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EXPLICIT REPRESENTATION OF INNOVATION PROCESSES WITH APPLICATION TO OPTIMAL INVESTMENT MODEL WITH MEMORY

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1. Introduction

We consider a financial market model driven by an $\mathbf{R}^n$-valued Gaussian process with stationary increments which is different from Brownian motion. Each component $Y(t)$ of the driving noise process is a Gaussian process with stationary increments defined by a continuous-time AR(∞)-type equation. The process $Y(t)$ has a good MA(∞)-type representation. The existence of such simultaneous good AR(∞) and MA(∞) representations enables us to apply a new method for the calculation of relevant conditional expectations, whence to obtain various explicit results for problems such as portfolio optimization. The class of $Y(t)$ includes the following special processes with two parameters p and q:

$$Y(t) = W(t) - \int_0^t \left(\int_{-\infty}^s pe^{-(q+p)(s-u)} dW(u) \right) ds \qquad (0 \le t \le T), \quad (1.1)$$

where p and q are real constants such that

$$0 < q < \infty, \qquad -q < p < \infty,$$

and $(W(t))_{t\in\mathbf{R}}$ is a one-dimensional Brownian motion.

For our filtration $\{\mathcal{F}_t\}_{0\le t\le T}$, we take the augmentation of the filtration generated by $(Y(t))_{0\le t\le T}$. It follows that $Y(t)$ is a (Gaussian) $(\mathcal{F}_t)$-semimartingale of the form

$$Y(t) = B(t) - \int_0^t \alpha(s) ds \qquad (0 \le t \le T), \quad (1.2)$$

where $(\alpha(t))_{0\le t\le T}$ is an $(\mathcal{F}_t)$-adapted process and $(B(t))_{0\le t\le T}$ is an $(\mathcal{F}_t)$-Brownian motion called the *innovation process*. In applications, we need a good representation of $\alpha(\cdot)$. We find that $\alpha(\cdot)$ has the following representation (Theorem 2.1):

$$\alpha(t) = \int_0^t k(t,s) dY(s) \qquad (0 \le t \le T), \quad (1.3)$$

where $k(t,s)$ is a deterministic function represented explicitly in terms of the AR(∞)-coefficient $a(\cdot)$ and the corresponding MA(∞)-coefficient $c(\cdot)$ of $Y(t)$. In the two-parameter case (1.1), $k(t,s)$ has a very simple form (Theorem 2.3). We can regard (1.2) with (1.3) as an explicit representation of the innovation process $(B(t))_{0\le t\le T}$ in terms of $(Y(t))_{0\le t\le T}$.

The representation (1.2) with (1.3) involves $Y(t)$ itself. In applications, we also need a representation of $Y(t)$ only in terms of the innovation process $(B(t))_{0\le t\le T}$. In the two-parameter case (1.1), we have the following representation of $\alpha(\cdot)$ (Theorem 2.4):

$$\alpha(t) = \int_0^t l(t,s)dB(s) \qquad (0 \le t \le T), \tag{1.4}$$

where $l(t,s)$ is an explicit elementary function.

Many authors consider financial market models in which the standard driving noise, that is, Brownian motion, is replaced by a different one such as fractional Brownian motion so that the model can capture *memory effect*. See, e.g., Comte and Renault [7, 8], Rogers [31], Heyde [16], Willinger et al. [33], Barndorff-Nielsen and Shephard [5], Barndorff-Nielsen et al. [4], Hu and Øksendal [18], Hu et al. [19], Elliott and van der Hoek [9], and Heyde and Leonenko [17]. In most of these references, driving noise processes are assumed to have stationary increments since this is a natural requirement of simplicity. Among such models, the model $\mathcal{M}$ driven by $Y(t)$'s, which are Gaussian processes with *stationary increments*, is possibly the simplest one. One advantage of $\mathcal{M}$ is that, by the semimartingale representations of $Y(t)$'s stated above, it admits *explicit calculations* as we see in Section 3. Another advantageous feature of the model $\mathcal{M}$ is that we can easily estimate its characteristic parameters from given market data.

2. Driving noise process with memory

2.1. Completely monotone kernels. Let $Y(t)$ be a continuous Gaussian process with stationary increments satisfying $Y(0) = 0$ and the following continuous-time AR(∞)-type equation:

$$\frac{dY}{dt}(t) + \int_{-\infty}^t a(t-s)\frac{dY}{dt}(s)ds = \frac{dW}{dt}(t), \tag{2.1}$$

where $(W(t))_{t\in\mathbf{R}}$ is a one-dimensional standard Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $W(0) = 0$, and dY/dt and dW/dt are the derivatives of $Y(t)$ and $W(t)$ respectively in the random distribution sense (see [1]).

Let $T \in (0,\infty)$. We need $Y(t)$ to be defined for $t \in \mathbf{R}$ to construct the process but once it is constructed, we may regard $Y(t)$ as being defined for $t \in [0,T]$. The integral on the left-hand side of (2.1) has the effect of incorporating memory into the dynamics of the process $Y(t)$.

In analogy with time series analysis, it is natural to introduce processes with memory by considering AR-type equations of the form (2.1). However, because of technical difficulties in continuous time, it is important to assume reasonable conditions. We assume that the delay kernel $a(\cdot)$ is a bounded, integrable, completely monotone function on $(0,\infty)$. Thus $a(\cdot)$ is of the form

$$a(t) = \int_0^\infty e^{-ts}\nu(ds) \qquad (t>0), \tag{2.2}$$

where ν is a finite Borel measure on $(0,\infty)$. Under this assumption, $Y(t)$ has a good MA(∞)-type representation

$$Y(t) = W(t) - \int_0^t \left\{ \int_{-\infty}^s c(s-u)dW(u) \right\} ds \tag{2.3}$$

(see [1]), where $c(\cdot)$ is a finite, completely monotone function on $(0,\infty)$ satisfying $\int_0^\infty c(t)dt < 1$ and

$$\left\{ 1 + \int_0^\infty e^{izt}a(t)dt \right\} \left\{ 1 - \int_0^\infty e^{izt}c(t)dt \right\} = 1 \qquad (\Im z > 0). \tag{2.4}$$

For our filtration $(\mathcal{F}_t)_{0\le t\le T}$, we take the augmentation of the filtration generated by $Y(t)$. We define $b(t,s)$ by

$$b(t,s) = -c(t+s) - \int_0^t a(u)c(t+s-u)du \qquad (t,s>0).$$

For $s,\tau \in (0,\infty)$, $t\in(0,\infty)$ and $n\in\mathbf{N}$, we put

$$\begin{cases} b_1(s,\tau;t) := b(s,\tau), \\ b_n(s,\tau;t) := \int_0^\infty b(s,u)b_{n-1}(t+u,\tau;t)du \qquad (n=2,3,\dots). \end{cases}$$

We also put

$$h(t,s,\tau) = \sum_{k=1}^\infty \{ b_{2k-1}(s,\tau;t) + b_{2k}(t-s,\tau;t) \} \qquad (t,\tau>0,\ 0<s<t).$$

It should be noticed that (2.3) is not a semimartingale representation of $Y(t)$ since the Brownian motion $W(t)$ is not $(\mathcal{F}_t)$-adapted. However, we have the next theorem.

Theorem 2.1 ([2]). *There exists a one-dimensional Wiener process $B(t)$, $0\le t\le T$, satisfying*

$$Y(t) = B(t) - \int_0^t \left\{ \int_0^s k(s,u)dY(u) \right\} ds \qquad (0\le t\le T), \tag{2.5}$$

and

$$\sigma(B(u): 0\le u\le t) = \sigma(Y(u): 0\le u\le t) \qquad (0\le t\le T), \tag{2.6}$$

where $k(\cdot,\cdot)$ *is given explicitly by*

$$k(t,s) = a(t-s) + \int_0^\infty h(t,s,\tau)a(t+\tau)d\tau \qquad (0 < s < t < \infty). \quad (2.7)$$

The Brownian motion $(B(t))_{0\le t\le T}$ is the so called *innovation process* associated with $(Y(t))_{0\le t\le T}$. The equality (2.5) with (2.7) gives an explicit representation of the innovation process $B(t)$ in terms of $Y(t)$. We can regard (2.5) as a semimartingale representation of $(Y(t))_{0\le t\le T}$ with respect to the filtration $(\mathcal{F}_t)_{0\le t\le T}$.

The proof of Theorem 2.1 is based on the Kailath-Shiryaev theorem (cf. [24]) and the next prediction formula for $Y(t)$.

Theorem 2.2 ([2]). *Let* $t > 0$. *We assume that* $f(\cdot)$ *is a bounded measurable function on* $[t,\infty)$ *satisfying*

$$\int_0^\infty f(t+s)^2 ds < \infty, \qquad \int_0^\infty \left\{ \int_0^\infty |f(t+u)c(u-s)du \right\}^2 ds < \infty.$$

Then we have

$$E\left[\int_t^\infty f(s)dY(s)|\mathcal{F}(t)\right] = \int_0^t \left(\int_0^\infty h(t,t-s,\tau)f(t+\tau)d\tau\right) dY(s). \quad (2.8)$$

2.2. **Driving noise process with two parameters.** The simplest non-trivial example of $a(\cdot)$ in the previous subsection is $a(t) = pe^{-qt}$ for $t > 0$ with

$$0 < q < \infty, \qquad -q < p < \infty. \quad (2.9)$$

In this case, since

$$\left(1 + \frac{p}{q-iz}\right)\left(1 - \frac{p}{p+q-iz}\right) = 1,$$

we have $c(t) = pe^{-(p+q)t}$, whence

$$Y(t) = W(t) - \int_0^t \left(\int_{-\infty}^s pe^{-(q+p)(s-u)} dW(u)\right) ds. \quad (2.10)$$

As remarked in the previous subsection, (2.10) is not a semimartingale representation of $Y(t)$ with respect to the filtration $(\mathcal{F}_t)_{0\le t\le T}$, which is the augmentation of the filtration generated by $Y(t)$. Let $(B(t))_{0\le t\le T}$ be the innovation process in Theorem 2.1. From Theorem 2.1, we obtain the next theorem.

Theorem 2.3 ([2]). *The process* $Y(t)$ *in* (2.10) *has the semimartingale representation* (2.5) *with*

$$k(s,u) = (2q+p)p\frac{(2q+p)e^{qu} - pe^{-qu}}{(2q+p)^2 e^{qs} - p^2 e^{-qs}} \qquad (0 < u < s) \qquad (2.11)$$

The semimartingale representation (2.5) of $Y(t)$ involves $Y(t)$ itself. However, using the resolvent $l(t,s)$ of $k(t,s)$ given by, for $0 \le s \le t \le T$,

$$\begin{cases} l(t,s) - k(t,s) + \int_s^t l(t,u)k(u,s)du = 0, \\ l(t,s) - k(t,s) + \int_s^t k(t,u)l(u,s)du = 0, \end{cases} \tag{2.12}$$

we can also give the following representation of $(Y(t))$:

$$Y(t) = B(t) - \int_0^t \left\{ \int_0^s l(s,u)dB(u) \right\} ds \qquad (0 \le t \le T). \tag{2.13}$$

Solving (2.12) with (2.11) explicitly, we get the next theorem.

Theorem 2.4 ([21]). *The process* $(Y(t))_{0\le t\le T}$ *in* (2.10) *has the semimartingale representation* (2.13) *with*

$$l(t,s) = pe^{-(p+q)(t-s)} \left\{ 1 - \frac{2pq}{(2q+p)^2 e^{2qs} - p^2} \right\} \qquad (0 \le s \le t \le T). \tag{2.14}$$

3. Optimal investment model with memory

3.1. **The model and problems.** In this section, we consider optimal investment problems for a financial market model with memory. This market model $\mathcal{M}$ consists of n risky and one riskless assets. The price of the riskless asset is denoted by $S_0(t)$ and that of the ith risky asset by $S_i(t)$. We put $S(t) = (S_1(t), \dots, S_n(t))'$, where A' denotes the transpose of a matrix A. The dynamics of the $\mathbf{R}^n$-valued process $S(t)$ are described by the stochastic differential equation

$$\begin{aligned} dS_i(t) &= S_i(t) \left[\mu_i(t)dt + \sum\nolimits_{j=1}^n \sigma_{ij}(t)dY_j(t) \right], \quad & t \ge 0, \\ S_i(0) &= s_i, & i = 1, \dots, n, \end{aligned} \tag{3.1}$$

while those of $S_0(t)$ by the ordinary differential equation

$$dS_0(t) = r(t)S_0(t)dt, \quad t \ge 0, \quad S_0(0) = 1, \tag{3.2}$$

where the coefficients $r(t) \ge 0$, $\mu_i(t)$, and $\sigma_{ij}(t)$ are continuous deterministic functions on $[0,\infty)$ and the initial prices s_i are positive constants. We assume that the $n \times n$ volatility matrix $\sigma(t) = (\sigma_{ij}(t))_{1\le i,j\le n}$ is nonsingular for $t \ge 0$.

We define the jth component $Y_j(t)$ of the $\mathbf{R}^n$-valued driving noise process $Y(t) = (Y_1(t), \dots, Y_n(t))'$ by the autoregressive type equation

$$\frac{dY_j(t)}{dt} = -\int_{-\infty}^t p_j e^{-q_j(t-s)} \frac{dY_j(s)}{ds} ds + \frac{dW_j(t)}{dt}, \quad t \in \mathbf{R}, \quad Y_j(0) = 0, \tag{3.3}$$

where $W(t) = (W_1(t), \ldots, W_n(t))'$, $t \in \mathbf{R}$, is an $\mathbf{R}^n$-valued standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, the derivatives $dY_j(t)/dt$ and $dW_j(t)/dt$ are in the random distribution sense, and p_j's and q_j's are constants such that

$$0 < q_j < \infty, \quad -q_j < p_j < \infty, \quad j = 1, \ldots, n \tag{3.4}$$

(see Section 2). Equivalently, we may define $Y_j(t)$ by the moving-average type representation

$$Y_j(t) = W_j(t) - \int_0^t \left[\int_{-\infty}^s p_j e^{-(q_j+p_j)(s-u)} dW_j(u) \right] ds, \quad t \in \mathbf{R} \tag{3.5}$$

(see Section 2). The components $Y_j(t)$, $j = 1, \ldots, n$, are Gaussian processes with stationary increments that are independent of each other. Each $Y_j(t)$ has short memory that is described by the two parameters p_j and q_j. In the special case $p_j = 0$, $Y_j(t)$ reduces to the Brownian motion $W_j(t)$.

Let $(\mathcal{F}_t)_{t \geq 0}$ be the augmentation of the filtration generated by the process $(Y(t))_{t \geq 0}$, which is the underlying information structure of the market model $\mathcal{M}$. We have the following two kinds of semimartingale representations of $Y(t)$:

$$Y_j(t) = B_j(t) - \int_0^t \left[\int_0^s k_j(s, u) dY_j(u) \right] ds, \quad t \geq 0, \quad j = 1, \ldots, n, \tag{3.6}$$

$$Y_j(t) = B_j(t) - \int_0^t \left[\int_0^s l_j(s, u) dB_j(u) \right] ds, \quad t \geq 0, \quad j = 1, \ldots, n, \tag{3.7}$$

where, for $j = 1, \ldots, n$, $(B_j(t))_{t \geq 0}$ is the innovation process, i.e., it is an $\mathbf{R}$-valued standard Brownian motion such that

$$\sigma(Y_j(s) : 0 \leq s \leq t) = \sigma(B_j(s) : 0 \leq s \leq t), \quad t \geq 0.$$

Notice that B_j's are independent of each other. The deterministic kernels $k_j(t, s)$ and $l_j(t, s)$ are given explicitly by

$$k_j(t, s) = p_j(2q_j + p_j) \frac{(2q_j + p_j)e^{q_j s} - p_j e^{-q_j s}}{(2q_j + p_j)^2 e^{q_j t} - p_j^2 e^{-q_j t}}, \quad 0 \leq s \leq t, \tag{3.8}$$

$$l_j(t, s) = e^{-(p_j+q_j)(t-s)} l_j(s), \quad 0 \leq s \leq t, \tag{3.9}$$

with

$$l_j(s) := p_j \left[1 - \frac{2p_j q_j}{(2q_j + p_j)^2 e^{2q_j s} - p_j^2} \right], \quad s \geq 0. \tag{3.10}$$

We have the equalities

$$\int_0^t k_j(t, s) dY_j(s) = \int_0^t l_j(t, s) dB_j(s), \quad t \geq 0, \quad j = 1, \ldots, n. \tag{3.11}$$

For the market model $\mathcal{M}$, we consider an agent with initial endowment $x \in (0,\infty)$ who invests, at each time t, $\pi_i(t)X^{x,\pi}(t)$ dollars in the ith risky asset for $i = 1,\dots,n$ and $[1-\sum_{i=1}^n \pi_i(t)]X^{x,\pi}(t)$ dollars in the riskless asset, where $X^{x,\pi}(t)$ denotes the agent's wealth at time t. The wealth process $X^{x,\pi}(t)$ is governed by the stochastic differential equation

$$\frac{dX^{x,\pi}(t)}{X^{x,\pi}(t)} = \left[1-\sum\nolimits_{i=1}^n \pi_i(t)\right]\frac{dS_0(t)}{S_0(t)} + \sum\nolimits_{i=1}^n \pi_i(t)\frac{dS_i(t)}{S_i(t)}, \quad X^{x,\pi}(0) = x. \tag{3.12}$$

Here, we choose the self-financing strategy $\pi(t) = (\pi_1(t),\dots,\pi_n(t))'$ from the admissible class

$$\mathcal{A}_T := \left\{\pi = (\pi(t))_{0\le t\le T} : \begin{array}{l}\pi \text{ is an } \mathbf{R}^n\text{-valued, progressively measurable} \\ \text{process satisfying } \int_0^T \|\pi(t)\|^2 dt < \infty \text{ a.s.}\end{array}\right\}$$

for the finite time horizon of length $T \in (0,\infty)$, where $\|\cdot\|$ denotes the Euclidean norm of $\mathbf{R}^n$. If the time horizon is infinite, we choose $\pi(t)$ from

$$\mathcal{A} := \{(\pi(t))_{t\ge 0} : (\pi(t))_{0\le t\le T} \in \mathcal{A}_T \text{ for every } T \in (0,\infty)\}.$$

Let $\alpha \in (-\infty,1)\setminus\{0\}$ and $c \in \mathbf{R}$. We consider the following three optimal investment problems for the model $\mathcal{M}$:

$$V(T,\alpha) := \sup_{\pi\in\mathcal{A}_T} \frac{1}{\alpha} E\left[(X^{x,\pi}(T))^\alpha\right], \tag{P1}$$

$$J(\alpha) := \sup_{\pi\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{\alpha T}\log E\left[(X^{x,\pi}(T))^\alpha\right], \tag{P2}$$

$$I(c) := \sup_{\pi\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\log P\left[X^{x,\pi}(T) \ge e^{cT}\right]. \tag{P3}$$

The goal of Problem P1 is to maximize the expected utility of wealth at the end of the finite horizon. This classical optimal investment problem dates back to Merton [26]. We refer to Karatzas and Shreve [22] and references therein for work on this and related problems. Our approach is based on a Cameron–Martin type formula. This formula holds under the assumption that a relevant Riccati type equation has a solution, and the key step of our arguments is to show the existence of such a solution.

The aim of Problem P2 is to maximize the growth rate of expected utility of wealth over the infinite horizon. This problem is studied by Bielecki and Pliska [6], and subsequently by other authors under various settings, including Fleming and Sheu [11, 12], Kuroda and Nagai [23], Pham [29, 30], Nagai and Peng [28], Hata and Iida [13], and Hata and Sekine [14, 15]. In our arguments, as in those for Problem P1, existence results on solutions to Riccati type equations play a key role. The result of Nagai and Peng [28] on the asymptotic behavior of solutions to Riccati equations is also an essential ingredient in our arguments.

The purpose of Problem P3 is to maximize the large deviation probability that the wealth grows at a higher rate than the given benchmark c. This problem is studied by Pham [29, 30], in which a significant result, that is, a duality relation between Problems P2 and P3, is established. Subsequently, this problem is studied by Hata and Iida [13] and Hata and Sekine [14, 15] under different settings. In Pham's approach to Problem P3, one needs an explicit expression of $J(\alpha)$. Since our solution to Problem P2 gives such an explicit expression of $J(\alpha)$, we can solve Problem P3.

3.2. **Optimal investment over a finite horizon.** In this subsection, we consider the finite horizon optimization problem P1 for the market model $\mathcal{M}$. We assume $\alpha \in (-\infty, 1) \setminus \{0\}$ and

$$0 < q_j < \infty, \quad 0 \le p_j < \infty, \qquad j = 1, \dots, n. \tag{3.13}$$

Thus $p_j > 0$ rather than $p_j > -q_j$ for $j = 1, \dots, n$.

Let $Y(t) = (Y_1(t), \dots, Y_n(t))'$ and $B(t) = (B_1(t), \dots, B_n(t))'$ be the driving noise and innovation processes, respectively, described in the previous subsection. We define an $\mathbf{R}^n$-valued deterministic function $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))'$ by

$$\lambda(t) := \sigma^{-1}(t)\left[\mu(t) - r(t)\mathbf{1}\right], \qquad t \ge 0, \tag{3.14}$$

where $\mathbf{1} := (1, \dots, 1)' \in \mathbf{R}^n$. For $k_j(t,s)$'s in (3.8), we put

$$k(t,s) := \mathrm{diag}(k_1(t,s), \dots, k_n(t,s)), \qquad 0 \le s \le t.$$

Let $\xi(t) = (\xi_1(t), \dots, \xi_n(t))'$ be the $\mathbf{R}^n$-valued process $\int_0^t k(t,s)dY(s)$, i.e.,

$$\xi_j(t) := \int_0^t k_j(t,s)dY_j(s), \qquad t \ge 0, \quad j = 1, \dots, n. \tag{3.15}$$

Let β be the conjugate exponent of α, i.e.,

$$(1/\alpha) + (1/\beta) = 1.$$

Notice that $0 < \beta < 1$ (resp. $-\infty < \beta < 0$) if $-\infty < \alpha < 0$ (resp. $0 < \alpha < 1$).

We put $l(t) := \mathrm{diag}(l_1(t), \dots, l_n(t))$, $p := \mathrm{diag}(p_1, \dots, p_n)$, and $q := \mathrm{diag}(q_1, \dots, q_n)$ with $l_j(t)$'s as in (3.10). We also put

$$\rho(t) = (\rho_1(t), \dots, \rho_n(t))', \quad b(t) = \mathrm{diag}(b_1(t), \dots, b_n(t))$$

with

$$\rho_j(t) := -\beta l_j(t)\lambda_j(t), \qquad t \ge 0, \quad j = 1, \dots, n, \tag{3.16}$$

$$b_j(t) := -(p_j + q_j) + \beta l_j(t), \quad t \ge 0, \quad j = 1, \dots, n. \tag{3.17}$$

We consider the following one-dimensional backward Riccati equations: for $j = 1, \dots, n$

$$\begin{aligned} &\dot{R}_j(t) - l_j^2(t)R_j^2(t) + 2b_j(t)R_j(t) + \beta(1-\beta) = 0, \\ &0 \le t \le T, \qquad R_j(T) = 0. \end{aligned} \tag{3.18}$$

The following lemma, especially (iii), is crucial in our arguments.

Lemma 3.1 ([20]). *Let $j \in \{1, \dots, n\}$.*

(1) *If $p_j = 0$, then* (3.18) *has a unique solution $R_j(t) \equiv R_j(t;T)$.*
(2) *If $-\infty < \alpha < 0$, then* (3.18) *has a unique nonnegative solution $R_j(t) \equiv R_j(t;T)$.*
(3) *If $p_j > 0$ and $0 < \alpha < 1$, then* (3.18) *has a unique solution $R_j(t) \equiv R_j(t;T)$ such that $R_j(t) \ge b_j(t)/l_j^2(t)$ for $t \in [0,T]$.*

In what follows, we write $R_j(t) \equiv R_j(t;T)$ for the unique solution to (3.18) in the sense of Lemma 3.1. Then $R(t) := \mathrm{diag}(R_1(t), \dots, R_n(t))$ satisfies the backward matrix Riccati equation

$$\begin{aligned} &\dot{R}(t) - R(t)l^2(t)R(t) + b(t)R(t) + R(t)b(t) + \beta(1-\beta)I_n = 0, \\ &0 \le t \le T, \qquad R(T) = 0, \end{aligned} \tag{3.19}$$

where I_n denotes the $n \times n$ unit matrix. For $j = 1, \dots, n$, let $v_j(t) \equiv v_j(t;T)$ be the solution to the following one-dimensional linear equation:

$$\begin{aligned} &\dot{v}_j(t) + [b_j(t) - l_j^2(t)R_j(t;T)]v_j(t) + \beta(1-\beta)\lambda_j(t) - R_j(t;T)\rho_j(t) = 0, \\ &0 \le t \le T, \qquad v_j(T) = 0. \end{aligned} \tag{3.20}$$

Then $v(t) \equiv v(t;T) := (v_1(t;T), \dots, v_n(t;T))'$ satisfies

$$\begin{aligned} &\dot{v}(t) + [b(t) - l^2(t)R(t;T)]v(t) + \beta(1-\beta)\lambda(t) - R(t;T)\rho(t) = 0, \\ &0 \le t \le T, \qquad v_j(T) = 0. \end{aligned} \tag{3.21}$$

For $j = 1, \dots, n$ and $(t,T) \in \Delta$, write

$$g_j(t;T) := v_j^2(t;T)l_j^2(t) + 2\rho_j(t)v_j(t;T) - l_j^2(t)R_j(t;T) - \beta(1-\beta)\lambda_j^2(t), \tag{3.22}$$

where

$$\Delta := \{(t,T) : 0 < T < \infty,\ 0 \le t \le T\}. \tag{3.23}$$

Recall that we have assumed $\alpha \in (-\infty, 1) \setminus \{0\}$ and (3.20). Here is the solution to Problem P1.

Theorem 3.2 ([20]). *For $T \in (0,\infty)$, the strategy $(\hat{\pi}_T(t))_{0 \le t \le T} \in \mathcal{A}_T$ defined by*

$$\hat{\pi}_T(t) := (\sigma')^{-1}(t)\left[(1-\beta)\lambda(t) - \{1 - \beta + l(t)R(t;T)\}\xi(t) + l(t)v(t;T)\right] \tag{3.24}$$

is the unique optimal strategy for Problem P1. The value function $V(T) \equiv V(T,\alpha)$ *in* (P1) *is given by*

$$V(T) = \frac{1}{\alpha}[xS_0(T)]^{\alpha} \exp\left[\frac{(1-\alpha)}{2}\sum\nolimits_{j=1}^{n} \int_0^T g_j(t;T)dt\right]. \tag{3.25}$$

3.3. **Optimal investment over an infinite horizon.** In this subsection, we consider the infinite horizon optimization problem P2 for the financial market model $\mathcal{M}$. Throughout this section, we assume (3.13) and the following two conditions:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T r(t)dt = \bar{r} \quad \text{with } \bar{r} \in \mathbf{R}, \tag{3.26}$$

$$\lim_{T\to\infty} \lambda(t) = \bar{\lambda} \quad \text{with } \bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)' \in \mathbf{R}^n. \tag{3.27}$$

Here recall $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))'$ from (3.14). In the main result of this subsection (Theorem 3.3), we will also assume $\alpha^* < \alpha < 1$, $\alpha \neq 0$, where

$$\alpha^* := \max(\alpha_1^*, \ldots, \alpha_n^*) \tag{3.28}$$

with

$$\alpha_j^* := \begin{cases} -\infty & \text{if } \ 0 \le p_j \le 2q_j, \\ -3 - \dfrac{8q_j}{p_j - 2q_j} & \text{if } \ 2q_j < p_j < \infty. \end{cases} \tag{3.29}$$

Notice that $\alpha^* \in [-\infty, -3)$.

To give the solution to Problem P2, we take the following steps:

(1) For the value function $V(T) \equiv V(T,\alpha)$ in (P1), we calculate the following limit explicitly:

$$\tilde{J}(\alpha) := \lim_{T\to\infty} \frac{1}{\alpha T} \log[\alpha V(T)]. \tag{3.30}$$

(2) For $\hat{\pi} \in \mathcal{A}$ in (3.37) below, we calculate the growth rate

$$J^*(\alpha) := \lim_{T\to\infty} \frac{1}{\alpha T} \log E\left[(X^{x,\hat{\pi}}(T))^{\alpha}\right], \tag{3.31}$$

and verify that $J^*(\alpha) = \tilde{J}(\alpha)$.

(3) Since the definition of $V(T)$ implies

$$\limsup_{T\to\infty} \frac{1}{\alpha T} \log E[(X^{x,\pi}(T))^{\alpha}] \le \tilde{J}(\alpha) \qquad \forall \pi \in \mathcal{A}, \tag{3.32}$$

we conclude that $\hat{\pi}$ is an optimal strategy for Problem P2 and that the optimal growth rate $J(\alpha)$ in (P2) is given by $J^*(\alpha) = \tilde{J}(\alpha)$.

Let $\alpha \in (-\infty, 1) \setminus \{0\}$ and β be its conjugate exponent. Let $j \in \{1, \dots, n\}$. For $b_j(t)$ in (3.17), we have $\lim_{t\to\infty} b_j(t) = \bar{b}_j$, where

$$\bar{b}_j := -(1-\beta)p_j - q_j.$$

Notice that $\bar{b}_j < 0$. We consider the equation

$$p_j^2 x^2 - 2\bar{b}_j x - \beta(1-\beta) = 0. \tag{3.33}$$

When $p_j = 0$, we write $\bar{R}_j$ for the unique solution $\beta(1-\beta)/(2q_j)$ of this linear equation. If $p_j > 0$, then

$$\bar{b}_j^2 + \beta(1-\beta)p_j^2 = (1-\beta)[(p_j+q_j)^2 - q_j^2] + q_j^2 \geq q_j^2 > 0,$$

so that we may write $\bar{R}_j$ for the larger solution to the quadratic equation (3.33). We write $R_j(t) = R_j(t;T)$ for the unique solution to (3.18) in the sense of Lemma 3.1. Let $j \in \{1, \dots, n\}$. For $\rho_j(t)$ in (3.16), we have $\lim_{t\to\infty} \rho_j(t) = \bar{\rho}_j$, where

$$\bar{\rho}_j := -\beta p_j \bar{\lambda}_j.$$

Let $v_j(t) \equiv v_j(t;T)$ be the solution to (3.20). Define $\bar{v}_j$ by

$$\left(\bar{b}_j - p_j^2 \bar{R}_j\right)\bar{v}_j + \beta(1-\beta)\bar{\lambda}_j - \bar{R}_j\bar{\rho}_j = 0. \tag{3.34}$$

For $j = 1, \dots, n$ and $-\infty < \alpha < 1$, $\alpha \neq 0$, we put

$$F_j(\alpha) := \frac{(p_j+q_j)^2\bar{\lambda}_j^2\alpha}{[(1-\alpha)(p_j+q_j)^2 + \alpha p_j(p_j+2q_j)]}, \tag{3.35}$$

and

$$\begin{aligned} G_j(\alpha) &:= (p_j+q_j) - q_j\alpha \\ &\quad - (1-\alpha)^{1/2}\left[(1-\alpha)(p_j+q_j)^2 + \alpha p_j(p_j+2q_j)\right]^{1/2}. \end{aligned} \tag{3.36}$$

Recall $\xi(t)$ from (3.14). Taking into account (3.24), we consider $\hat{\pi} = (\hat{\pi}(t))_{t\geq 0} \in \mathcal{A}$ defined by

$$\hat{\pi}(t) := (\sigma')^{-1}(t)\left[(1-\beta)\lambda(t) - (1-\beta+p\bar{R})\xi(t) + p\bar{v}\right], \qquad t \geq 0, \tag{3.37}$$

where $\bar{R} := \mathrm{diag}(\bar{R}_1, \dots, \bar{R}_n)$, $\bar{v} := (\bar{v}_1, \dots, \bar{v}_n)'$, and $p := \mathrm{diag}(p_1, \dots, p_n)$.

Recall that we have assumed (3.13), (3.26) and (3.27). Recall also α^* from (3.28) with (3.29). Here is the solution to Problem P2.

Theorem 3.3 ([20]). *Let $\alpha^* < \alpha < 1$, $\alpha \neq 0$. Then $\hat{\pi}$ is an optimal strategy for Problem P2 with limit in* (3.31). *The optimal growth rate $J(\alpha)$ in* (P2) *is given by*

$$J(\alpha) = \bar{r} + \frac{1}{2\alpha}\sum_{j=1}^n F_j(\alpha) + \frac{1}{2\alpha}\sum_{j=1}^n G_j(\alpha), \tag{3.38}$$

where F_j's and G_j's are as in (3.35) *and* (3.36).

3.4. **Large deviations probability control.** We study the large deviations probability control problem P3 for the market model $\mathcal{M}$. Throughout this subsection, we assume (3.13), (3.26) and (3.27). We also assume

$$\text{either } \bar{\lambda} \neq (0,\dots,0)' \text{ or } (p_1,\dots,p_n) \neq (0,\dots,0). \tag{3.39}$$

For $x \in (0,\infty)$ and $\pi \in \mathcal{A}$, let $L^{x,\pi}(T)$ be the growth rate defined by

$$L^{x,\pi}(T) := \frac{\log X^{x,\pi}(T)}{T}, \qquad T > 0.$$

We have $P\left(L^{x,\pi}(T) \geq c\right) = P\left(X^{x,\pi}(T) \geq e^{cT}\right)$. Following Pham [29, 30], we consider the optimal logarithmic moment generating function

$$\Lambda(\alpha) := \sup_{\pi\in\mathcal{A}} \limsup_{T\to\infty} \log E[\exp(\alpha T L^{x,\pi}(T))], \qquad 0 < \alpha < 1.$$

Since $\Lambda(\alpha) = \alpha J(\alpha)$ for $\alpha \in (0,1)$, it follows from Theorem 3.3 that

$$\Lambda(\alpha) = \bar{r}\alpha + \frac{1}{2}\sum_{j=1}^{n} F_j(\alpha) + \frac{1}{2}\sum_{j=1}^{n} G_j(\alpha), \qquad 0 < \alpha < 1,$$

where F_j's and G_j's are as in (3.35) and (3.36).

Proposition 3.4 ([20]). *We have*

$$\frac{d\Lambda}{d\alpha}(0+) = \bar{c}, \qquad \lim_{\alpha\uparrow 1} \frac{d\Lambda}{d\alpha}(\alpha) = \infty,$$

where

$$\bar{c} := \bar{r} + \frac{1}{4}\sum_{j=1}^{n} \frac{p_j^2}{p_j + q_j} + \frac{1}{2}\|\bar{\lambda}\|^2.$$

For $\alpha \in (0,1)$, we write $\hat{\pi}(t;\alpha)$ for the optimal strategy $\hat{\pi}(t)$ in (3.37). Recall $I(c)$ from (P3). From Theorem 3.3, Proposition 3.4, and Pham [29, Theorem 3.1], we immediately obtain the following solution to Problem P3:

Theorem 3.5 ([20]). *We have*

$$I(c) = -\sup_{\alpha\in(0,1)} \left[\alpha c - \Lambda(\alpha)\right], \qquad c \in \mathbf{R}.$$

Moreover, if $\alpha(d) \in (0,1)$ is such that $\dot{\Lambda}(\alpha(d)) = d \in (\bar{c},\infty)$, then, for $c \geq \bar{c}$, the sequence of strategies

$$\hat{\pi}^m(t) := \hat{\pi}(t; \alpha(c + \tfrac{1}{m})),$$

is nearly optimal in the sense that

$$\lim_{m\to\infty} \limsup_{T\to\infty} \frac{1}{T} \log P\left(X^{x,\hat{\pi}^m}(T) \geq e^{cT}\right) = I(c), \qquad c \geq \bar{c}.$$

We turn to the problem of deriving an optimal strategy for the problem (P3), rather than a nearly optimal sequence, when $c < \bar{c}$. We define $\pi_0 \in \mathcal{A}$ by

$$\hat{\pi}_0(t) := (\sigma')^{-1}(t)\left[\lambda(t) - \xi(t)\right], \qquad t \geq 0,$$

where recall $\xi(t)$ from (3.15).

Theorem 3.6 ([20]). *For $c < \bar{c}$, $\hat{\pi}_0$ is optimal for Problem P3 with limit, i.e.,*

$$\lim_{T\to\infty} \frac{1}{T} \log P\left[X^{x,\hat{\pi}_0}(T) \geq e^{cT}\right] = I(c), \qquad c < \bar{c}.$$

Remark. From Theorem 10.1 in Karatzas and Shreve [22, Chapter 3], we see that $\hat{\pi}_0$ is the log-optimal or growth optimal strategy in the sense that

$$\sup_{\pi\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T} \log X^{x,\pi}(T) = \limsup_{T\to\infty} \frac{1}{T} \log X^{x,\hat{\pi}_0}(T) \quad \text{a.s.}$$

We note that $\lim_{\alpha\downarrow 0} \hat{\pi}(t;\alpha) = \hat{\pi}_0(t)$ a.s. for $t \geq 0$.

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