数理解析研究所講究録
1485 巻 2006 年 22-32

Dynamics of type 3 polynomial self maps of degree 2 of $\mathbf{C}^2$

東海大学・理学部 内村桂輔 (Keisuke Uchimura)
Department of Mathematics
Tokai University

Abstract

We study the dynamics of endomorphisms $f_c(x,y) = (x^2+cy, y^2+cx)$ of $\mathbf{C}^2$. The author studied the dynamics of non-holomorphic maps $g_c(z) = z^2 + c\bar{z}$. The endomorphism $f_c(x,y)$ is a kind of extension of $g_c(z)$.

Let Γ_c and $K(f_c)$ be the critical set and the set of points with bounded orbits of $f_c(x,y)$. We show the followings :

$$\text{(A)} \quad \text{If } c > 2+2\sqrt{2} \text{ or } c < -2 \quad \text{then } K(f_c) \cap \Gamma_c = \phi,$$

and $K(f_c)$ is a Cantor set which lies on the real plane $(x = \bar{y})$.

$$\text{If} \quad -2 \leq c \leq 4 \quad \text{then} \quad K(f_c) \cap \Gamma_c \neq \phi.$$

(B) If c is a complex number near 2, then $K(f_c) \cap \Gamma_c$ consists of three points and the forward orbits of three points approach to three saddles. All periodic points other than these three saddles are repelling. These give affirmative answers to Hubbard's conjectures.

(C) When $c = -2$, the dynamics of $f_{-2}(x,y)$ are analoguous to those of a typical quadratic polynomial map (Chebyshev map) $p_{-2}(z) = z^2 - 2$. The invariant measure μ of $f_{-2}(x,y)$ can be written exactly as it can be written for $p_{-2}(z)$. External rays and foliations of $f_{-2}(x,y)$ defined by Bedford and Jonsson have the similar properties as the typical external rays of $p_{-2}(z)$.

1 Introduction

Uchimura [Uc1, Uc2, Uc3] studied the dynamics of the maps

$$g_c(z) = z^2 + c\bar{z} : \mathbf{C} \to \mathbf{C}.$$

The dynamics of the maps are analogous to those of quadratic maps

$$g_c(z) = z^2 + cz.$$

Uchimura [Uc3] shows the following:
(1) $K(g_c)$ is connected with the simply connected complement in $\hat{\mathbf{C}}$ if and only if $-2 \le c \le 4$.
(2) If $c < -2$, $K(g_c)$ is a Cantor set. g_c restricted to $K(g_c)$ is topological conjugate to the shift on 4 symbols.

Ueda[Ue] shows that any holomorphic map on $\mathbf{P}^2$ of degre 2 is equivalent to one of the following maps:
(1) $(x : y : z) \longmapsto (x^2 : y^2 : z^2)$
(2) $(x : y : z) \longmapsto (x^2 + yz : y^2 : z^2)$
(3) $(x : y : z) \longmapsto (x^2 + cyz : y^2 + cxz : z^2)$
(4) $(x : y : z) \longmapsto (x^2 + cxy + y^2 : z^2 + xy : yz)$.

f is equivalent to $g \iff$ There exist linear maps L_1 and L_2 such that $L_2^{-1} \circ f \circ L_1 = g$.

The type (3) on $\mathbf{C}^2$ is written as

$$f_c(x, y) = (x^2 + cy, y^2 + cx).$$

The function f_c restricted on the plane $\{x = \bar{y}\}$ is

$$g(z) = z^2 + c\bar{z} \quad \text{when} \quad c \in \mathbf{R}.$$

The map

$$f_c(x, y) = (x^2 + cy, y^2 + cx).$$

admits an invariant line $\{x = y\}$ on which it acts as the quadratic polynomial

$$f_c(z) = z^2 + cz.$$

Forness and Sibony [FS] defined analogy with Mandelbrot set

$$M := \{c \in \mathbf{C} : K(f_c) \cap Crit(f_c) \neq \phi\}.$$

In our case,

the critical set $Crit(f_c)$ is written as $xy = \dfrac{c^2}{4}$.

We parametrize the critical set as follows :

$$x = -\frac{c}{2}t, \quad y = -\frac{c}{2t}, \quad t \in \mathbf{C} - \{0\}.$$

We have

$$\text{if} \quad \max\{|x|, |y|\} > |c| + 1, \quad \text{then} \quad f^n(x, y) \to \infty.$$

$Re\ c < 1$	$1 < Re\ c$
$\mathbf{C}^2$ dynamics?	$\mathbf{C}^2$dynamics?
There exist attractive periodic points of period $k \leq 2$	Hyperbolic periodic points of period $k \leq 2$ are saddles or repelling.
c is near 0 $K(f_c) \cap Crit(f_c) \supset$ a topological disk centered at $t = 1$ $J_2 \simeq S^1 \times S^1$ (Denker and Heinemann [DH])	c is near 2 $K(f_c) \cap Crit(f_c)$ $= 3$ points J_2 is real 1-dimensional and connected.

2 Critical sets and K sets.

We show the properties of the sets $K(f_c)$ of points with bounded orbits and ctitical set $Crit(f_c)$

Theorem 2.1 If $c > 2 + 2\sqrt{2}$ or $c < -2$, then
(1) $K(f_c) \cap Crit(f_c) = \phi$,
(2) $K(f_c) = K(g_c) = \textit{support}\ \mu \subset \{x = \bar{y}\}$,
(3) $K(f_c)$ is a Cantor set.

Proposition 2.2 If $-2 \leq c \leq 4$, then

$$K(f_c) \cap Crit(f_c) \neq \phi.$$

Sketch of the proof of Theorem 2.1 (1)
We show

$$f_c^n(Crit(f_c)) \to \infty \quad (n \to \infty).$$

Indeed. Let $(U_n(t), V_n(t)) := f_c^n(-\frac{c}{2}t, -\frac{c}{2t})$.

Then we have the followings :
(1) $U_n(t)$ is holomorphic and has no zero in $\mathbf{D} - \{0\}$, and so $\mid U_n(t) \mid$ has a minimum value on the unit circle.
(2) By [Uc3], we have

$$\text{if} \quad c < -2 \quad \text{then} \quad U_n(e^{i\theta}) \to \infty. \qquad \Box$$

Sketch of the proof of Theorem 2.1 (2) and (3)

Fornaess and Sibony [FS] show that

$$\text{if} \quad K(f) \cap Crit(f) = \phi, \quad \text{then}$$

repelling periodic points are dense in $K(f)$ and $K(f) = supp\ \mu$.
We show that $\sharp$\{periodic points of period n of g_c\} $= 4^n$.

By [Uc3], it follows that $K(g_c)$ is a Cantor set. Combining these results, Theorem 2.1 (2) and (3) follow. □

For the details of the proofs, see [Uc4].

3 Hubbard's conjectures

We consider J. Hubbard's conjectures concerning the dynamics of $f_c(x, y)$;
(1) In the case $c = 2$, the saddle $(-1, -1)$ is isolated.
(2) Let $U := \{c \in \mathbf{C} :$ the set $K(f_c) \cap Crit(f_c)$ consists of three points.$\}$. Then, U is a nonempty set containing 2 in its interior.

We give affirmative answers to these conjectures. Let

$$D(z_0, r) = \{z \in \mathbf{C} : \mid z_0 - z \mid < r\}.$$

Theorem 3.1 There exists a small positive number δ such that for any $c \in D(2, \delta)$, a saddle $(-c+1, -c+1)$ is isolated in the non-wondering set of f_c.

Theorem 3.2 There exists a small positive number δ such that if

$$c = 2 + \epsilon \in D(2, \delta), \quad \text{then}$$

$$K(f_c) \cap Crit(f_c) = \{(-1 - \frac{\epsilon}{2}, -1 - \frac{\epsilon}{2}), ((-1 - \frac{\epsilon}{2})\omega, (-1 - \frac{\epsilon}{2})\omega^2),$$
$$((-1 - \frac{\epsilon}{2})\omega^2, (-1 - \frac{\epsilon}{2})\omega)\},$$

where ω is a cubic root of unity.

From these theorems, we have the following theorem.

Theorem 3.3 There exists a small positive number δ such that if

$$c \in D(2, \delta), \quad \text{then any periodic point other than}$$

$$(-c+1, -c+1), (\omega(-c+1), \omega^2(-c+1)) \quad \text{and} \quad (\omega^2(-c+1), \omega(-c+1))$$

is repelling .

We consider the typical case $c = 2$. Theorem 3.2 says that points.

$$K(f_2) \cap Crit(f_2) \quad \text{is equal to the set of 3 points.}$$

Since $f_2(x, y)$ is symmetric with a rotation (ω, ω^2), the intersection is essentially one point.

Sketch of the proof of Theorem 3.1
Let

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x+1+\epsilon \\ y+1+\epsilon \end{pmatrix}.$$

Then we have the followings. Let (ξ, η) be an element in $\overline{B((0,0), 0.1)}$.

$$\text{If} \quad \eta \neq 0, \quad \text{then} \quad f^n(x,y) \to \infty.$$

$$\text{If} \quad \eta = 0, \quad \text{then} \quad f^n(x,y) \to (-c+1, -c+1). \qquad \Box$$

Sketch of the proof of Theorem 3.2
We show that only the orbit of the point at $t = 1$ is bounded.
case 1 $\quad t \in B(1, 0.068)$.

$$\text{We have} \quad (-\frac{c}{2}t, -\frac{c}{2t}) \in B((-1,-1), 0.1).$$

For any point p in the set

$$\{(-\frac{c}{2}t, -\frac{c}{2t}) \in B((-c+1, -c+1) : t \neq 1\},$$

$$\text{we have} \qquad f^n(p) \to \infty \quad (n \to \infty).$$

case 2 $\qquad t$ is in the set

$$\{t \in \mathbf{C} : \frac{1}{4} \leq |\, t \,| \leq 1, \quad 0 \leq \arg t \leq \frac{2}{3}\pi\} - D(1, 0.068) - D(\omega, 0.068).$$

We have

$$f^n(-\frac{c}{2}t, -\frac{c}{2t}) \to \infty.$$

Indeed. Let

$$(U_3(t), V_3(t)) := f^3(-\frac{c}{2}t, -\frac{c}{2t}).$$

Then

$$|\, U_3(t) \,|^2 + |\, V_3(t) \,|^2 \geq 19 \geq 2(|\, c \,| + 1)^2. \qquad \Box$$

Sketch of the proof of Theorem 3.3
We consider the post critical set

$$P(f_c) = \overline{\bigcup_{n=1}^{\infty} f_c^n(Crit(f_c))}.$$

If $c \in D(2, J)$ then we have the followings;

(1) $P(f_c) \cap f_c^{-2}(G) = \phi$, where $G := D(0, |\, c \,| + 1) \times D(0, |\, c \,| + 1) \setminus W$.

Here W is a certain closed set.

(2) $f_c^{-2}(G) \subset\subset G$,

(3) any component of $f^{-2}(G)$ is Kobayashi hyperbolic.

Combining (1), (2) and (3), we have that any periodic point other than 3 saddles are repelling. □
For the details of the proofs, see [Uc4].

4 External rays and foliations of $f_{-2}(x, y)$

The map $f_{-2}(x, y)$ admits an invariant plane $\{x = \bar{y}\}$ on which it acts as $g(z) = z^2 - 2\bar{z}$. $z^2 - 2\bar{z}$ is a Chebyshev polynomial in two variables. (See [K]). Uchimura [Uc1] studied the dynamics of $g(z)$. It shows that $K(g)$ is the closed domain S bounded by Steiner' hypocycloid, and g is invariant and chaotic on a Sierpinsky gasket in $K(g)$. Ueda [Ue] showed the followings for

$$f_{-2}(x, y) = (x^2 - 2y, y^2 - 2x).$$

(1) $\Psi(u, v) := (u + v + \frac{1}{uv}, \frac{1}{u} + \frac{1}{v} + uv) = (x, y)$

$$\tilde{f}(u, v) := (u^2, v^2).$$

Then $f_{-2} \circ \Psi = \Psi \circ \tilde{f}$.

(2) $k(f_{-2}) = S \quad \subset \{x = \bar{y}\}$.

Koornwinder [K] showed that there exists a diffeomorphism φ from int(S) to $int(T^2/S_3)$, where T^2 is a torus and S_3 is the symmetric group. Let (ϕ, θ) be a point in T^2. The symmetric group S_3 is represented as follows.

$$S_3 : \tau_0(\phi, \theta) = (\phi, \theta), \qquad \tau_1(\phi, \theta) = (\phi, \phi - \theta), \qquad \tau_2(\phi, \theta) = (-\phi + \theta, \theta),$$

$$\tau_3(\phi,\theta) = (-\phi,-\theta), \qquad \tau_4(\phi,\theta) = (-\theta,\phi-\theta), \qquad \tau_5(\phi,\theta) = (\theta-\phi,-\phi).$$

Bedford and Jonsson [BJ] studied external rays of regular polynomial endomorphisms of $\mathbf{C}^k$. We use their definitions and results, in our setting.

$$f_{-2}(x,y) = (x^2-2y, y^2-2x).$$

$$F(x:y:z) = (x^2-2yz : y^2-2xz : z^2)$$

$$\Pi := \mathbf{P}^2 - \mathbf{C}^2.\text{(the line at infinity)}$$

$$F\mid_{\Pi}: (x:y:0) \to (x^2:y^2:0)$$

$$J_\Pi = \{(x:y:z) :\mid x\mid=\mid y\mid\}$$

The stable set of J_Π for f is defined by

$$W^s(J_\Pi, f) := \{x \in \mathbf{P}^2 : d(f^n x, J_\Pi) \to 0\}.$$

Bedford and Jonsson show the followings.

(1) There exists a Boettcher coordinate Φ such that

$$\Phi : W^s(J_\Pi, f_{-2}) \to W^s(J_\Pi, f_h)$$

conjugating f_{-2} to f_h, where

$$f_h(x,y) = (x^2,y^2).$$

(2) $W^s(J_\Pi, f_{-2})$ is foliated by stable disks W_a.

(3) The restriction Φ_a of Φ to the stable disk W_a is biholomorphism onto a unit disk $\mathbf{D}_a$ for all $a \in J_\Pi$.

External ray $R(a,\theta)$ is defined by

$$R(a,\theta) = \{\Phi_a^{-1}(re^{i\theta}) : r > 1\}.$$

(4) There exists an endpoint map e

$$e : \bigcup_{a\in J_\Pi} W_a \to J_2 \quad \text{such that} \quad e_*\nu = \mu.$$

Nakane [N] observed that the map Ψ defined by Ueda is essentially the inverse of Böttcher coordinate Φ. On W_a,

$$\Phi(u,v) = \Phi(t, at), \quad \mid t\mid > 1.$$

He shows the followings :

(1) The stable disk W_a is written as

$$x = re^{-2\pi i\theta} + \frac{1}{r}e^{-2\pi i(\theta-\phi)} + e^{2\pi i\phi},$$

$$y = re^{2\pi i(\phi-\theta)} + \frac{1}{r}e^{2\pi i\theta} + e^{-2\pi i\phi} \quad (r > 1),$$

where $a = e^{2\pi i\phi}$. We denote the above point by $R(r, \phi, \theta)$. Then,

$$J_2 = S \subset \{x = \bar{y}\}.$$

(2) Each point $z \in J_2$ is the landing point of exactly 1, 3, or 6 external rays, if z is a cusp point on ∂S, z is a non-cusp point on ∂S or $z \in intS$, respectively.

Using these results, we study the structure of foliations. The stable set $W^s(J_\Pi, f_{-2})$ is foliated by stable disks W_a. We now show the structure of foliations. Metaphorically speaking, the structure is described as "three mouths eat a sandwich". Since the stable disk W_a is a topological disk, we may consider W_a as a mouth. The J_2 set may be considered as a sandwich.

Theorem 4.1 *For any point z in $int(S)$, there exist three stable disks W_a such that boundaries of three disks intersect at z. At the point, two external rays on each W_a land from opposite directions.*

Proof.

Since external rays land on S, ∂W_a lies on S. The intersection ∂W_a and S is a geodesic γ. That is, by a diffeomorphism φ from $int(S)$ to in (T^2/S_3), $\varphi(\gamma)$ becomes a segment in T^2. By considering reflections on T^2, we see that ∂W_a is a 2-fold covering on γ. For each point z of ∂W_a, two external rays $R(r, \phi, \theta)$ and $R(r, \tau_1(\phi, \theta))$ in W_a ($a = e^{2\pi i\phi}$) land at z. We consider two point $R(r, \phi, \theta)$ and $R(r, \phi, \phi - \theta)$. We see that these two points are "symmetric" about the real plane $\{x = \bar{y}\}$. Indeed.

(1) The midpoint of $R(r, \phi, \theta)$ and $R(r, \phi, \phi - \theta)$ lies on the plane $\{x = \bar{y}\}$.

(2) The segment connecting the two points is perpendicular to $\{x = \bar{y}\}$.

This shows a relationship between two external rays $R(\phi, \theta)$ and $R(\tau_1(\phi, \theta))$ in W_a. Also for each of two couples $(R(\tau_3(\phi, \theta)), R(\tau_4(\phi, \theta)))$ and $(R(\tau_2(\phi, \theta)), R(\tau_5(\phi, \theta)))$, we have the same relation. □

We compare external rays of $f_{-2}(x, y)$ with those of $P_{-2}(z) = z^2 - 2$. The map $P_{-2}(z)$ is a typical quadratic polynomial map and is called as a Chebyshev map.

External rays of $P_{-2}(z)$ are written as

$$R(r, \phi) : u = re^{2\pi i\phi} + \frac{1}{r}e^{2\pi i(-\phi)}$$

$$R(r, -\phi) : v = re^{2\pi i(-\phi)} + \frac{1}{r}e^{2\pi i\phi}, \quad r > 1.$$

Since $v = \bar{u}$, $R(r, \phi)$ and $R(r, -\phi)$ are "symmetrical" about the real axis.

Hence we can say that external rays of $f_{-2}(x, y)$ have the similar properties as those of $P_{-2}(z)$. The symmetric groups S_2 acts on external rays of $P_{-2}(z)$. On the other hand, the symmetric group S_3 acts on external rays of $f_{-2}(x, y)$.

On the plane $\{x = \bar{y}\}$, we have the following result.

Corollary 4.2. *$(W_a \cap \{x = \bar{y}\}) \cup \partial W_a$ is a line on the real plane $\{x = \bar{y}\}$. $W_a \cap \{x = \bar{y}\}$ lies in the exterior of the hypocycloid $\partial S$and ∂W_a lies in the interior and the boundary.*

Lastly, we show the exact form of the maximal entropy measure μ

$$\mu := (\frac{1}{2\pi} dd^c G) \cap (\frac{1}{2\pi} dd^c G).$$

We know that supp μ is equal to the closed domains S bounded by hypocycloid in $\{x = \bar{y}\}$.

Theorem 4.3 *On the plane $\{x = \bar{y}\}$, the maximal entropy measure μ is written as*

$$\mu = (\frac{2}{\pi})^2 \frac{dx_1 dx_2}{\sqrt{-x^2\bar{x}^2 + 4x^3 + 4\bar{x}^3 - 18x\bar{x} + 27}} \qquad (x = x_1 + ix_2, \quad x_1, x_2 \in \mathbf{R}).$$

This is an extension of the invariant measure

$$\mu = \frac{1}{\pi} \frac{dx}{\sqrt{(x+1)(3-x)}}$$

$$\text{for} \qquad q_{-2}(x) = x^2 - 2x \quad \text{on} \quad [-1, 3].$$

Sketch of the proof.
Briend and Duval [BD] shows that the repelling periodic points are equidistributed in supp μ i.e.

$$\mu_n := \frac{1}{d^{nk}} \sum \delta_y \quad \text{where} \quad f^n(y) = y, \quad \text{and} \quad y \quad \text{is repelling,}$$

converges weakly to μ.

We can show the following properties:

(1) Any periodic point of $f_{-2}(x, y)$ lies on the domain S and is repelling.

(2) We consider periodic points in S.

There is a diffeomorphism φ from $\mathrm{int}(S)$ to $\mathrm{int}(T^2/S_3)$. We transform the space (T^2/S_3) by the linear transformation T :

$$\begin{pmatrix} s \\ t \end{pmatrix} = T\begin{pmatrix} \phi \\ \theta \end{pmatrix}, \quad s = (\phi+\theta)/2, \quad t = \sqrt{3}(\phi-\theta)/2, \quad (\phi,\theta) \in T^2/S_3.$$

The image of (T^2/S_3) under T is a domain bounded by a equilateral triangle. We can show that the periodic points are equidistributed in the triangle on the (s,t) plane.

By [BJ] , we deduce that the invariant measure on the triangle in the (s,t) plane is Lebesgue measure. Pullback of Labesgue measure under $\varphi \circ T$ yields

$$\mu = \left(\frac{2}{\pi}\right)^2 \frac{dx_1 dx_2}{\sqrt{-x^2\bar{x}^2 + 4x^3 + 4\bar{x}^3 - 18x\bar{x} + 27}} \qquad \square.$$

References

[BJ] E. Bedford and M. Jonsson, *Dynamics of regular polynomial endomorphisms of* $\mathbf{C}^k$, Amer. J. Math. 122 (2000), 153-212.

[BD] J. -Y. Briend and J. Duval, *Exposants de Liapounoff et distribution des points periodiques d'un endomorphisme de* $\mathbf{CP}^k$,
Acta Math. 182 (1999), 143-157.

[DH] M. Denker and S. Heinemann, *Jordan tori and polynomial endomorphisms in* $\mathbf{C}^2$, Fund. Math. 157 (1998), 139-159.

[FS2] J. Fornaess and N. Sibany, *Dynamics of* $\mathbf{P}^2$ *(Examples),*
Contemp. Math. 269 (2001), 47-85.

[K] T. H. Koornwinder, *Orthogonal polynomials in two variables which are eigen functions of two algebraically independent partial differential operators ,*
III, IV. Indeg. Math. 36 (1974), 357-381.

[N] S. Nakane, *External says for a regular polynomial endomorphism of* $\mathbf{C}^2$ *associated with Chebyshev mappings,* preprint (2004).

[Uc1] K. Uchimura, *The dynamical systems associatted with Chebyshev polynomials in two variables,* Int. J. Bifurcation and Chaos 12B, (6) (1996), 2611-2618.

[Uc2] K. Uchimura, *Attractive basins of certain non-holomorphic maps,* Int. J. Bifurcation and Chaos 8, (7) (1998), 1517-1526.

[Uc3] K. Uchimura, *The sets of points with bounded orbits for generalized Chebyshev mappings,* Int. J. Bifurcation and Chaos 11, (1) (2001), 91-107.

[Uc4] K. Uchimura, *Dynamics of symmetric polynomial self map of* $\mathbf{C}^2$, preprint series, Dept. Math. Tokai Univ. no 027, (2004).

[Ue] T. Ueda, *On critically finite maps on the complex projective space,* RIMS Kokyuroku 1087 (1999), 132-138.