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Localized patterns in space reversible systems

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1 Introduction

Our interest of this paper is to study the stationary problem of the quintic Swift-Hohenberg equation

$$\frac{\partial u}{\partial t}=\left\{\nu-\left(1+\frac{\partial^2}{\partial x^2}\right)^2\right\}u+\mu u^3-u^5 \tag{1}$$

on the unbounded domain $x\in\mathbb{R}$, where ν and μ are parameters. In particular, we would like to focus on localized patterns with multi-bumps which appear in this equation.

First of all, let us explain the motivation of this work. Figure 1 is a numerical result of a bifurcation diagram of stationary solutions under the periodic boundary condition $u(x+L)=u(x)$ for $\mu=3$. Here we take a sufficiently large system size L. Several important properties included in the bifurcation diagram are summarized as follows: (i) A pure periodic solution bifurcates from the trivial solution $u=0$ as a subcritical pitchfork type at $\nu\approx 0$. (ii) A mixed mode bifurcation branch emerges as a secondary bifurcation from the pure mode branch. (iii) The mixed mode branch has a snaking structure, i.e. it repeats saddle-node bifurcations at $\nu\approx-1.735$ and $\nu\approx-1.110$ along the branch. (iv) Wave profiles of the stationary solutions on the snaking branch form localized patterns. Some of these patterns are shown in Figure 2. We observe that the number of bumps of a localized pattern increases when we choose a numerical solution on upper layers of the snaking branch. (v) The number of turnings on the snaking branch increases when we take a larger system size. Consequently, we observe new localized patterns which have more bumps.

These numerical observations, especially (v), imply that the stationary problem of (1) may have infinitely many localized pattern solutions for some parameter region in $\nu<0,\mu>0$. Moreover, there may exist a heteroclinic cycle between the trivial solution and a periodic solution since we can numerically construct a localized solution which has a quite large region of the periodic structure as shown in [10]. In other words, the existence of the heteroclinic cycle may generate infinitely many localized pattern solutions in the parameter region where the snaking bifurcation branch appears. Our motivation of this work is to study these arguments from the mathematical point of view.

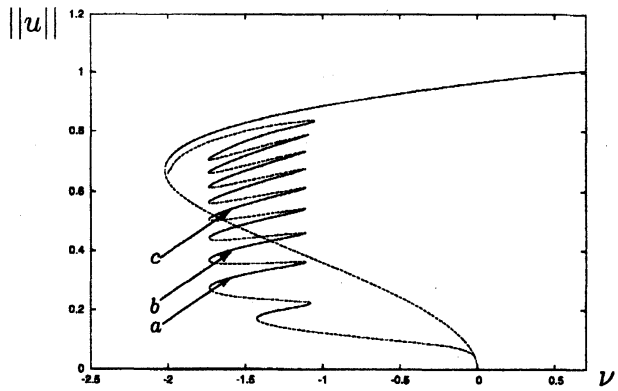


Figure 1: Bifurcation diagram at $\mu = 3$

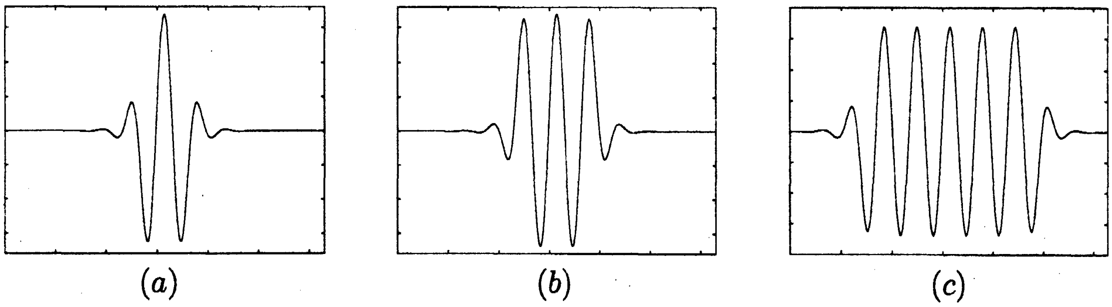


Figure 2: Localized patterns on the snaking branch

In the paper [6], they study the bifurcation structure around the trivial solution under a periodic boundary condition by using a center manifold reduction. From their analysis, the existence of pure periodic solutions and the existence of secondary bifurcations are consistent with the numerical result shown in Figure 1. However, the snaking structure of the mixed mode branch is not clear by their analysis although they relate an imperfection of pitchfork bifurcations to "Z"-shape for the mixed mode branch. Instead, they prove the existence of numerically obtained localized pattern solutions by a computer assisted proof based on the Conley index theory [14].

There are several related works about this problem. In the paper [10], the stationary localized solutions of the equation (1) have been first observed by numerical simulations in their search of an equation which has stable localized solutions. We refer to [10] for the physical background of this equation. Their numerical results show the existence of stable localized solutions not only in one space dimensional case but also in two dimensional case. Furthermore, they heuristically explain the relation between the existence of stable localized patterns and the coexistence of the stable trivial solution and stable spatially periodic solutions. In the papers [2] and [13], they treat a similar equation, which has a quadratic term instead of the quintic term in (1), and study an approximate problem derived from a truncation of higher order terms of a normal form. Their analysis shows the existence of homoclinic orbits connecting the trivial solution and the existence of heteroclinic orbits connecting the trivial solution and periodic solutions in the truncated normal form. In [13], they also show some numerical results related to snaking bifurcation branches as is shown in Figure 1.

The object of this work is to discuss the persistence of these homoclinic and heteroclinic orbits obtained in a truncated normal form for (1) under the addition of higher order terms. Especially, our main result in this paper is to prove the existence of heteroclinic orbits connecting the trivial solution and periodic solutions in the original stationary problem (1). This paper is organized as follows.

In Section 2, we formulate the stationary problem of the quintic Swift-Hohenberg equation into a Hamiltonian system. The Hamiltonian-Hopf bifurcation is observed at $\nu = 0$ and a normal form is derived around this singular point. We show that, in a truncated normal form, there exists a curve l with codimension one in the parameter space (μ, ν) such that a heteroclinic cycle connecting the trivial solution and a periodic solution exists on the curve. Furthermore, in two regions called I and II (see Figure 4) separated by the curve, we have homoclinic orbits connecting the trivial solution and homoclinic orbits connecting periodic solutions, respectively, in a truncated normal form.

The persistence of these connecting orbits is studied in Section 3. In Section 3.1, we consider how the remainder terms of the normal form affect the homoclinic orbits derived in the truncated normal form. In the paper [7], they study the persistence of symmetric homoclinic orbits in reversible systems. As a corollary of their result, it is concluded that the homoclinic orbits in the truncated normal form persist under the addition of the remainder terms. We next study the persistence of the heteroclinic orbits by using the Melnikov theory (e.g. [3][4][12]). We show that there exists a parameter region sufficiently close to the curve l such that these heteroclinic orbits persist under the addition of the remainder terms.

Finally, we discuss the relationship between the existence of the heteroclinic orbit and that of infinitely many localized pattern solutions. Under a generic assumption of nondegeneracy, it is concluded that there exist infinitely many localized pattern solutions converging to the heteroclinic orbit. We also argue some related works about the snaking bifurcation structure and infinitely many localized patterns at the end of this paper.

2 Normal form analysis

We derive a normal form for the stationary problem of the equation (1) by using the arguments in [1][9]. Let us introduce a new coordinate defined by

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} := \begin{pmatrix} \frac{3}{2\sqrt{2}} & 0 & 0 & \frac{-1}{2\sqrt{2}} \\ 0 & \frac{3}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} u \\ u_x \\ -2u_x - u_{xxx} \\ u_{xx} \end{pmatrix}.$$

Then the stationary problem of the equation (1) can be expressed by a Hamiltonian system

$$\frac{dz}{dx} = J\nabla_z H(z), \tag{2}$$

where $z = (z_1, z_2, z_3, z_4)^t$, $\nabla_z = (\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4})$ and

$$H(z) = (z_2z_3 - z_1z_4) + \frac{1}{2}(z_3^2 + z_4^2) + \frac{\nu}{16}(2z_1 + z_4)^2 + \frac{\mu}{256}(2z_1 + z_4)^4 - \frac{1}{3072}(2z_1 + z_4)^6. \tag{3}$$

In addition, the symplectic matrix J is given by

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

where we denote the $n \times n$ identity matrix by I_n.

At first, let us study the linearization around the trivial solution $z = 0$. The eigenvalues λ of the linearized matrix

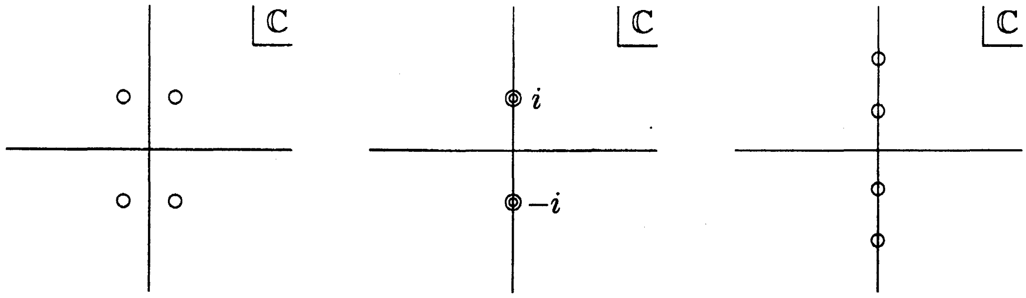
$$A(\nu) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 + \frac{\nu}{4} & 0 & 0 & 1 + \frac{\nu}{8} \\ -\frac{\nu}{2} & 0 & 0 & 1 - \frac{\nu}{4} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

are determined by $\lambda^2 = -1 \pm \sqrt{\nu}$ and Figure 3 shows the location of these eigenvalues in the complex plane for each ν. From this figure, the trivial solution $z = 0$ is a center type for $0 < \nu < 1$ and Lyapunov's center theorem (e.g. [9]) guarantees the existence of two families of periodic orbits around $z = 0$. The eigenvalues collide on the imaginary axis and the linearized matrix $A(0)$ becomes nilpotent at $\nu = 0$, called as Hamiltonian-Hopf [8] or 1:1 resonance [7] point. For $\nu < 0$, the trivial solution turns out to be a saddle-focus fixed point and a snaking bifurcation branch is numerically observed in this region (see Figure 1).

In order to transform the Hamiltonian (3) into a normal form, let us consider the adjoint problem of the linearized Hamiltonian vector field, i.e.

$$\frac{dz}{dx} = A(0)^t z = J\nabla_z H_0^t(z),$$

$$H_0^t(z) = z_1z_4 - z_2z_3 - \frac{1}{2}(z_1^2 + z_2^2),$$



$\nu < 0$ $\nu = 0$ $0 < \nu < 1$

Figure 3: Eigenvalues of the linearized matrix A.

where $H_0^t(z)$ corresponds to the Hamiltonian of the adjoint problem. We note that $I := z_1^2 + z_2^2$ and $K := z_1 z_4 - z_2 z_3$ satisfy

$$\{H_0^t, I\} = \{H_0^t, K\} = 0,$$

where $\{\bullet, \bullet\}$ expresses the Poisson bracket. Then, the normal form theory [1][9] enables us to construct a Lie transformation to the following normal form

$$\begin{aligned} H(z) = -K + \frac{1}{2}(z_3^2 + z_4^2) + \frac{\nu}{8}(I + K) + a_1 I^2 + a_2 IK + a_3 K^2 + b_1 I^3 + b_2 I^2 K + b_3 I K^2 + b_4 K^3 \\ + c_1 I^4 + c_2 I^3 K + c_3 I^2 K^2 + c_4 I K^3 + c_5 K^4 + o(z^8), \qquad (4) \end{aligned}$$

where $|\nu|$ is assumed to be small. Note that the coefficients a_i, b_i, c_i depend on μ and can be explicitly calculated. In the paper, the coefficients a_1, b_1, c_1 become important and are given by

$$\begin{aligned} a_1 &= \frac{3}{128}\mu, \\ b_1 &= \frac{121}{32768}(\mu^2 - \beta^2), \quad \beta = \sqrt{\frac{640}{363}}, \\ c_1 &= \frac{22203}{16777216}(\mu^2 - \gamma^2)\mu, \quad \gamma = \sqrt{\frac{50816}{22203}}. \end{aligned}$$

Let us decompose the normal form (4) into two parts defined by

$$\begin{aligned} H_{in}(z) &:= -K + \frac{1}{2}(z_3^2 + z_4^2) + \frac{\nu}{8}(I + K) + a_1 I^2 + a_2 IK + a_3 K^2 + b_1 I^3 + b_2 I^2 K + b_3 I K^2 + b_4 K^3, \\ H_{non}(z) &:= H(z) - H_{in}(z). \end{aligned}$$

For the rest of this section, we study the truncated normal form H_{in} and its corresponding Hamiltonian dynamical system.

Let us introduce a new coordinate (r, θ, R, Θ) such as

$$z_1 = r\cos\theta, \qquad R = z_3\cos\theta + z_4\sin\theta,$$
$$z_2 = r\sin\theta, \qquad \Theta = -z_3 r\sin\theta + z_4 r\cos\theta$$

in order to simplify the vector field. We note that $K = \Theta$ and $\{H_{in}, K\} = 0$. Thus, K is a first integral for the Hamiltonian system defined by H_{in}. Since our interest is to study an orbit which connects to the trivial solution, let us investigate the dynamics on the surface defined by $K = 0$. It is described as follows

$$\frac{dr}{dx} = R,$$
$$\frac{dR}{dx} = -r\left(6b_1r^4 + 4a_1r^2 + \frac{\nu}{4}\right). \tag{5}$$

Note that a nontrivial fixed point (r, R) of the equation (5) corresponds to a periodic solution with the angular velocity $\frac{\partial H_{in}}{\partial\Theta}(r, R, \Theta = 0)$ in the z-coordinate.

We study the dynamics of (5) for each parameter value (μ, ν). Figure 4 shows the bifurcation diagram for $\beta > \mu > 0, \nu < 0$ and Figure 5 describes the vector field (5) in (r, R) phase plane on I, II, and l. Two curves l and $\tilde{l}$ determined by

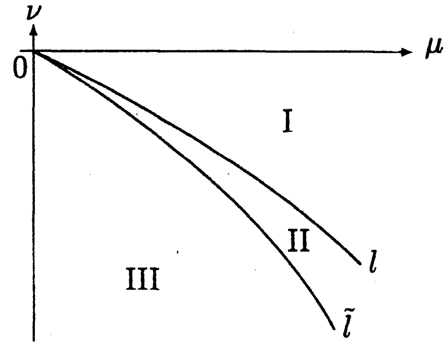


Figure 4: Bifurcation diagram for (μ, ν) parameter space.

$l : \nu = 2a_1^2/b_1$ and $\tilde{l} : \nu = 8a_1^2/3b_1$, respectively, decompose the parameter space into three regions shown by I, II, III. Except for III, we have two nontrivial fixed points r_-, r_+ $(r_- < r_+)$ determined by the following

$$r_\mp^2 = \frac{1}{6b_1}\left(-2a_1 \pm \sqrt{4a_1^2 - (3/2)b_1\nu}\right).$$

By checking eigenvalues of a linearized matrix at each fixed point, we show that r_- is a center type and r_+ is a saddle type. It is also observed that, on the curve l, there exists a separatrix consisting of heteroclinic orbits between the trivial solution and the fixed point r_+. Moreover, homoclinic orbits connecting the trivial solution in the region I and homoclinic orbits connecting the fixed point r_+ in the region II

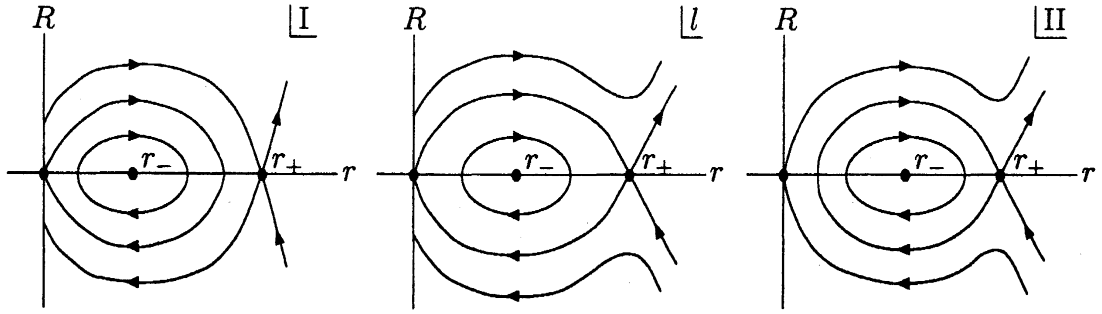


Figure 5: Dynamics on the surface of $K = 0$

exist, respectively. On the curve $\tilde{l}$, two fixed points r_-, r_+ collide and disappear in the region III.

Let us finally remark that the other cases ($\nu > 0$ and $\mu < 0, \nu < 0$) can be also studied easily by the similar way, however the dynamics is rather simple and our interest of study is not in these regions. Therefore, we omit to discuss these cases in this paper.

3 Persistence of connecting orbits

This section deals with the persistence of homoclinic and heteroclinic orbits obtained by the truncated normal form H_{in} under the addition of the higher order term H_{non}. That is to say we consider the existence of these connecting orbits in the original dynamical system (2).

3.1 Persistence of homoclinic orbits

Let us consider the persistence of the homoclinic orbits appearing in the region I and II. We recall that a vector field

$$\frac{dx}{dt} = f(x), \quad x, f(x) \in \mathbb{R}^n \tag{6}$$

is called reversible with respect to a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ with $S^2 = I_n$ if the vector field satisfies $f(Sx) = -Sf(x)$. A distinctive feature of reversible systems is that if $x(t)$ is a solution of (6) then so $Sx(-t)$ is. We call an orbit $\gamma := \{x(t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^n$ symmetric if $S\gamma = \gamma$. For a general reference of reversible systems, we refer to [11].

It should be noted that the dynamical system (2) is reversible with respect to an involution $S : (z_1, z_2, z_3, z_4) \mapsto (z_1, -z_2, -z_3, z_4)$. In addition, $\mathrm{Fix}(S) := \{z \in \mathbb{R}^4 \mid Sz = z\}$ forms a two dimensional subspace in $\mathbb{R}^4$. For example, it is easy to check that homoclinic orbits in the region I with $R(0) = 0$ and $\theta(0) = 0, \pi$ are symmetric. The persistence of symmetric homoclinic orbits in reversible systems has been discussed on the general setting in the paper [7]. Thus, we briefly explain the persistence property of the homoclinic orbits obtained in Section 2 by the argument in [7].

At first, let us identify $\mathbb{R}^4$ with $\mathbb{C}^2$ through complex variables $A := z_1 + iz_2$ and $B := z_3 + iz_4$ in order to easily observe a rotation symmetry mentioned later. It should be noted that the involution S explained above is represented by $S : (A, B) \mapsto (\bar{A}, -\bar{B})$. We introduce a scaling which characterizes the dynamics studied in Section 2. Since the homoclinic orbits derived in the truncated normal form have the order $O(|\nu|^{\frac{1}{4}})$ (see (5)), let us consider the following scaling

$$A \to |\nu|^{\frac{1}{4}} A, \quad B \to |\nu|^{\frac{1}{4}} B$$

and set $\epsilon := |\nu|^{\frac{3}{2}}$. Then, from the normal form (4), the system for $Y := (A, B)^t$ can be described as follows

$$\frac{dY}{dx} = F(Y) + \tilde{F}(Y, \epsilon), \tag{7}$$

where $F(Y) = (F_1(Y), F_2(Y))^t$ is given by

$$\begin{aligned} F_1(Y) &= -iA + B + iAP(Y), \\ F_2(Y) &= -iB + iBP(Y) + AQ(Y), \\ P(Y) &= \frac{\nu}{8} + \sqrt{|\nu|}(a_2 I + 2a_3 K) + |\nu|(b_2 I^2 + 2b_3 IK + 3b_4 K^2), \\ Q(Y) &= -\frac{\nu}{4} - \sqrt{|\nu|}(4a_1 I + 2a_2 K) - |\nu|(6b_1 I^2 + 4b_2 IK + 2b_3 K^2) \end{aligned}$$

and $\tilde{F}(Y, \epsilon)$ is composed of the higher order terms of the normal form H_{non} with $O(\epsilon)$ and satisfies $\tilde{F}(Y, 0) = 0$. In this sence, $\tilde{F}(Y, \epsilon)$ is regarded as a perturbation to the unperturbed system

$$\frac{dY}{dx} = F(Y) \tag{8}$$

which possesses the homoclinic orbits studied in Section 2. Thus, our problem is to investigate the persistence of these homoclinic orbits under the addition of the perturbation term $\tilde{F}(Y, \epsilon)$.

Let us denote a rotation group by $R_\phi : (A, B) \mapsto (Ae^{i\phi}, Be^{i\phi})$. Then, it is easy to check that the unperturbed system (8) has a rotation symmetry, $F(R_\phi Y) = R_\phi F(Y)$. In the following, we discuss the persistence of symmetric homoclinic orbits connecting the trivial solution. Let $q_0(x)$ be a symmetric homoclinic orbit with the property that $q_0 := q_0(0) \in \mathrm{Fix}(S)$ and $\lim_{x \to \pm\infty} q_0(x) = 0$. Then, an unstable manifold of $Y = 0$ with respect to the unperturbed system (8) can be described by

$$W_0^u(0) = \{R_\phi q_0(x) \mid x \in \mathbb{R}, \phi \in [0, 2\pi]\}.$$

Obviously, $q_0 \in \mathrm{Fix}(S) \cap W_0^u(0)$. Then, the tangent space of $W_0^u(0)$ at q_0 consists of

$$T_{q_0} W_0^u(0) = \mathrm{Span}\{Lq_0, \dot{q}_0(0)\},$$

where $L = iI_2$. Hence, from the properties $SL = -LS$ and $S\dot{q}_0(0) = -\dot{q}_0(0)$, it is shown that the intersection of $W_0^u(0)$ and $\mathrm{Fix}(S)$ is transverse.

Let $W_\epsilon^u(0)$ be an unstable manifold of $Y = 0$ with respect to the perturbed system (7). Due to the transverse intersection of $W_0^u(0)$ and $\mathrm{Fix}(S)$, there exists a unique point $q_\epsilon \in \mathrm{Fix}(S) \cap W_\epsilon^u(0)$ for each ϵ such that $q_\epsilon \to q_0$ as $\epsilon \to 0$. Let us denote by $q_\epsilon(x)$ the solution passing through q_ϵ at $x = 0$. Then, the reversibility leads to $\lim_{x\to\infty} q_\epsilon(x) = \lim_{x\to\infty} Sq_\epsilon(-x) = 0$ and $q_\epsilon(x)$ also stays on a stable manifold $W_\epsilon^s(0)$ of $Y = 0$ for (7). It means that the symmetric homoclinic orbits persist for small ϵ in the system (7), i.e. the original system (2).

By the similar manner, the persistence of the symmetric homoclinic orbits derived in the region II can be shown. Finally, we have the following as a corollary of [7].

Corollary 1 *For small $|\nu|$, there exist symmetric homoclinic orbits in the region I and II shown in Figure 4.*

3.2 Persistence of heteroclinic orbits

This section is devoted to discussing the persistence of heteroclinic orbits obtained on the curve l in Section 2. For this purpose, we adopt the method of Melnikov, which is a well-known technique to measure the distance between a stable and an unstable manifolds. See [3][4][12] and references therein for the details of Melnikov's method.

At first, let us introduce scaling parameters $\epsilon = (\epsilon_1, \epsilon_2)$ as $\epsilon_1 = |\nu|^{\frac{3}{2}}$ and $\epsilon_2 = \nu - \frac{2a_1^2}{b_1}$. Note that not only ϵ_1, which is already used in Section 3.1, but also ϵ_2 is needed in order to focus on the neighborhood of the curve l in the parameter space. Let us denote the set of non-negative real numbers by $\mathbb{R}_+ := \{r \in \mathbb{R} \mid r \geq 0\}$ and define a polar coordinate $(r_0, r_1, \psi_0, \psi_1) \in \mathbb{R}_+ \times \mathbb{R}_+ \times S^1 \times S^1$ such that

$$z_1 + iz_2 = |\nu|^{\frac{1}{4}} r_0 e^{-i\left(t+\psi_0-\frac{|\nu|^{3/2}}{2}\right)}, \quad z_3 + iz_4 = |\nu|^{\frac{1}{4}} r_1 e^{-i\left(t+\psi_1+\frac{|\nu|^{3/2}}{2}\right)}.$$

Then we have the following dynamical system for $W = (r_0, r_1, \psi_0, \psi_1)^t$:

$$\frac{dW}{dx} = f(W) + \tilde{f}(W, \epsilon), \tag{9}$$

where $f(W) = (f_{r_0}(W), f_{r_1}(W), f_{\psi_0}(W), f_{\psi_1}(W))^t$ is given by

$$\begin{aligned}
f_{r_0}(W) &= r_1 \cos(\psi_1 - \psi_0), \\
f_{r_1}(W) &= r_0 \cos(\psi_1 - \psi_0) Q(W), \\
f_{\psi_0}(W) &= \frac{r_1}{r_0} \sin(\psi_1 - \psi_0) - P(W), \\
f_{\psi_1}(W) &= -\frac{r_0}{r_1} \sin(\psi_1 - \psi_0) Q(W) - P(W),
\end{aligned}$$

$$P(W) = \frac{a_1^2}{4b_1} + \sqrt{|\nu|}(a_2 I + 2a_3 K) + |\nu|(b_2 I^2 + 2b_3 IK + 3b_4 K^2),$$
$$Q(W) = -\frac{a_1^2}{2b_1} - \sqrt{|\nu|}(4a_1 I + 2a_2 K) - |\nu|(6b_1 I^2 + 4b_2 IK + 2b_3 K^2) \quad (10)$$

with $I = r_0^2$ and $K = -r_0 r_1 \sin(\psi_1 - \psi_0)$. The second term $\tilde{f}(W, \epsilon)$ is regarded as a higher order term with $O(\epsilon)$ and the leading terms of $\tilde{f}(W, \epsilon)$ with respect to ϵ are composed of

$$\tilde{f}_{r_0}(W, \epsilon) = -r_1 \sin(\psi_1 - \psi_0)\epsilon_1 + o(\epsilon^2),$$
$$\tilde{f}_{r_1}(W, \epsilon) = \frac{a_1^2}{2b_1} r_0 \sin(\psi_1 - \psi_0)\epsilon_1 + r_0 \cos(\psi_1 - \psi_0)\tilde{Q}(W, \epsilon) + o(\epsilon^2),$$
$$\tilde{f}_{\psi_0}(W, \epsilon) = \frac{r_1}{r_0} \cos(\psi_1 - \psi_0)\epsilon_1 - \tilde{P}(W, \epsilon) + o(\epsilon^2),$$
$$\tilde{f}_{\psi_1}(W, \epsilon) = \frac{a_1^2}{2b_1}\frac{r_0}{r_1} \cos(\psi_1 - \psi_0)\epsilon_1 - \frac{r_0}{r_1} \sin(\psi_1 - \psi_0)\tilde{Q}(W, \epsilon) - \tilde{P}(W, \epsilon) + o(\epsilon^2),$$
$$\tilde{P}(W, \epsilon) = \frac{\epsilon_2}{8} + \epsilon_1(c_2 I^3 + 2c_3 I^2 K + 3c_4 IK^2 + 4c_5 K^3),$$
$$\tilde{Q}(W, \epsilon) = -\frac{\epsilon_2}{4} - \epsilon_1(8c_1 I^3 + 6c_2 I^2 K + 4c_3 IK^2 + 2c_4 K^3). \quad (11)$$

Essentially, the dynamics determined by the Hamiltonian H_{in} in Section 2 is equivalent to that for the unperturbed system

$$\frac{dW}{dx} = f(W). \quad (12)$$

Let us define $G(I, K) := \int Q(W)dI$ and $J := r_1^2 - G(I, K)$. Then it is easy to check that J and K form independent first integrals of the unperturbed system. As is discussed in Section 2, we can reduce (12) into the following system

$$\left(\frac{dI}{dx}\right)^2 = 4[I\{G(I, K) + J\} - K^2],$$
$$\frac{d(\psi_1 - \psi_0)}{dx} = \frac{K}{Ir_1^2}[IQ(W) + G(I, K) + J] \quad (13)$$

by using these first integrals. Obviously, fixed points in (13) correspond to periodic solutions in (12). Moreover, these fixed points are obtained as double roots of the following function

$$g(I, J, K) := I\{G(I, K) + J\} - K^2$$

with respect to I. We denote the derivative of $g(I, J, K)$ with respect to I by $\tilde{g}(I, J, K) := \frac{\partial g}{\partial I}(I, J, K)$.

If we fix $K = 0$, the dynamics studied in (5) can be represented in (13). Especially, we obtain double roots I_-, I_+ given by

$$I_- = -\frac{a_1}{6b_1\sqrt{|\nu|}}, \quad I_+ = -\frac{a_1}{2b_1\sqrt{|\nu|}},$$

where $a_1 > 0$, $b_1 < 0$ in our setting. These two fixed points correspond to r_-, r_+ obtained in Section 2, respectively. Let us denote by J_-, J_+ the values of J for I_-, I_+, respectively. That is to say, $I_\pm$ turn out to be the roots of $g(I_\pm, J_\pm, 0) = \tilde{g}(I_\pm, J_\pm, 0) = 0$. In addition, due to $g(I, J_+, 0) > 0$ for $0 < I < I_+$, we conclude the existence of the heteroclinic orbit $q(x)$ from the trivial solution to I_+, which corresponds to the heteroclinic orbit shown in Figure 5 for (r, R)-coordinate.

Now let us study the derivative

$$\Xi(I,J,K) := \begin{pmatrix} \frac{\partial g}{\partial I}(I,J,K) & \frac{\partial g}{\partial J}(I,J,K) \\ \frac{\partial \tilde{g}}{\partial I}(I,J,K) & \frac{\partial \tilde{g}}{\partial J}(I,J,K) \end{pmatrix}.$$

From the explicit form of $Q(W)$, the determinant of $\Xi(I, J, K)$ can be expressed by

$$\det\left[\Xi(I,J,K)\right] = 6\sqrt{|\nu|}a_1 I^2 + |\nu|(16b_1 I^3 + 6b_2 I^2 K) + J.$$

Due to $I_+ \neq 0$ and $J_+ = 0$, the determinant at $(I_+, J_+, 0)$ is given by

$$\det\left[\Xi(I_+, J_+, 0)\right] = 2\sqrt{|\nu|}I_+^2(3a_1 + 8\sqrt{|\nu|}b_1 I_+) = -2\sqrt{|\nu|}I_+^2 a_1 \neq 0.$$

Thus, it is concluded that the double root $(I_+, J_+, 0)$ of $g(I, J, K) = 0$ can be continuated for small $|K|$. More precisely, there exist functions $I(K)$ and $J(K)$ for small $|K|$ with $I(0) = I_+$ and $J(0) = J_+$ such that $g(I(K), J(K), K) = \tilde{g}(I(K), J(K), K) = 0$.

Let us denote by $R_{\Omega_K x} W_K$ the one parameter family of periodic orbits obtained in the above argument. Here R_ϕ is a representation of the rotation group in the polar coordinate and the angular velocity Ω_K is determined by $\Omega_K = f_{\psi_0}(W)|_{I=I(K),K}$. It should be noted that

$$\psi_1 - \psi_0 = -\operatorname{sgn}(K)\frac{\pi}{2} \tag{14}$$

for the periodic orbits, where $\operatorname{sgn}(K)$ represents the sign of K. Consequently, W_K is determined by $r_0^2 = I(K)$, $r_1^2 = J(K) + G(I(K), K)$, and (14). It should be remarked that the ω-limit set and the α-limit set of the heteroclinic orbit $q(x)$ consist of $\omega(q(x)) = \{R_{\Omega_0 x} W_0 | x \in \mathbb{R}\}$ and $\alpha(q(x)) = \{0\}$.

From the argument in [7], the one parameter family of the periodic orbits $R_{\Omega_K x} W_K$ persists under small perturbation. Let us denote these periodic orbits for (9) by $X_K(x, \epsilon)$ which satisfies $X_K(x, \epsilon) \to R_{\Omega_K x} W_K$ as $\epsilon \to 0$. We define an unstable manifold of the trivial solution and a stable manifold of the periodic orbit $X_K(\cdot, \epsilon)$ for (9) as

$$\begin{aligned} W_\epsilon^u(0) &:= \left\{\xi \in \mathbb{R}_+ \times \mathbb{R}_+ \times S^1 \times S^1 \mid \alpha(\xi) = 0\right\}, \\ W_\epsilon^s(X_K(\cdot, \epsilon)) &:= \left\{\xi \in \mathbb{R}_+ \times \mathbb{R}_+ \times S^1 \times S^1 \mid \omega(\xi) = l_K(\epsilon)\right\}, \end{aligned}$$

where $l_K(\epsilon) := \{X_K(x, \epsilon) \mid x \in \mathbb{R}\}$. The direct calculation in the unperturbed system shows that the unstable manifold and the stable manifold have the same dimension, i.e. $\dim(W_0^u(0)) = \dim(W_0^s(X_K(\cdot, 0))) = 2$. Furthermore, $W_0^u(0)$ and $W_0^s(X_0(\cdot, 0))$

have an intersection and, due to the rotation symmetry in the unperturbed system, the intersection can be represented by

$$\Gamma := \{R_\phi q(x) \mid x \in \mathbb{R},\ \phi \in S^1\} = W_0^u(0) \cap W_0^s(X_0(\cdot, 0)).$$

Thus, the intersection of these invariant manifolds is degenerate and gives the two dimensional manifold Γ parametrized by (x, ϕ).

Let us characterize one heteroclinic orbit

$$\begin{aligned} h(x; x_0, \phi_0) &= (r_0(x; x_0, \phi_0), r_1(x; x_0, \phi_0), \psi_0(x; x_0, \phi_0), \psi_1(x; x_0, \phi_0)), \\ h(0; x_0, \phi_0) &= R_{\phi_0} q(x_0) \end{aligned}$$

on Γ in the unperturbed system. We set $p_0 = R_{\phi_0} q(x_0)$ and abbreviate x_0, ϕ_0 in the notation $h(x) = h(x; x_0, \phi_0)$ unless any confusion occurs. Recall that K is a first integral in the unperturbed system. It means $\psi_1 - \psi_0 = 0$ along the heteroclinic orbits on Γ.

We introduce the following two vectors

$$\begin{aligned} e_{p_0}^1(x) &:= -r_0(x; x_0, \phi_0) Q(h(x)) \frac{\partial}{\partial r_0} + r_1(x; x_0, \phi_0) \frac{\partial}{\partial r_1}, \\ e_{p_0}^2(x) &:= -r_0(x; x_0, \phi_0) r_1(x; x_0, \phi_0) \left(\frac{\partial}{\partial \psi_1} - \frac{\partial}{\partial \psi_0} \right). \end{aligned} \tag{15}$$

Then, it is easy to check that $e_{p_0}^i(x), i = 1, 2$, are orthogonal to Γ at $h(x) \in \Gamma$. Especially, a cross section

$$\Sigma_{p_0} := \operatorname{Span}\{e_{p_0}^1, e_{p_0}^2\}$$

at p_0 and Γ intersect transversely, where $e_{p_0}^i := e_{p_0}^i(0), i = 1, 2$.

Let $p_\epsilon^u, p_{\epsilon,K}^s$ be the intersections of $\Sigma_{p_0} \cap W_\epsilon^u(0)$, $\Sigma_{p_0} \cap W_\epsilon^s(X_K(\cdot, \epsilon))$, respectively. We measure the separation of the invariant manifolds by

$$d_i(x_0, \phi_0, \epsilon, K) := \left(p_\epsilon^u - p_{\epsilon,K}^s\right) \cdot e_{p_0}^i, \quad i = 1, 2.$$

Obviously, it is concluded that if $d_i(x_0, \phi_0, \epsilon, K) = 0, i = 1, 2$, then $W_\epsilon^u(0) \cap W_\epsilon^s(X_K(\cdot, \epsilon)) \neq \emptyset$. Let us define

$$\begin{aligned} M_{ij}(x_0, \phi_0) &:= \frac{\partial d_i}{\partial \epsilon_j}(x_0, \phi_0, 0, 0), \\ M_{i3}(x_0, \phi_0) &:= \frac{\partial d_i}{\partial K}(x_0, \phi_0, 0, 0) \end{aligned}$$

for $i, j = 1, 2$. Then $d_i(x_0, \phi_0, \epsilon, K)$ can be expressed by

$$\begin{pmatrix} d_1(x_0, \phi_0, \epsilon, K) \\ d_2(x_0, \phi_0, \epsilon, K) \end{pmatrix} = \begin{pmatrix} M_{11}(x_0, \phi_0) & M_{12}(x_0, \phi_0) & M_{13}(x_0, \phi_0) \\ M_{21}(x_0, \phi_0) & M_{22}(x_0, \phi_0) & M_{23}(x_0, \phi_0) \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ K \end{pmatrix} + O\left((\epsilon + K)^2\right). \tag{16}$$

The following two lemmas are crucial for the main result of this paper.

Lemma 2 *$M_{ij}(x_0,\phi_0), i,j=1,2$, can be expressed by*

$$M_{ij}(x_0,\phi_0)=\int_{-\infty}^{\infty} e^i_{p_0}(x)\cdot\frac{\partial\tilde{f}}{\partial\epsilon_j}(h(x),0)\,dx.$$

Lemma 3 *For any $(x_0,\phi_0)\in\mathbb{R}\times S^1$, the following holds:*

$$\det\begin{pmatrix} M_{11}(x_0,\phi_0) & M_{12}(x_0,\phi_0) \\ M_{21}(x_0,\phi_0) & M_{22}(x_0,\phi_0)\end{pmatrix}\neq 0.$$

We would like to refer [5] for the proofs of Lemma 2, 3. From these lemmas, the implicit function theorem guarantees that there exist unique functions $\epsilon(x,\phi,K)=(\epsilon_1(x,\phi,K),\epsilon_2(x,\phi,K))$ such that $d_i(x,\phi,\epsilon(x,\phi,K),K)=0, i,j=1,2$, for a small neighborhood of $(x_0,\phi_0,K=0)$. Since the initial point (x_0,ϕ_0) is arbitrary and the functions $(\epsilon_1(x,\phi,K),\epsilon_2(x,\phi,K))$ are unique, the domain of the functions with respect to the first two components (x,ϕ) can be extended to the whole space $\mathbb{R}\times S^1$. Obviously, there also exists an another heteroclinic orbit at $(\epsilon_1(x,\phi,K),\epsilon_2(x,\phi,K))$ from the periodic solution $X_K(x,\epsilon)$ to the trivial solution, due to the reversibility of the dynamical system (9). Furthermore, we note that $\epsilon(x,\phi,K)\not\equiv 0$ since $W_0^u(0)\cap W_0^s(X_K(\cdot,0))=\emptyset$ for $K\neq 0$. As a conclusion, we obtain the following theorem.

Theorem 4 *There exists a heteroclinic cycle connecting the trivial solution and the periodic orbit $X_K(\cdot,\epsilon)$ in the parameter region $(\epsilon_1(x,\phi,K),\epsilon_2(x,\phi,K))$ for $(x,\phi)\in\mathbb{R}\times S^1$ and small $|K|$.*

4 Discussion

In summary, from the argument in Section 3, we prove the existence of homoclinic orbits connecting the trivial solution in the parameter region I shown in Figure 4 and the existence of homoclinic orbits connecting periodic solutions in the region II (Corollary 1). Moreover, it is concluded in Theorem 4 that there exists a parameter region close to the curve l such that heteroclinic orbits connecting the trivial solution and periodic solutions do exist. Recall that, as is mentioned in Section 1, numerical results shown in Figure 1, 2 and in [10] suggest that the heteroclinic orbits may relate the existence of infinitely many localized pattern solutions. Now let us consider its underlying mechanism for this problem.

Suppose that the intersection proved in Theorem 4 is nondegenerate, i.e.

$$\dim\left[T_pW_\epsilon^u(0)\cap T_pW_\epsilon^s(X_K(\cdot,\epsilon))\right]=1$$

for some $p\in W_\epsilon^u(0)\cap W_\epsilon^s(X_K(\cdot,\epsilon))\neq\emptyset$. Let us note that this is the most generic situation for the intersect of the unstable and the stable manifolds. Under this generic condition, as is discussed in [13], we can relate the existence of the heteroclinic orbit to that of the infinitely many symmetric homoclinic orbits due to λ-lemma [4] and the reversible property. Roughly speaking, the λ-lemma implies a heteroclinic tangle composed of $W_\epsilon^u(0)$ and $W_\epsilon^s(X_K(\cdot,\epsilon))$. Then, this heteroclinic tangle leads to infinitely many intersection points of $W_\epsilon^u(0)$ and $\mathrm{Fix}(S)$, when $W_\epsilon^u(0)$ approaches $W_\epsilon^s(X_K(\cdot,\epsilon))$. Finally, from the argument in Section 3.1, these intersection points construct symmetric homoclinic orbits.

However, in general, it is not an easy problem to check whether the intersection of unstable and stable manifolds is nondegenerate or not. Especially, our strategy in this paper is based on the normal form analysis and the problem is related to exponentially small splittings of separatrices (e.g. [4]). The analysis to this direction is left for future work.

On the other hand, there have been much interest and progress in studying localized patterns in dissipative PDEs, especially in reaction-diffusion equations. Let us note that the assumptions we need in the whole story of this paper are Hamiltonian-Hopf bifurcations and space reversibility. Therefore, it is possible to generalize the result to reaction-diffusion systems with these assumptions.

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