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A stationary solution for a fluid dynamical model of semiconductor

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Dedicated to Professor Tai-Ping Liu on his 60th birthday

1 Introduction

We study the existence and the asymptotic stability of a stationary solution to the initial boundary value problem for a one-dimensional hydrodynamic model of semiconductors. In the previous researches [1] and [2], this problem is considered under the assumption that a doping profile is flat. However, this assumption is too narrow, from the physical point of view. In the present paper we briefly discuss the asymptotic stability of the stationary solution without this assumption on the doping profile. For the detailed discussion on this research, please see the paper [9].

The motion of electrons in semiconductors is governed by the system of equations

$$\rho_t + (\rho u)_x = 0, \tag{1.1a}$$
$$(\rho u)_t + \left(\rho u^2 + p(\rho)\right)_x = \rho\phi_x - \rho u, \tag{1.1b}$$
$$\phi_{xx} = \rho - D. \tag{1.1c}$$

We study this system over the bounded domain $\Omega := (0,1)$ for the spatial variable x. Here, the unknown functions ρ, u and ϕ stand for the electron density, the electron velocity and the electrostatic potential, respectively. Thus, a product $j := \rho u$ means the current density. The pressure p is assumed to be a function of the electron density ρ given by

$$p = p(\rho) = K\rho^\gamma, \tag{1.2}$$

where the constants K and γ are supposed to satisfy $K > 0$ and $\gamma \geq 1$. In the physical point of view, the case $\gamma = 1$ is important. The doping profile $D \in \mathcal{B}^0(\overline{\Omega})$ is a function of the spatial variable x and satisfy

$$\inf_{x\in\Omega} D(x) > 0. \tag{1.3}$$

We prescribe the initial and the boundary data as

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \tag{1.4}$$
$$\rho(t, 0) = \rho_l > 0, \quad \rho(t, 1) = \rho_r > 0, \tag{1.5}$$
$$\phi(t, 0) = 0, \quad \phi(t, 1) = \phi_r > 0, \tag{1.6}$$

where ρ_l, ρ_r and ϕ_r are constants. In addition, we assume that the compatibility conditions on $\rho(t,x)$ with orders 0 and 1 hold at $(t,x) = (0,0)$ and $(t,x) = (0,1)$. Namely,

$$\rho(0,0) = \rho_l, \quad \rho(0,1) = \rho_r, \quad (\rho u)_x(0,0) = 0, \quad (\rho u)_x(0,1) = 0. \tag{1.7}$$

This initial boundary value problem is studied in the region where the subsonic condition (1.8a) and positivity of the density (1.8b) hold

$$\inf_{x\in\Omega} \left(p'(\rho) - u^2\right) > 0, \tag{1.8a}$$

$$\inf_{x\in\Omega} \rho > 0. \tag{1.8b}$$

Thus, the initial data is supposed to satisfy these conditions:

$$\inf_{x\in\Omega} \left(p'(\rho_0(x)) - u_0^2(x)\right) > 0, \quad \inf_{x\in\Omega} \rho_0(x) > 0. \tag{1.9}$$

Here, note that the subsonic condition is equivalent to that one characteristic speed λ_1 of the hyperbolic equations (1.1b), (1.1c) is negative and another characteristic λ_2 is positive, that is,

$$\lambda_1 := u - \sqrt{p'(\rho)} < 0, \quad \lambda_2 := u + \sqrt{p'(\rho)} > 0. \tag{1.10}$$

The subsonic condition means that two boundary conditions (1.5), (1.6) are sufficient for the well-posedness of this initial boundary value problem, since λ_1 is negative and λ_2 is positive.

The stationary solution is a solution to (1.1) independent of time variable t, satisfying the same boundary conditions (1.5) and (1.6). Hence, the stationary solution $(\tilde{\rho},\tilde{u},\tilde{\phi})$ satisfies the system of equations

$$(\tilde{\rho}\tilde{u})_x = 0, \tag{1.11a}$$

$$\left(\tilde{\rho}\tilde{u}^2 + p(\tilde{\rho})\right)_x = \tilde{\rho}\tilde{\phi}_x - \tilde{\rho}\tilde{u}, \tag{1.11b}$$

$$\tilde{\phi}_{xx} = \tilde{\rho} - D, \tag{1.11c}$$

and boundary conditions

$$\tilde{\rho}(0) = \rho_l > 0, \quad \tilde{\rho}(1) = \rho_r > 0, \tag{1.12}$$

$$\tilde{\phi}(0) = 0, \quad \tilde{\phi}(1) = \phi_r > 0. \tag{1.13}$$

The strength of the boundary data

$$\delta_b := |\rho_r - \rho_l| + |\phi_r| \tag{1.14}$$

plays a crucial role in showing the unique existence and the asymptotic stability of the stationary solution. The unique existence of the stationary solution (ρ, u, ϕ), satisfying (1.11), (1.12) and (1.13), is summarized in the next lemma.

Lemma 1.1. *Suppose that the doping profile $D(x)$ and the boundary data satisfy* (1.3), (1.5) *and* (1.6). *Then, for an arbitrary ρ_l, there exists a positive constant δ_1 such that if $\delta_b \le \delta_1$, the stationary problem* (1.11), (1.12) *and* (1.13) *has a unique solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})(x) \in \mathcal{B}^2(\overline{\Omega})$ satisfying the condition* (1.8).

The proof of the existence of the stationary solution is given by the Schauder fixed-point theorem. On the other hand, the uniqueness is shown by the maximum principle.

In order to discuss the asymptotic stability of the stationary solution, we introduce function spaces

$$\mathfrak{X}_i^j([0,T]) := \bigcap_{k=0}^{i} C^k([0,T]; H^{j+i-k}(\Omega)) \quad \text{for} \quad i,\ j = 0,1,2,$$
$$\mathfrak{X}_i([0,T]) := \mathfrak{X}_i^0([0,T]).$$

The stability of the stationary solution is given by the following theorem.

Theorem 1.2. *Let $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ be the stationary solution of* (1.11), (1.12) *and* (1.13). *Suppose that the initial data (ρ_0, u_0) belongs to the function space $H^2(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy conditions* (1.5), (1.6), (1.7) *and* (1.9). *Then there exists a positive constant δ_2 such that if $\delta_b + \|(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_2 \le \delta_2$, the initial boundary value problem* (1.1), (1.4), (1.5) *and* (1.6) *has a unique solution $(\rho, u, \phi)(t,x)$ in the function space $\mathfrak{X}_2([0,\infty))$. Moreover, the solution $(\rho, u, \phi)(t,x)$ verifies the additional regularity $\phi - \tilde{\phi} \in \mathfrak{X}_2^2([0,\infty))$ and a decay estimate*

$$\|(\rho - \tilde{\rho}, u - \tilde{u})(t)\|_2 + \|(\phi - \tilde{\phi})(t)\|_4 \le C\|(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_2 e^{-\alpha t}, \tag{1.15}$$

where C and α are certain positive constants, independent of time t.

Related results. The hydrodynamic model of semiconductors was introduced by Bløtekjær [8]. Recently, many engineers and mathematicians study this model. For the derivation of this model, the book [6] is a good reference.

From physical and engineering point of view, it is very interesting to consider the hydrodynamic model of semiconductors with the Dirichlet boundary condition. Degond and Markowich [1] investigated the stationary solution to the hydrodynamic model with the Dirichlet boundary condition. They proved the existence of the stationary solution satisfying the subsonic condition. Li, Markowich and Mei [2] also considered the existence and the asymptotic stability of the stationary solution. However, they assumed that the doping profile is flat, that is, $|D - \rho_l| \ll 1$. This assumption is too narrow to cover actual semiconductor-devices. For instance the typical example of the doping profile, drawn in [7], does not satisfies this assumption. Matsumura and Murakami removed this assumption in the paper [3]. Precisely, they proved the asymptotic stability of the stationary solution without the assumption that the doping profile is flat. However, their boundary condition is periodic in space variable. Therefore, the asymptotic stability of the stationary solution under the Dirichlet boundary condition for the non-flat doping profile had not been proved. Recently, the problem is solved by the authors in [9]. This short note is devoted to discussion on the outline of this paper.

Notation. For a nonnegative integer $l \geq 0$, $H^l(\Omega)$ denotes the l-th order Sobolev space in the L^2 sense, equipped with the norm $\|\cdot\|_l$. We note $H^0 = L^2$ and $\|\cdot\| := \|\cdot\|_0$. $C^k([0,T]; H^l(\Omega))$ denotes the space of the k-times continuously differentiable functions on the interval $[0,T]$ with values in $H^l(\Omega)$. For a nonnegative integer $k \geq 0$, $\mathcal{B}^k(\overline{\Omega})$ denotes the space of the functions whose derivatives up to k-th order are continuous and bounded over $\overline{\Omega}$, equipped with the norm $|\cdot|_i$.

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2 The outline of the proof of Theorem 1.2

This section is devoted to the brief discussion on the proof of Theorem 1.2. The existence of the time local solution is proved by the similar method as in [4] and [5].

Lemma 2.1. *Suppose the initial data $(\rho_0, u_0) \in H^2(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy* (1.9), (1.5) ,(1.6) *and* (1.7). *Then there exists a constant $T_1(\|\rho_0\|_2, \|u_0\|_2) > 0$, such that the initial boundary value problem* (1.1), (1.4), (1.5) *and* (1.6) *has a solution $(\rho, u, \phi)(t,x) \in \mathfrak{X}_2([0,T_1])$. Moreover, the solution $(\rho, u)(t,x)$ satisfies the condition* (1.8) *for an arbitrary $t \in [0, T_1]$.*

In order to complete the proof of Theorem 1.2, we have to derive an a-priori uniform estimate with respect to the time variable t. For this purpose, it is convenient to regard the solution (ρ, u, ϕ) as a perturbation from the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$. Hence, we introduce new unknown functions as

$$\psi(t,x) := \rho(t,x) - \tilde{\rho}(x), \quad \eta(t,x) := u(t,x) - \tilde{u}(x), \quad \omega(t,x) := \phi(t,x) - \tilde{\phi}(x). \tag{2.1}$$

Multiplying $1/\rho$ by (1.1b), we have

$$u_t + uu_x + (h(\rho))_x = \phi_x - u, \quad h(\xi) := \int_1^\xi \frac{p'(\zeta)}{\zeta}\, d\zeta. \tag{2.2}$$

Similarly, we have from (2.2) that

$$\tilde{u}\tilde{u}_x + (h(\tilde{\rho}))_x = \tilde{\phi}_x - \tilde{u}. \tag{2.3}$$

Subtracting (1.11a) from (1.1a), (2.3) from (2.2) and (1.11c) from (1.1c), respectively, we obtain the equations for the perturbation (ψ, η, ω)

$$\psi_t + ((\tilde{\rho} + \psi)(\tilde{u} + \eta) - \tilde{\rho}\tilde{u})_x = 0, \tag{2.4a}$$

$$\eta_t + \frac{1}{2}\left((\tilde{u} + \eta)^2 - \tilde{u}^2\right)_x + (h(\tilde{\rho} + \psi) - h(\tilde{\rho}))_x - \omega_x + \eta = 0, \tag{2.4b}$$

$$\omega_{xx} = \psi. \tag{2.4c}$$

The initial and the boundary data to the system (2.4) are derived from (1.4), (1.5), (1.6), (1.12) and (1.13) as

$$\psi(x,0) = \psi_0(x) := \rho_0(x) - \tilde{\rho}(x), \quad \eta(x,0) = \eta_0(x) := u_0(x) - \tilde{u}(x), \tag{2.5}$$

$$\psi(t,0) = \psi(t,1) = 0, \quad t \geq 0, \tag{2.6}$$

$$\omega(t,0) = \omega(t,1) = 0, \quad t \geq 0. \tag{2.7}$$

Since $(\tilde{\rho}, \tilde{u}, \tilde{\phi}) \in \mathfrak{X}_2([0,T])$ and ω satisfies (2.4c), the local existence of the solution (ψ, η, ω) to the initial boundary value problem (2.4), (2.5), (2.6) and (2.7) follows from Lemmas 1.1 and 2.1.

Corollary 2.2. *Suppose that the initial data (ψ_0, η_0) belongs to $H^2(\Omega)$ and $(\tilde{\rho} + \psi_0, \tilde{u} + \eta_0)$ satisfy (1.9). Then there exists a constant $T_2(\|\psi_0\|_2, \|\eta_0\|_2) > 0$ such that the initial boundary value problem (2.4), (2.5), (2.6) and (2.7) has a unique solution $(\psi, \eta, \omega) \in \mathfrak{X}_2([0,T_2]) \times \mathfrak{X}_2([0,T_2]) \times \mathfrak{X}_2^2([0,T_2])$.*

Owing to the above Corollary 2.2, it suffices to derive the a-priori uniform estimate (2.8) in order to complete the proof of the existence of the solution globally in time.

Proposition 2.3. *Let $(\psi, \eta, \omega)(t,x) \in \mathfrak{X}_2([0,T]) \times \mathfrak{X}_2([0,T]) \times \mathfrak{X}_2^2([0,T])$ be a solution to (2.4), (2.5), (2.6) and (2.7). Then there exists a positive constant ϵ_0 such that if $N(T) + \delta_b \leq \epsilon_0$, the following estimate holds for an arbitrary $t \in [0,T]$.*

$$\|(\psi,\eta)(t)\|_2^2 + \|\omega(t)\|_4^2 + \int_0^t \|(\psi,\eta)(\tau)\|_2^2 + \|\omega(\tau)\|_4^2 \, d\tau \leq C\|(\psi,\eta)(0)\|_2^2, \tag{2.8}$$

where C is a positive constant independent of T.

To show the estimate (2.8), it is convenient to use notations

$$N(t) := \sup_{0 \leq \tau \leq t} \|(\psi,\eta)(\tau)\|_2, \quad M^2(t) := \int_0^t \|\psi_x(\tau)\|^2 + \|\eta_x(\tau)\|^2 \, d\tau.$$

First, we derive the basic estimate to obtain the a-priori estimate (2.8). To this end, we define the energy form $\mathcal{E}$ as

$$\mathcal{E} := \frac{1}{2}\rho(u - \tilde{u})^2 + \Psi(\rho, \tilde{\rho}) + \frac{1}{2}(\phi - \tilde{\phi})_x^2, \quad \Psi(\rho, \tilde{\rho}) := \int_{\tilde{\rho}}^{\rho} h(\xi) - h(\tilde{\rho}) \, d\xi. \tag{2.9}$$

Notice that $\mathcal{E}$ is equivalent to $|(\psi,\eta,\omega_x)|^2$ if $|(\psi,\eta,\omega_x)|<c$, since $G(\rho,\tilde{\rho})$ is equivalent to $|\psi|^2$. Namely, there exist positive constants c_1 and C_1 such that

$$c_1|(\psi,\eta,\omega)|^2 \le \mathcal{E} \le C_1|(\psi,\eta,\omega)|^2 \tag{2.10}$$

for $|(\psi,\eta,\omega)|\le c$. Multiply the equation (2.4b) by $(\rho u-\tilde{\rho}\tilde{u})$ and then apply the integration by parts, to see the energy form $\mathcal{E}$ satisfies the following equation

$$\mathcal{E}_t+\tilde{\rho}\eta^2 = R_{1x}+R_2, \tag{2.11a}$$

$$R_1 := \omega\omega_{xt}+\omega(\rho\phi-\tilde{\rho}\tilde{\phi})-(h(\rho)-h(\tilde{\rho}))(\rho u-\tilde{\rho}\tilde{u})+(h(\rho)-h(\tilde{\rho}))(\psi\tilde{u}), \tag{2.11b}$$

$$\begin{aligned} R_2 := &-\left\{\frac{1}{2}(u^2-\tilde{u}^2)(\rho u-\tilde{\rho}\tilde{u})\right\}_x - u\psi\eta+(\rho u-\tilde{\rho}\tilde{u})_x\eta\tilde{u} \\ &+\left\{\frac{1}{2}(u^2-\tilde{u}^2)_x-\omega_x+\eta\right\}\psi\tilde{u}-(h(\rho)-h(\tilde{\rho}))\,(\psi\tilde{u})_x. \end{aligned} \tag{2.11c}$$

Applying the Sobolev inequality and using the equation (2.4c), we have the following estimate

$$|R_2|\le C(N(T)+\delta_b)\,|(\psi,\psi_x,\eta,\eta_x)|^2. \tag{2.12}$$

On the other hand, we have

$$\int_0^1 R_x\,dx=0$$

thanks to the boundary conditions (2.6) and (2.7). Therefore, the integration of the equation (2.9) over the domain $\Omega\times[0,t]$ shows the next lemma.

Lemma 2.4. *Assume the same conditions as in Proposition 2.3 hold. Then there exists a positive constant ϵ_0 such that if $N(T)+\delta_b\le\epsilon_0$, the following estimate holds for an arbitrary $t\in[0,T]$*

$$\|(\psi,\eta,\omega_x)(t)\|^2+\int_0^t\|(\psi,\eta,\omega_x)(\tau)\|^2\,d\tau \le C\|(\psi,\eta,\omega_x)(0)\|^2+C(N(T)+\delta_b)M^2(t), \tag{2.13}$$

where C is a positive constant independent of T.

The estimates of the higher order derivatives of (ψ,η,ω), stated in the next lemma, are obtained by an energy method. For details, see [9].

Lemma 2.5. *Assume the same conditions as in Proposition 2.3 hold. Then there exists a positive constant ϵ_1 such that if $N(T)+\delta_b\le\epsilon_1$, the following estimate holds for an arbitrary $t\in[0,T]$ and an integer $i=1,2$*

$$\begin{aligned} &\|(\partial_t^i\psi,\partial_t^i\eta_t,\partial_t^i\omega_x)(t)\|^2+\int_0^t\|(\partial_t^i\psi,\partial_t^i\eta,\partial_t^i\omega_x)(\tau)\|^2\,d\tau \\ &\qquad\le C\left(A_i^2(0)+A_{i-1}^2(t)+\int_0^t A_{i-1}^2(\tau)\,d\tau\right), \end{aligned} \tag{2.14}$$

$$A_i^2(t):=\sum_{j=0}^{i}\|(\partial_t^j\psi,\partial_t^j\eta)(t)\|^2,\quad \textit{for}\quad i=1,2, \tag{2.15}$$

where C is a positive constant independent of T.

Notice that $c\|(\psi,\eta)(t)\|_i \le A_i^2(t) \le C\|(\psi,\eta)(t)\|_i$ $(i = 1, 2)$, which holds from the equations (2.4a) and (2.4b) under the condition that $N(T) + \delta_b$ is sufficiently small. In addition, the estimate $\|\omega\|_4 \le C\|\psi\|_2$ follows from (2.4c). Hence we obtain the a-priori estimate (2.8) by using Lemma 2.4 and Lemma 2.5 for the small quantity $N(T) + \delta_b$. Consequently, we complete the proof of Proposition 2.3 and thus we see that Theorem 1.2 holds.

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