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Elliptic Stochastic PDEs with polynomial perturbations having a correspondence to Euclidean QFT

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Abstract

Elliptic stochastic partial differential equations (SPDE) with polynomial perturbation terms are studied using results by S. Kusuoka and A.S. Üstünel and M. Zakai concerning transformation of measures on abstract Wiener space. These interactions of the polynomial type arise in (Euclidean) quantum field theory.

1 Introduction

We study elliptic stochastic partial (pseudo) differential equations (SPDE) heuristically written as follows

$$\Big(-\frac{\partial^2}{\partial t^2}+(-\Delta_3+m^2)^3\Big)\psi(x)+\lambda:\psi^3(x):=\Big(-\frac{\partial^2}{\partial t^2}+(-\Delta_3+m^2)^3\Big)^{\frac{1}{2}}\dot{W}(x), \tag{1.1}$$

$$x\equiv(t,\vec{x})\in\boldsymbol{R}\times\boldsymbol{R}^3,$$

where Δ_d is the d-dimensional Laplace operator, W is an isonormal Gaussian process on $\boldsymbol{R}^4$, $\lambda\geq 0$ is some given number and $:\psi^3:$ is the cubic Wick power of ψ.

In order to understand an importance and a motivation of the setting of (1,1), we start with the review of (1.2) below for general $d\in\mathbf{N}$, which has been considered in [AY1] in a framework of change of variable formula on Nelson's Euclidean free field:

$$(-\Delta_d+m^2)\psi(x)+\lambda:\psi^3(x):=(-\Delta_d+m^2)^{\frac{1}{2}}\dot{W}(x),\qquad x\equiv(t,\vec{x})\in\boldsymbol{R}\times\boldsymbol{R}^{d-1}, \tag{1.2}$$

where W is an isonormal Gaussian process on $\boldsymbol{R}^d$. We have to recall that Nelson's Euclidean free field is a Gaussian random variable ϕ_ω taking values in $\mathcal{S}'(\boldsymbol{R}^d)$ defined on a probability space $(\Omega,\mathcal{F},P)$ such that

$$E[<\varphi_1,\phi.><\varphi_2,\phi.>]=\int_{\boldsymbol{R}^d}((-\Delta_d+1)^{-1}\varphi_1)(x)\varphi_2(x)dx,\qquad \text{for real } \varphi_1,\varphi_2\in\mathcal{S}(\boldsymbol{R}^d).$$

We can give $<\varphi,\phi_\omega>_{\mathcal{S},\mathcal{S}'}$ a stochastic integral expression by using the isonarmal Gaussian process W on $\mathbf{R}^d$ as follows:

$$<\varphi,\phi_\omega>_{\mathcal{S},\mathcal{S}'}=\int_{\boldsymbol{R}^d}((-\Delta_d+1)^{-\frac{1}{2}}\varphi)(x)dW_\omega(x). \tag{1.3}$$

By (1.3) the random field ϕ_ω is symbolically written by

$$\phi_\omega=(-\Delta_d+1)^{-\frac{1}{2}}\dot{W}_\omega,$$

or we can write this as a linear elliptic SPDE such that

$$-\Delta_d\phi_\omega+\phi_\omega=(-\Delta_d+1)^{\frac{1}{2}}\dot{W}_\omega. \tag{1.4}$$

Hence, (1.4) is the SPDE corresponding to Nelson's Euclidean free field, and (1.2) is an SPDE given by putting a cubic perturbation term to (1.4).

In [AY1], for $d = 2$ an existence of a random field ϕ that satisfies (1.2) and its explicit expression have been given by applying a change of variable formula on an abstract Wiener space. But, however, for $d \geq 3$ in the framework of abstract Wiener space it is not possible to consider and give a solution of (1.2). Thus, as a substitute of (1.2) for $d = 4$ we shall consider (1.1) here. In Theorem 2.5 we give a solution of (1.1) explicitly.

2 Formulation and results

Let $m > 0$ be some given mass that will be fixed in the sequel. For each real $\alpha \in \mathbf{R}$, let J^α be the pseudo differential operator of which symbol j^α is given by

$$j^\alpha(\tau, \xi) \equiv \left(\tau^2 + (|\xi|^2 + m^2)^3\right)^{-\alpha}, \qquad (\tau, \xi) \in \mathbf{R} \times \mathbf{R}^3.$$

Then the operator J^α is interpreted as

$$J^\alpha \equiv \left(-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3\right)^{-\alpha} \qquad \text{on} \quad \mathcal{S}(\mathbf{R}^4),$$

where $\mathcal{S}(\mathbf{R}^4)$ is the Schwartz space of rapidly decreasing function on $\mathbf{R}^4$. In particular, for $\alpha > 0$ we denote the kernel representation of J^α by $J^\alpha(x - y)$, x, $y \in \mathbf{R}^4$ such that

$$(J^\alpha \varphi)(x) = \int_{\boldsymbol{R}^4} J^\alpha(x-y)\varphi(y)dy \qquad \text{for} \quad \varphi \in \mathcal{S}(\mathbf{R}^4).$$

Denoting $x \equiv (t, \vec{x}) \in \mathbf{R} \times \mathbf{R}^3$, this is defined by the Fourier inverse transform:

$$J^\alpha(x) = (2\pi)^{-4} \int_{\boldsymbol{R}^3} \int_{\boldsymbol{R}} e^{\sqrt{-1}(t\cdot\tau + \vec{x}\cdot\xi)} \left(\tau^2 + (|\xi|^2 + m^2)^3\right)^{-\alpha} d\tau d\xi \in L^1(\boldsymbol{R}^4),$$

where and throughout this paper if there is no indication of a *measure*, then $L^p(\boldsymbol{R}^d)$ $(p \geq 1)$ is understood as the L^p space on $\boldsymbol{R}^d$ with respect to the **Lebesgue** *measure* on $\boldsymbol{R}^d$.

For each a, $b > 0$ let $B^{a,b}$ be the linear subspace of $\mathcal{S}'(\boldsymbol{R}^2)$ defined by

$$B^{a,b} = \{(|x|^2 + 1)^{\frac{b}{4}} J^{-a} f : \ f \in L^2(\boldsymbol{R}^4)\}, \tag{2.1}$$

Then, $B^{a,b}$ is a separable Hilbert space with the scalar product

$$< u|v >= \int_{\boldsymbol{R}^4} J^a((|x|^2+1)^{-\frac{b}{4}} u(x))\, J^a((|x|^2+1)^{-\frac{b}{4}} v(x)) dx, \qquad u,\, v \in B^{a,b}. \tag{2.2}$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and consider an isonormal Gaussian process $W = \{W(h), h \in L^2_{real}(\boldsymbol{R}^4)\}$, where L^2_{real} is the real L^2 space. Hence, W is a centered Gaussian family of random variables on $(\Omega, \mathcal{F}, P)$ such that

$$E[W(h)W(g)] = \int_{\boldsymbol{R}^2} h(x)\, g(x) dx, \qquad h,\, g \in L^2_{real}(\boldsymbol{R}^4),$$

where E denotes the expectation with respect to the probability measure P.

Let $\eta_1 \in C_0^\infty(\boldsymbol{R}^4)$ be such that $\eta_1(x) = \eta_1(y)$ for $|x| = |y|$ and

$$0 \leq \eta_1(x) \leq 1, \qquad \eta_1(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2, \end{cases} \tag{2.3}$$

and let $\eta_k(x) = \eta_1(\frac{x}{k}) \in C_0^\infty(\boldsymbol{R}^4)$, $k = 1, 2, 3, \ldots.$ Also define $\rho \in C_0^\infty(\boldsymbol{R}^4)$ as follows:

$$\rho(x) = \begin{cases} C\exp(-\dfrac{1}{1-|x|^2}) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases},$$

where the constant C is taken to satisfy

$$\int_{\boldsymbol{R}^2} \rho(x)dx = 1. \tag{2.4}$$

Let

$$\rho_k(x) = k^4\rho(kx), \qquad k = 1,2,3,\dots.$$

For $\alpha > 0$ we define $J_k^\alpha \in \mathcal{S}(\boldsymbol{R}^4)$, $k = 1,2,3,\dots$ by

$$J_k^\alpha(x) = \int_{\boldsymbol{R}^2} J^\alpha(y)\rho_k(x-y)dy. \tag{2.5}$$

Also

$$F_k^\alpha(x;y_1,\dots,y_p) = (\eta_k(x))^p J_k^\alpha(x-y_1)\cdots J_k^\alpha(x-y_p), \tag{2.6}$$

and

$$F^\alpha(x;y_1,\dots,y_p) = J^\alpha(x-y_1)\cdots J^\alpha(x-y_p), \quad p = 1,2,3,\dots. \tag{2.7}$$

Then we see that the function F_k^α and F^α are symmetric in the last p variables $(y_1,\dots,y_p)$ and

$$F_k^\alpha \in \mathcal{S}((\boldsymbol{R}^4)^{p+1}), \qquad F_k^\alpha(x;y_1,\dots,y_p) = 0 \quad \text{for} \quad |x| \geq 2k. \tag{2.8}$$

For each $\alpha > 0$, $p \geq 1$ and $k \geq 1$ we define the random variable $:_k \phi_{\alpha,\omega}^p :$ as a multiple stochastic integral such that

$$:_k \phi_{\alpha,\omega}^p : (x) = \int_{(\boldsymbol{R}^4)^p} F_k^\alpha(x;y_1,\dots,y_p)dW_\omega(y_1)\cdots dW_\omega(y_p). \tag{2.9}$$

In particular for $p = 1$,

$$\phi_\omega(\cdot) \equiv \phi_{\frac{1}{2},\omega}(\cdot) \equiv \int_{(\boldsymbol{R}^4)} J^{\frac{1}{2}}(\cdot - y)dW_\omega(y) \tag{2.10}$$

is well defined as a $B^{a,b}$-valued random variable ($\forall a > 0$, $\forall b > 4$). Let μ be the probability law of $\phi_\omega = \phi_{\frac{1}{2},\omega}$. Precisely, μ is a Borel probability measure on $B^{a,b}$ such that

$$\mu(A) = P(\{\omega | \phi_\omega \in A\}), \quad A \in \mathcal{B}(B^{a,b}) \quad (a > 0,\ b > 2). \tag{2.11}$$

By an obvious modification of [AY1] we have the following of which proof is omitted here.

Theorem 2.1 *i) Let $a > 0$ and $b > 4$. For each $p \in \mathbf{N}$ and $k \in \mathbf{N}$ let $\tau_k = \tau_{(p),k}$ be the measurable map from $B^{a,b}$ to $B^{a,b}$ defined by*

$$\tau_k(\psi)(x) \quad = \quad p!(\eta_k(x))^p \sum_{n=0}^{[\frac{p}{2}]} \frac{(-\frac{1}{2}c_k)^n}{n!(p-2n)!}\Big(< J_k(x - \cdot), (J^{-\frac{1}{2}}\psi)(\cdot) >_{S,S'}\Big)^{p-2n}, \quad \text{for} \quad \psi \in B^{a,b}, \tag{2.12}$$

where

$$c_k = \int_{\boldsymbol{R}^4} (J_k^{\frac{1}{2}}(y))^2 dy.$$

Then

$$P\left(\{\omega \,|\, \tau_k(\phi_\omega)(x) =:_k \phi_\omega^p : (x) \quad \forall x \in \boldsymbol{R}^4\}\right) = 1, \tag{2.13}$$

the $B^{a,b}$-valued measurable functions $\{\tau_k(\psi)\}$ on $(B^{a,b}, \mathcal{B}^\mu, \mu)$ form a Cauchy sequence in the Banach space $L^2(B^{a,b} \to B^{a,b}; \mu)$, and there exists a $\mathcal{B}(B^{a,b})/\mathcal{B}^\mu$-measurable function $\tau = \tau_{(p)} \in L^2(B^{a,b} \to B^{a,b}; \mu)$ such that

$$\lim_{k\to\infty} \int_{B^{a,b}} \|\tau_k(\psi) - \tau(\psi)\|^2_{B^{a,b}}\mu(d\psi) = 0. \tag{2.14}$$

Moreover one has

$$\tau(\phi_\omega) =: \phi_{\frac{1}{2},\omega}^p : \qquad P-a.s. \quad \omega \in \Omega, \tag{2.15}$$

where $: \phi_{\frac{1}{2},\omega}^p :$ is the p-th Wick power of the $B^{a,b}$-valued random variable ϕ_ω. ∎

In this section we are setting $x \equiv (t, \vec{x})$, $\xi \equiv (\tau, \vec{\xi}) \in \mathbf{R} \times \mathbf{R}^3$ and defining the Fourier and Fourier inverse transform as follows:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbf{R}^4} e^{-ix\xi} f(x) dx, \qquad \mathcal{F}^{-1}[\hat{f}](x) = \int_{\mathbf{R}^4} e^{ix\xi} \hat{f}(\xi) \bar{d}\xi$$

for $f \in \mathcal{S}(\mathbf{R}^4 \to \mathbf{C})$, where $\bar{d}\xi = (2\pi)^{-4} d\xi$. Now, for given fixed $m > 0$ let $H^\gamma = H^\gamma(\boldsymbol{R}^4)$ be the Hilbert space on $\boldsymbol{R}^4$ such that

$$H^\gamma(\boldsymbol{R}^4) = \left\{ \phi \in \mathcal{S}'(\boldsymbol{R}^4) \mid \int_{\boldsymbol{R}^4} |\mathcal{F}\phi|^2(t, \vec{x}) \left(t^2 + (|\vec{x}|^2 + m^2)^3\right)^\gamma dt d\vec{x} < \infty \right\}.$$

The inner product of $H^\gamma(\boldsymbol{R}^4)$ is given by

$$< u, v >_{H^\gamma} = (2\pi)^{-4} \int_{\boldsymbol{R}^4} (\mathcal{F}u)(t, \vec{x}) \, (\mathcal{F}v)(t, \vec{x}) \left(t^2 + (|\vec{x}|^2 + m^2)^3\right)^\gamma dt d\vec{x}.$$

Then, from the definition (2.11) of the probability measure μ, we see that

$$\begin{aligned} &\int_{B^{a,b}} e^{\sqrt{-1} < \psi, \varphi >_{\mathcal{S}', \mathcal{S}}} \mu(d\psi) \\ &= \int_\Omega \exp\left[\sqrt{-1} \int_{\boldsymbol{R}^4} \left(\int_{\boldsymbol{R}^4} \varphi(x) J^{\frac{1}{2}}(x - y) dx \right) dW_\omega(y)\right] P(d\omega) \\ &= \exp\left(-\frac{1}{2} \|\varphi\|^2_{H^{-1}}\right) = \exp\left(-\frac{1}{2} \|J^1 \varphi\|^2_{H^1}\right). \end{aligned} \tag{2.16}$$

The inclusion map $i : H^{-1} \to B^{a,b}$ defined by

$$i(h) = J^1 h, \qquad h \in H^{-1} \tag{2.17}$$

is continuous and $i(H^{-1}) = H^1$ is dense in $B^{a,b}$. By this we can identify H^{-1} with H^1, and we have the following continuous injection:

$$(B^{a,b})^* \hookrightarrow H^{-1} \cong H^1 \hookrightarrow B^{a,b}.$$

Setting

$$\mathcal{H} = H^{-1},$$

we thus have the abstract Wiener space $(B^{a,b}, i(\mathcal{H}), \mu)$ with the Cameron-Martin space

$$i(\mathcal{H}) = J^1 H^{-1} = H^1. \tag{2.18}$$

In the sequel, without giving the definitions, we will use the terminologies and notations on abstract Wiener spaces (cf., eg., [UZ], [Nu], [AY1]). The following theorem is also an obvious modification of [AY1].

Theorem 2.2 (polynomial $H - C^1$ maps) *For $a > 0$, $b > 4$, let $(B^{a,b}, i(\mathcal{H}), \mu)$ be the abstract Wiener space defined above, and denote the "Gross-Sobolev derivative" and "divergence" operators on $(B^{a,b}, i(\mathcal{H}), \mu)$ by ∇ and δ, respectively ([UZ]). For $M \geq 0$ let η_M be the space-cut-off such that $\eta_M(x) = \eta_1(\frac{x}{M})$. Then the map $u_p(\psi) = \eta_M \tau_{(p)}(\psi)$ ($\mathcal{H}$-valued Wiener functional) is an element of $\boldsymbol{D}_{2,k}(\mathcal{H})$ ($\forall k \geq 1$), and the following holds:*

$$\begin{aligned} \nabla u_p(\psi)(x, y) &= p \left\langle \eta_M, \tau_{(p-1)}(\psi)(\cdot) J^0(\cdot - x) J^0(\cdot - y) \right\rangle_{\mathcal{S}, \mathcal{S}'} \\ &\in L^2(\mathcal{H} \otimes \mathcal{H}; \mu). \end{aligned}$$

The divergence of u_p is given by

$$\delta u_p(\psi) = < \eta_M, \tau_{(p+1)}(\psi) >_{\mathcal{S}, \mathcal{S}'} \qquad \mu - a.s. \quad \psi \in B^{a,b}. \tag{2.19}$$

Let $B(p)$ be such that $\mu(B(p)) = 1$ and $B(p) + H^1 \subset \overline{B}(p)$, then

$$\nabla u_p(\psi + i(h))(x,y) = p\sum_{q=0}^{p-1} \binom{p-1}{q} \Big\langle \eta_M, (J^0(i(h)))^q \tau_{(p-1-q)}(\psi)(\cdot) \times J^0(\cdot - x)J^0(\cdot - y)\Big\rangle_{S,S'}, \quad \forall\psi \in B(p), \forall h \in \mathcal{H}. \tag{2.20}$$

u_p is an $H - C^1$ map on $(B^{a,b}, i(\mathcal{H}), \mu)$:

$$\mathcal{H} \ni h \longmapsto \nabla u_p(\psi + i(h)) \in \mathcal{H} \otimes \mathcal{H} \quad \textit{is continuous for all } \psi \in B(p), \tag{2.21}$$

where $J^0(x) = \delta_{\{0\}}(x)$ (with $\delta_{\{0\}}$ the Dirac point measure at $\{0\}$). ■

Definition 1 *For $u \in D_{2,1}(\mathcal{H})$ and $\lambda \in \boldsymbol{R}$ we define*

$$\Lambda_{\lambda u}(\psi) = \det_2(I_{\mathcal{H}} + \lambda\nabla u(\psi)) \exp(-\lambda\delta u(\psi) - \frac{\lambda^2}{2}|u(\psi)|^2_{\mathcal{H}}), \tag{2.22}$$

where $\det_2(I_{\mathcal{H}}+\lambda\nabla u(\psi))$ denotes the Carleman-Fredholm determinant of the Hilbert-Schmidt operator $\lambda\nabla u(\psi) \in \mathcal{H}\otimes\mathcal{H}$ and $|\ \ |_{\mathcal{H}}$ denotes the norm of the Hilbert space $\mathcal{H}$. ■

Thus, in the present framework of the abstract Wiener space, equation (1.1) can be rewriten as

$$\Big(-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3\Big)\psi(x) + \lambda\eta_M(x)\tau_{(3)}(\psi)(x) = \Big(-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3\Big)^{\frac{1}{2}} \dot{W}(x), \tag{2.23}$$

$$x \equiv (t, \vec{x}) \in \boldsymbol{R} \times \boldsymbol{R}^3.$$

In the abstract Wiener space framework, by using change of variable formulas we can specify a solution of (2.23) in the following manner. To discuss the problem generally, we let S be a topological space and $\mathcal{B}(S)$ be its Borel σ-field. Let μ be a complete probability measure on $(S, \overline{\mathcal{B}(S)}^{\mu})$, and suppose that T is a measurable map such that $T : (S, \overline{\mathcal{B}(S)}^{\mu}) \longmapsto (S, \mathcal{B}(S))$, where

$$\overline{\mathcal{B}(S)}^{\mu} = \text{"the completion of } \mathcal{B}(S) \text{ with respect to } \mu\text{"}.$$

A *signed* measure ν on $(S, \overline{\mathcal{B}(S)}^{\mu})$ will be called as a "*Girsanov measure on $(S, \overline{\mathcal{B}(S)}^{\mu})$ associated with μ and T*" if and only if it satisfies

$$\int_S f(T\psi)d\nu(\psi) = \int_S f(\phi)d\mu(\phi) \tag{2.24}$$

$$\text{for any bounded measurable } f : (S, \mathcal{B}(S)) \longmapsto (\boldsymbol{R}, \mathcal{B}(\boldsymbol{R})).$$

In particular if such a signed measure ν is a *probability* measure on $(S, \overline{\mathcal{B}(S)}^{\mu})$, then this will be called the "*Girsanov probability measure on $(S, \overline{\mathcal{B}(S)}^{\mu})$ associated with μ and T*". The key idea of the interpretation of (2.24) to the SPDE's discussed here is the following:

If a "*Girsanov probability measure ν on $(S, \overline{\mathcal{B}(S)}^{\mu})$ associated with μ and T* " exists, then by (2.24) the probability law of $T\phi$ under ν is μ. In other words, for a random variable ψ taking values in S with probability law ν there exists a random variable ϕ with probability law μ, and the relation

$$T\psi = \phi$$

holds.

We apply this relation to our actual problem. Let μ be the probability law of $\mathcal{S}'(\mathbf{R}^4)$ valued random variable ϕ_ω defined by (2.10), then μ is a complete probability measure on $(B^{a,b}, \mathcal{B}^{\mu})$. Let T be the map defined on $B^{a,b}$ such that

$$T(\psi) = \psi + J^1(\lambda\eta_M\tau_{(3)}(\psi)) \qquad \psi \in B^{a,b}.$$

We may set $S = B^{a,b}$ and $\mathcal{B}(S) = \mathcal{B}(B^{a,b})$ in the above general discussion. Hence, if there exists ν which is a " *Girsanov probability measure on* $(B^{a,b}, \mathcal{B}^{\mu})$ *associated with* μ *and* T", then for a $B^{a,b}$-valued random variable ψ with the probability law ν, there corresponds an $\mathcal{S}'(\mathbf{R}^4)$ valued random variable ϕ on $(B^{a,b}, \mathcal{B}^{\mu}, \nu)$ of which probability law is identical with μ such that

$$T\psi = \phi,$$

or, explicitly

$$\psi + J^1(\lambda \eta_M \tau_{(3)}(\psi)) = \phi,$$

and equivalently

$$\psi + \Big(-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3\Big)^{-1}(\lambda \eta_M \tau_{(3)}(\psi)) = \phi.$$

Since the probability law of ϕ is μ, it can be expressed by $\phi = J^{\frac{1}{2}}\dot{W}$ for some isonormal Gaussian process W on $\boldsymbol{R}^4$. Then, by operating $-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3$ to both sides of the last equation, we see that this is equivalent to (2.23):

$$\Big(-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3\Big)\psi(x) + \lambda \eta_M(x)\tau_{(3)}(\psi)(x) = \Big(-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3\Big)^{\frac{1}{2}}\dot{W}(x), \qquad (2.25)$$

By this way we can reduce the existence problem of the solution of the SPDE (2.25) to the existence problem of the corresponding *Girsanov probability measure* ν satisfying (2.24). Thus, in the present framework to get a solution of (1.1), it suffices to show that the existence of a measure ν which is a " *Girsanov probability measure on* $(B^{a,b}, \mathcal{B}^{\mu})$ *associated with* μ *and* T". The following Lemmas 2.3 and 2.4 guarantee the existence of such ν. Proofs of these Lemmas are very similar to the corresponding results given in [AY1] and are omitted here. In short, Lemma 2.3 can be proven through the same manner as the proof of the *Key Lemma* in [AY1], namely by making use of the fact that δu and ∇u are the 4-th and 2nd Wick power of ψ respectively, this can be shown by applying Nelson's exponential bounds.

Lemma 2.3 (Key lemma for cubic power perturbation) *Take* $\lambda > 0$ *and* $\epsilon > 0$ *to satisfy* $\lambda(1+\epsilon) < \frac{2}{9L}$, *where* $L = \int_{\boldsymbol{R}^2}(J^1(x))^2 dx$. *Then for*

$$u(\psi) = u_3(\psi) = \eta_M \tau_{(3)}(\psi),$$

the following holds

$$\exp\{-\lambda\delta u + \frac{1+\epsilon}{2}\lambda^2\|\nabla u\|_2^2\} \in \cap_{q<\infty} L^q(\mu), \qquad (2.26)$$

where $\| \,\|_2$ *denotes the Hilbert-Schmidt norm* $\| \,\|_{\mathcal{H}\otimes\mathcal{H}}$. ■

Define

$$\Lambda_{\lambda u}(\psi) \equiv \left|\det{}_2(I_{H^{-1}} + 3\lambda\eta_M(x) : \psi^2(x) : \delta_{\{x\}}(y))\right| \qquad (2.27)$$
$$\times \exp\Big\{-\lambda\int_{\boldsymbol{R}^4}\eta_M(x) : \psi^4(x) : dx - \frac{\lambda^2}{2}\int_{\boldsymbol{R}^2}(J^{\frac{1}{2}}(\eta_M : \psi^3 :)(x))^4 dx\Big\}.$$

Lemma 2.4 *Let* $a > 0$ *and* $b > 4$. *Under the assumption of Lemma 2.3, the following holds:*

$$\Lambda_{\lambda u_3} \in \cap_{q<\infty} L^q(\mu), \qquad E^{\mu}[\Lambda_{\lambda u_3}] = 1. \qquad (2.28)$$

Let

$$D = \{y \in B^{a,b} \,|\, \det{}_2(I_{\mathcal{H}} + \lambda\nabla u_3(y)) \neq 0\},$$

and let $N(\psi, D)$ *denote the cardinality of the set* $T^{-1}\{\psi\}\cap D$ *for* $T(\psi) = \psi + i(\lambda u_3(\psi))$, *then* $N(\psi, D)$ *is a measurable function and the following holds:*

$$\mu(\{\psi \,|\, 1 \le N(\psi, D) < \infty\}) = 1. \qquad (2.29)$$

■

Finally, from the above Lemmas we have the following main result of which proof is also very similar (almost only by changing the notations) to the main Theorem in [AY1]. We omit the proof also.

Theorem 2.5 (Solution for the space-cut-off cubic perturbation) *Take $\lambda \geq 0$ to satisfy $\lambda < \frac{2}{9L}$ for $L < \infty$ given in Lemma 2.3. For any fixed positive number M let $\eta_M(x) = \eta_1(\frac{x}{M})$, and define*

$$T_3(\psi) = \psi + i(\lambda u_3(\psi)), \qquad u_3(\psi) = \eta_M \tau_{(\beta,3)}(\psi) \tag{2.30}$$

and

$$d\nu_3 = q \circ T_3 |\Lambda_{\lambda u_3}| d\mu \qquad \text{for } q \text{ such that}$$

$$q(\psi) = \begin{cases} \dfrac{1}{N(\psi, D)} & \text{if } N(\psi, D) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

Then $\Lambda_{\lambda u_3}\mu$ is a (signed) Girsanov measure and ν_3 is a Girsanov probability measure on $(B^{a,b}, \mathcal{B}^\mu)$ associated with μ and T_3 :
i)

$$E^\mu[f \circ T_3 \Lambda_{\lambda u_3}] = E^\mu[f], \quad E^\nu[f \circ T_3] = E^\mu[f] \; \forall f \in C_b(B^{a,b}). \tag{2.31}$$

ii) ν_3 gives a solution of (2.32) below in the following sense. If ψ is a $B^{a,b}$-valued random variable with probability law ν_3, then the following holds for some isonormal Gaussian process W on $\boldsymbol{R}^4$:

$$\Big(-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3\Big)\psi(x) + \lambda\eta_M(x) : \psi^3(x) := \Big(-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3\Big)^{\frac{1}{2}} \dot{W}(x), \tag{2.32}$$

$$x \equiv (t, \vec{x}) \in \boldsymbol{R} \times \boldsymbol{R}^3,$$

■

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