数理解析研究所講究録
1492 巻 2006 年 120-121

ORBIT SPACES OF HYPERSPACES

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A Peano continuum is a connected and locally connected, compact, metrizable space that contains more than one point. By the Hilbert cube we mean the infinite countable power $[0,1]^{\infty}$ of the closed unit interval. A Hilbert cube manifold is a separable metrizable space that admits an open cover by sets homeomorphic to open subsets of the Hilbert cube.

Let G be compact Lie group acting (continuously) on a Peano continuum X. We denote by $\exp X$ the G-space of all nonempty compact subsets of X endowed with the Hausdorff metric topology and the induced action of G.

Here we present the following results and some related open problems.

Theorem 0.1. *Let G be a compact Lie group acting nontransitively on the Peano continuum X. Then the orbit space $(\exp X)/G$ is homeomorphic to the Hilbert cube.*

Theorem 0.2. *Let G be a compact Lie group acting on the Peano continuum X, and let $\exp_0 X = (\exp X)\setminus\{X\}$. Then the orbit space $(\exp_0 X)/G$ is a Hilbert cube manifold.*

Conjecture 0.3. *Let G be a compact Lie group acting transitively on the Peano continuum X. Then the orbit space $(\exp X)/G$ is not homeomorphic to the Hilbert cube.*

Recall that for an integer $n \geq 2$, the Banach-Mazur compactum $BM(n)$ is the set of isometry classes of n-dimensional Banach spaces topologized by the famous Banach-Mazur metric.

Corollary 0.4. *Let $O(n)$ denote the orthogonal group and $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$. Then for all $n \geq 2$, the orbit space $(\exp \mathbb{S}^{n-1})/O(n)$ is homeomorphic to the Banach-Mazur compactum $BM(n)$.*

Below we assume that $n \geq 2$ is an integer. Let $\mathbb{B}^n$ be the closed unit ball of $\mathbb{R}^n$ and let $C(\mathbb{B}^n)$ denote the subspace of $\exp \mathbb{B}^n$ consisting of all nonempty compact convex subsets $A \subset \mathbb{B}^n$ such that $A \cap \mathbb{S}^{n-1} \neq \emptyset$.

Theorem 0.5. (1) *$C(\mathbb{B}^n)$ is homeomorphic to the Hilbert cube.*
(2) *$C(\mathbb{B}^n)$ is an $O(n)$-AR.*
(3) *The orbit space $C(\mathbb{B}^n)/O(n)$ is homeomorphic to the Banach-Mazur compactum $BM(n)$.*

Let $SO(n)$ be the special orthogonal group. Consider the $SO(n)$-invariant subset $\operatorname{Sym}\mathbb{S}^{n-1} \subset \exp \mathbb{S}^{n-1}$ consisting of all the sets $A \in \exp \mathbb{S}^{n-1}$ such that A is symmetric with respect to an $(n-1)$-dimensional linear subspace L_A of $\mathbb{R}^n$. It is an intriguing problem to understand the topological structure of $\operatorname{Sym}\mathbb{S}^{n-1}$. In particular, we ask ask the following:

Question 0.6. (1) *Is* $\operatorname{Sym}\mathbb{S}^{n-1}$ *homeomorphic to the Hilbert cube?*

(2) *Is* $\operatorname{Sym}\mathbb{S}^{n-1}$ *an* $SO(n)$*-AR? (an AR?)*

(3) *What is the topological structure of the orbit space* $(\operatorname{Sym}\mathbb{S}^{n-1})/\mathrm{SO}(n)$*?*

Of course, similar questions can be asked about the hyperspaces of all the sets $A \in C(\mathbb{B}^n)$ (respectively, $A \in \exp\mathbb{B}^n$) such that A is symmetric with respect to some $(n-1)$-dimensional linear subspace L_A of $\mathbb{R}^n$.

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