17] away from roots of unity. The two linearly independent solutions are bound to satisfy a Wronskian equation,

$$q^{S^z} Q^+(zq^2)Q^-(z) - q^{-S^z} Q^+(z)Q^-(z) = \text{const.}\ (zq^2-1)^M, \tag{45}$$

where the degree of the polynomials now obey

$$\deg Q^\pm = \frac{M}{2} \mp S^z \ .$$

Note that the Wronskian equation (45) implies the six-vertex Bethe ansatz equations; see [24, 25] for details.

While we have discussed here the factorization of the auxiliary matrix Q_μ into the factors $Q^\pm$ only on the level of eigenvalues, there is a simplified construction which assigns to each of the factors a proper operator; see [25] for details.

5 Concluding Remarks

We would like to stress once more that the representation theoretic construction of Q-operators and the derivation of the TQ equation presented in Section 3 apply to the case of "generic" q. Although in this overview the representation π^p in Definition (15) has only been specified for $q^N = 1$ the case $q^N \neq 1$ has been discussed in [28]; see also [23] for a discussion resolving the convergence problems with an infinite-dimensional representation π^p. At the moment we are still missing the analogue of (36) and (37) when $q^N \neq 1$. The method can also be extended to more complicated models than the six-vertex one. For instance, those associated with higher rank algebras. However, one has then to account for the possibility of a more complicated decomposition of the tensor products, similar to the one we encountered in the derivation of (37).

The other obvious target for a generalization of this method is the eight-vertex model. The investigation [4, 5, 6] of Fabricius and McCoy has already extended in great detail our knowledge of the spectrum of the eight-vertex model and Baxter's 1972 solution of the TQ equation. Their findings match closely the six-vertex picture summarized in this article; see also the corresponding article in this proceedings volume [7]. However, because of the intricate algebraic form of Baxter's 1972 eight-vertex Q-operator [8] it is difficult to take directly the trigonometric limit and obtain a well-defined six-vertex Q-operator. At the moment we can therefore match the eight and six-vertex results *on the level of eigenvalues* only. Then the factor Q^+ in (40) should be identified as the six-vertex analogue of Baxter's Q-operator in [8]. The other factor, Q^-, corresponds to the same eight-vertex Q but when it is evaluated in a different regime. Notice that through a recent refined construction presented in [25] the factorization (40) can also be made on the level of operators for the six-vertex model. What we are missing at the moment is a feasible elliptic construction of the Q-operator which allows one to carry out the trigonometric limit directly in the definition of the operator and to derive the TQ equation as well as the eight-vertex analogue of the essential identity (41) discussed in [4, 5] in a similar manner as presented here.

ACKNOWLEDGEMENT. The author would like to thank M. Shiraishi and T. Miwa for the opportunity to present his results at the RIMS workshop "Progress in Exactly Solvable Lattice Models", Kyoto, July 2004 and M. Jimbo and B.M. McCoy for numerous insightful discussions. This work has been financially supported by the EPSRC Grant GR/R93773/01 and a University Research Fellowship of the Royal Society.

References

[1] T. Deguchi, K. Fabricius and B. M. McCoy, *J. Stat. Phys.* **102** (2004) 701

[2] K. Fabricius and B. M. McCoy, *J. Stat. Phys.* **103** (2001) 647

[3] K. Fabricius and B. M. McCoy MathPhys Odyssey 2001 (Progress in Mathematical Physics 23) ed M. Kashiwara and T. Miwa (Boston: Birkhauser) pp 119

[4] K. Fabricius and B. M. McCoy, *J. Stat. Phys.* **111** (2003) 323

[5] K. Fabricius and B. M. McCoy, *Publ. RIMS* **40** (2004) 905

[6] K. Fabricius and B. M. McCoy, *New Developments in the Eight Vertex Model II. Chains of odd length.* cond-mat/0410113

[7] K. Fabricius and B. M. McCoy, *Root of unity symmetries in the 8 and 6 vertex models.* cond-mat/0411419

[8] R. J. Baxter, *Ann. Phys., NY* **70** (1972) 193

[9] R. J. Baxter, *Ann. Phys., NY* **76** (1973) 1–24; 25–47; 48–71

[10] R. J. Baxter Exactly Solved Models in Statistical Mechanics (London: Academic Press) 1982

[11] E.H. Lieb *Phys. Rev.* **162** (1967) 162; *Phys. Rev. Lett.* **18** (1967) 1046; *Phys. Rev. Lett.* **19** (1967) 108

[12] B. Sutherland, *Phys. Rev. Lett.* **19** (1967) 103

[13] V. Fridkin, Y. Stroganov, D. Zagier, *J. Phys. A: Math. Gen.* **33** (2000) L125

[14] Y. Stroganov, *J. Phys. A: Math. Gen.* **34** (2001) L179

[15] V.V. Bazhanov, V.V. Mangazeev, Eight-vertex model and non-stationary Lame equation, hep-th/0411094

[16] I. Krichever, O. Lipan, P. Wiegmann, A. Zabrodin, *Comm. Math. Phys.* **188** (1997) 267

[17] G. P. Pronko and Y. Stroganov, *J. Phys. A: Math. Gen.* **32** (1999) 2333

[18] G. Kuperberg, *Internat. Math. Res. Notices* (1996) 139

[19] A.V. Razumov and Yu. G. Stroganov, *J. Phys. A: Math. Gen.* **34** (2001) 3185; On refined enumerations of some symmetry classes of ASMs, math-ph/0312071.

[20] M. T. Batchelor, J. de Gier, B. Nienhuis, *J. Phys. A: Math. Gen.* **34** (2001) L265

[21] C. Korff *J. Phys. A: Math. Gen.* **36** (2003) 5229

[22] C. Korff *J. Phys. A: Math. Gen.* **37** (2004) 385

[23] C. Korff *J. Phys. A: Math. Gen.* **37** (2004) 7227

[24] C. Korff *Auxiliary matrices on both sides of the equator.* math-ph/0408023 (to appear in J. Phys. A: Math. Gen. 37, December issue)

[25] C. Korff *Solving Baxter's TQ equation via representation theory.* Proceedings "Non-Commutative Geometry and Representation Theory in Mathematical Physics", July 2004, Karlstads Universitet, Karlstad, Sweden; math-ph/0411034

[26] V. V. Bazhanov and Yu. G. Stroganov *J. Stat. Phys.* **59** (1990) 799

[27] A. Antonov and B. Feigin *Phys. Lett. B* **392** (1997) 115

[28] M. Rossi and R. Weston *J. Phys. A: Math. Gen.* **35** (2002) 10015.

[29] V. Bazhanov , S. Lukyanov and A. Zamolodchikov, *Comm. Math. Phys.* **200** (1999) 297

[30] P. Dorey, R. Tateo *J. Phys. A: Math. Gen.* **32** (1999) L419;
V. Bazhanov, S. Lukyanov and A. Zamolodchikov *J. Stat. Phys.* **102** (2001) 567; *Adv. Theor. Math. Phys.* **7** (2003) 711

[31] V. Chari and A. Pressley *A Guide to Quantum Groups*, Cambridge Univ. Press, 1994

[32] V.G. Drinfel'd *Quantum groups.* In: A.M. Gleason (ed.) Proceedings of the 1986 International Congress of Mathematics, Berkeley, pp.798-820. Providence, RI: AMS 1987

[33] M. Jimbo Lett. Math. Phys. 10 (1985) 63

[34] C. de Concini and V. Kac. Representations of Quantum Groups at Roots of 1. In *Operator algebras, unitary representations, enveloping algebras, and invariant theory,* eds. A. Connes et al., Progress in Mathematical Physics 92, Birkhäuser, 1990, pp 471; C. De Concini, V. Kac, C. Procesi J. Amer. Math. Soc. **5** (1992) 151

[35] V. Chari and A. Pressley *Representation Theory* **1** (1997) 280

[36] P. Roche and D. Arnaudon *Lett. Math. Phys.* **17** (1989) 295

[37] C. Korff, *J. Phys. A: Math. Gen.* **37** (2004) 1681