数理解析研究所講究録
1501 巻 2006 年 181-194

Multiplication table and topology of real hypersurfaces.

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ABSTRACT. *This is a review article on the multiplication table associated to the complete intersection singularities of projection. We show how the logarithmic vector fields appear as coefficients to the Gauss-Manin system (Theorem 2.7). We examine further how the multiplication table on the Jacobian quotient module calculates the logarithmic vector fields tangent to the discriminant and the bifurcation set (Proposition 3.3). As applications, we establish signature formulae for Euler characteristics of real hypersurfaces (Theorem 4.2) by means of these fields.*

1 Introduction

This is a review article on the multiplication table associated to the isolated complete intersection singularities (i.c.i.s.) of projection and notions tightly related with them. The notion of i.c.i.s. of projection has been picked up among general i.c.i.s. by Viktor Goryunov [3], [4] as good models to which many arguments on the hypersurface singularities can be applied (see for example Theorem 2.1, Lemma 2.5). All isolated hypersurface singularities can be considered as special cases of the i.c.i.s. of projection. Many of important quasihomogeneous i.c.i.s. are also i.c.i.s. of projection.

The main aim of this article is to transmit the message that the multiplication tables defined on different quotient rings calculate important data both on analytic and topological characterisation of the i.c.i.s. of projection. We show that the multiplication table on the Jacobian quotient module in $\mathcal{O}^k_{\tilde{X}\times S}$ calculates the logarithmic vector fields (i.e. the coefficients to the Gauss-Manin system defined for the period integrals) tangent to the discriminant and the bifurcation set (Proposition 3.3) of the i.c.i.s. of projection.. This idea is present already in the works by Kyoji Saito [13] and James William Bruce [2] for the case of hypersurface singularities (i.e. $k=1$).

On the other hand, as applications, we establish signature formulae for Euler characteristics of real hypersurfaces (Theorem 4.2) by means of logarithmic vector fields. These are paraphrase of results established by Zbigniew Szafraniec [14].

The author expresses his gratitude to Aleksandr Esterov who drew his attention to the utility of multiplication table. The main part of this work has been accomplished during author's stay at the International Centre for Theoretical Physics (Trieste) and Hokkaido University where the author enjoyed fruitful working condition. The author expresses his deep gratitude to the concerned institutions and to Prof.Toru Ohmoto who gave him an occasion to report part of results at RIMS (Kyoto) conference.

AMS Subject Classification: 14M10 (primary), 32B10, 14P05 (secondary).
Key words and phrases: complete intersections, Gauss-Manin system, real algebraic sets.
Partially supported by ICTP (Trieste), JSPS grant in aid of Prof. Toru Ohmoto (Hokkaido Univ.).

2 Complete intersection of projection

Let us consider a $k-$tuple of holomorphic germs

$$\vec{f}(x,u) = (f_1(x,u), \cdots, f_k(x,u)) \in (\mathcal{O}_X)^k \tag{2.1}$$

in the neighbourhood of the origin for $X = (\mathbf{C}^{n+1}, 0)$. This is a 1- parameter deformation of the germ

$$\vec{f}^{(0)}(x) = (f_1(x,0), \cdots, f_k(x,0)) \in (\mathcal{O}_{\tilde{X}})^k \tag{2.2}$$

for $\tilde{X} = (\mathbf{C}^n, 0)$.

After [3] we introduce the notion of R_+equivalence of projection. Let $p : \mathbf{C}^{n+1} \longrightarrow \mathbf{C}$ be a nondegenerate linear projection i.e. $dp \neq 0$.

Definition 1 *We call the diagram*

$$Y \hookrightarrow \mathbf{C}^{n+1} \longrightarrow^{p} \mathbf{C},$$

the projection of the variety $Y \hookrightarrow \mathbf{C}^{n+1}$ on the line. Two varieties Y_1, Y_2 belong to the same R_+ equivalence class of projection if there exists a biholomorphic mapping from $\mathbf{C}^{n+1}$ to $\mathbf{C}^{n+1}$ that preseves the projection and induces a translation $p \rightarrow p + const$ on the line.

In this way, we are led to the definition of an equivalence class up to the following ideals,

$$T_f = \mathcal{O}_X\langle\frac{\partial \vec{f}}{\partial x_1}, \cdots, \frac{\partial \vec{f}}{\partial x_n}\rangle + \vec{f}^* (m_{\mathbf{C}^k,0}) \cdot (\mathcal{O}_X)^k \tag{2.3}$$

and

$$T_f^+ := T_f + \mathbf{C}\frac{\partial \vec{f}}{\partial u} \tag{2.4}$$

that is nothing but the tangent space to the germ of R_+ equivalence class of projection. We introduce the spaces

$$Q_f := (\mathcal{O}_X)^k / T_f, \tag{2.5}$$

$$Q_f^+ := (\mathcal{O}_X)^k / T_f^+. \tag{2.6}$$

We remark that though T_f^+ is not necessarily an ideal the quotien Q_f^+ can make sense. Assume that Q_f is a finite dimensional $\mathbf{C}$ vector space. In this case, we call the number $\tau := dim_{\mathbf{C}} Q_f^+$ the R_+- codimension of projection $\mu := dim_{\mathbf{C}} Q_f$ the multiplicity of the critical point $(x,u) = 0$ of the height function u on $X_0 := \{(x,u) \in X; f_1(x,u) = \cdots = f_k(x,u) = 0\}$. We denote by $\langle \vec{e}_1(x,u), \cdots, \vec{e}_\tau(x,u)\rangle$ the basis of the $\mathbf{C}$-vector space Q_f^+. If $\tau < \infty$, it is easy to see that $\vec{f}(x,u) = 0$ (resp. $\vec{f}(x,0) = 0$) has isolated singularity at $0 \in X$ (resp. $0 \in \tilde{X}$). Let us consider a R_+- versal deformation of $\vec{f}^{(0)}(x)$

$$\vec{F}(x,u,t) = \vec{f}^{(0)}(x) + \vec{e}_0(x,u) + t_1\vec{e}_1(x,u) + \cdots + t_\tau \vec{e}_\tau(x,u), \tag{2.7}$$

with $\vec{e}_0(x,u) = \vec{f}(x,u) - \vec{f}(x,0)$. We consider the deformation of X_0 as follows

$$X_t := \{(x,u) \in \mathbf{C}^{n+1}; \vec{F}(x,u,t) = \vec{0}\}, \tag{2.8}$$

that is also a $(\tau + 1)$-dimensional deformation of the germ $\tilde{X}_0 := \{x \in \tilde{X}; f_1(x,0) = \cdots = f_k(x,0) = 0\}$. The following fact is crucial for further arguments.

Theorem 2.1 *([3], Theorem 2.1) For the k-tuple of holomorphic germs* (2.1) *with* $0 < \mu < +\infty$, *we have the equality* $\mu = \tau + 1$.

Futher, in view of the Theorem 2.1 we make use of the notation, $S = (\mathbf{C}^{\tau+1}, 0) = (\mathbf{C}^{\mu}, 0)$,$s = (u,t) \in S$, $s_0 = u, s_i = t_i, 1 \leq i \leq \tau$.

Let $I_{C_0} \subset \mathcal{O}_X$ be the ideal generated by $k \times k$ minors of the marix $(\frac{\partial \vec{f}(x,u)}{\partial x_1}, \cdots, \frac{\partial \vec{f}(x,u)}{\partial x_n})$.

Proposition 2.2 *([3]) We have the equality*

$$\mu = dim_{\mathbf{C}} Q_f = dim \frac{\mathcal{O}_X}{\mathcal{O}_X(f_1(x,u), \cdots, f_k(x,u)) + I_{C_0}}.$$

Let us denote by $Cr(\vec{F})$ the set of critical locus of the projection $\pi : \bigcup_{t \in \mathbf{C}^\tau} X_t \rightarrow S$. That is to say

$$Cr(\vec{F}) = \{(x,u,t); (x,u) \in X_t, rank(\frac{\partial \vec{F}(x,s)}{\partial x_1}, \cdots, \frac{\partial \vec{F}(x,s)}{\partial x_n}) < k\}. \tag{2.9}$$

We denote by $D \subset S$ the image of projection $\pi(Cr(\vec{F}))$ which is usually called discriminant set of the deformation X_t of projection. It is known that for the R_+-versal deformation, D is defined by a principal ideal in $\mathcal{O}_S$ generated by a single defining function $\Delta(s)$ [9]. Under this situation we define $\mathcal{O}_S-$ module of vector fields tangent to the discriminant D which is a sub-module of Der_S the vector fields on S with coefficients from $\mathcal{O}_S$.

Definition 2 *We define the* logarithmic vector fields *associated to D as follows,*

$$Der_S(log\ D) = \{\vec{v} \in Der_S; \vec{v}(\Delta) \in \mathcal{O}_S \cdot \Delta\}.$$

We call that a meromorphic p−form ω with a simple pole along D belongs to the $\mathcal{O}_S$module of the logarithmic differential forms $\Omega^p_S(log\ D)$ *associated to D iff the following two conditions are satisfied*

$$1) \Delta \cdot \omega \in \Omega^p_S,$$

$$2) d\Delta \cdot \omega \in \Omega^{p+1}_S,$$

or equivalently

$$\Delta \cdot d\omega \in \Omega^{p+1}_S.$$

For the $\mathcal{O}_S$-module of the logarithmic differential forms the following fact is known.

Theorem 2.3 *(See [11] for the case* $k = 1$, *[9], [1] for the case k general) The module* $Der_S(log\ D)$ *is a free* $\mathcal{O}_S$*-module of rank* μ. *Furthermore there exists a* μ*-tuple of vectors* $\vec{v}_1, \cdots, \vec{v}_\mu \in Der_S(log\ D)$ *such that*

$$\Delta(s) = det(\vec{v}_1, \cdots, \vec{v}_\mu).$$

Proposition 2.4 *(see [16] for the case $k=1$, [3] for general k)*
For every $\vec{v}_j \in Der_S(log\ D)$, $1 \leq j \leq \mu$, there exists its lifting $\hat{\vec{v}}_j \in Der_{\tilde{X}\times S}$ tangent to the critical set $Cr(\vec{F})$. More precisely, the following decomposition holds,

$$\vec{v}_j(F_q(x,s)) = \sum_{p=1}^{n} h_{j,p}(x,s)\frac{\partial F_q}{\partial x_p} + \sum_{r=1}^{k} a_{jq}^{(r)}(x,s)F_r + b_{j,q}(x,s,\vec{F}),\ 1 \leq q \leq k$$

for some $h_{s,j}(x,s) \in \mathcal{O}_{\tilde{X}\times S}$, $b_{j,q}(x,s,\vec{F}) \in \mathcal{O}_{\tilde{X}\times S} \otimes_{\mathcal{O}_{\tilde{X}\times S}} m_S^2$. In this notation,

$$\hat{\vec{v}}_j = \vec{v}_j - \sum_{p=1}^{n} h_{j,p}(x,s)\frac{\partial}{\partial x_p}.$$

Conversely, to every vector field $\hat{\vec{v}}_j \in Der_{\tilde{X}\times S}$ tangent to the critical set $Cr(\vec{F})$ we can associate a vector field $\vec{v}_j \in Der_S(log\ D)$ as its push down.

This is a direct consequence of the preparation theorem.

Lemma 2.5 *([3]) The discriminant $\Delta(s)$ defined in Theorem 2.3 can be expressed by a Weierstrass polynomial,*

$$\Delta(s) = u^{\mu} + d_1(t)u^{\mu-1} + \cdots + d_{\mu}(t),$$

with $d_1(t) = \cdots = d_{\mu}(0) = 0$.

From this lemma we deduce immediately the existence of an "Euler" vector field even for non-quasihomogeneous $\tilde{f}(x,u)$ that plays essential rôle in the construction of the higher residue pairing by K.Saito[12].

Lemma 2.6 *(For $k=1$, see [12] (1.7.5)) There is a vector field $\vec{v}_1 = (u+\sigma_1^0(t))\frac{\partial}{\partial u} + \sum_{i=1}^{\tau} \sigma_1^i(t)\frac{\partial}{\partial s_i} \in Der_S(log\ D)$ such that*

$$\vec{v}_1(\Delta(s)) = \mu\Delta(s).$$

Proof It is clear that for a vector field $\vec{v}_1 \in Der_S(log\ D)$ with the component $(u+\sigma_1^0(t))\frac{\partial}{\partial u}$ whose existence is guaranteed by Theorem 3,1 [3] , the expression $\vec{v}_1(\Delta(s))$ must be divisible by $\Delta(s)$. In calculating the term of $\vec{v}_1(\Delta(s))$ that may contain the factor u^{μ}, we see that

$$\vec{v}_1(\Delta(s)) = \mu u^{\mu} + \tilde{d}_1(t)u^{\mu-1} + \cdots + \tilde{d}_{\mu}(t).$$

Thus we conclude that $\tilde{d}_i(t) = \mu d_i(t)$, $1 \leq i \leq \mu$. **Q.E.D.**

Now we introduce the filtered $\mathcal{O}_S$-module of fibre integrals $\mathcal{H}^{(\vec{\lambda})}$ for a multi-index $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k) \in (\mathbf{Z}_{<0})^k$.

$$I_{\phi}^{\vec{\lambda}}(s) = \int_{t(\gamma)} \phi(x,s)F_1(x,s)^{\lambda_1}\cdots F_k(x,s)^{\lambda_k}dx,$$

for $\phi(x,s) \in \mathcal{O}_{\tilde{X}\times S}$. Let us denote by $X^{(q)} := \{x \in \tilde{X}; F_q(x,s) = 0\}$ a smooth hypersurface defined for $s \notin D$. We define the Leray's tube operation isomorphism (see [17], [7]),

$$\begin{array}{cccc} t: & H_{n-k}(\cap_{q=1}^{k} X^{(q)}) & \rightarrow & H_n(\tilde{X} \setminus \cup_{q=1}^{k} X^{(q)}), \\ & \gamma & \mapsto & t(\gamma). \end{array}$$

The concrete construction of the operation t can be described as follows. First we consider the coboundary isomorphism of the compact homology groups,

$$\delta : H_{n-k}(\cap_{q=1}^{k} X^{(q)}) \to H_{n-k+1}(\cap_{q=2}^{k} X^{(q)} \setminus X^{(1)}).$$

A compact cycle γ in $\cap_{q=1}^{k} X^{(q)}$ is mapped onto a cycle $\delta(\gamma)$ of one higher dimension that is obtained as a S^1 bundle over γ. Repeated application of δ yields an interated coboundary homomorphism,

$$H_{n-k}(\cap_{q=1}^{k} X^{(q)}) \to^{\delta} H_{n-k+1}(\cap_{q=2}^{k} X^{(q)} \setminus X^{(1)}) \to^{\delta} \cdots$$

$$\cdots \to^{\delta} H_{n-1}(X^{(k)} \setminus \cup_{q=1}^{k-1} X^{(q)}) \to^{\delta} H_n(\tilde{X} \setminus \cup_{q=1}^{k} X^{(q)}).$$

The Leray's tube operation is a $k-$time iterated δ homomorphism i.e. $t = \delta^m$. The Froissart decomposition theorem ([7], §6-3) shows that the collection of all cycles of $H_n(\tilde{X} \setminus \cup_{q=1}^{k} X^{(q)})$ are obtained by the application of iterated δ homomorphism operations to the cycles from $H_{n-p}(\tilde{X} \cap X^{(q_1)} \cap X^{(q_2)} \cdots \cap X^{(q_p)})$, $p = 0, \cdots, k$.

Let us denote by Φ the $\mathbf{C}$ vector space $\frac{\mathcal{O}_{X_0}}{I_{C_0}}$ whose $\mathbf{C}-$ dimension is equal to μ after the Proposition 2.2. We denote its basis by $(\phi_0(x,u), \cdots, \phi_\tau(x,u))$

Now let us introduce a notation of the multi-index $-\mathbf{1} = (-1, \cdots -1) \in (\mathbf{Z}_{<0})^k$. We consider a vector of fibre integrals $\mathbf{I}_\Phi :=^t (I_{\phi_0}^{(-1)}(s), \cdots, I_{\phi_\tau}^{(-1)}(s))$. Then following theorem for $k = 1$ has been anounced in [13] (4.14) without proof.

Theorem 2.7 *1.For every $\vec{v} \in Der_S(log\ D)$, we have the following inclusion relation*

$$\vec{v}(\mathcal{H}^{(-1)}) \hookrightarrow \mathcal{H}^{(-1)}.$$

That is to say for every $\vec{v}_j \in Der_S(log\ D)$, there exists a $\mu \times \mu$ matrix with holomorphic entries $B_j(s) \in End(\mathbf{C}^\mu) \otimes \mathcal{O}_S$ such that

$$\vec{v}_j(\mathbf{I}_\Phi) = B_j(s)\mathbf{I}_\Phi, 1 \leq j \leq \mu.$$

2. The vector of fibre integrals $\mathbf{I}_\Phi$ satisfies the following Pfaff system of Fuchsian type

$$d\mathbf{I}_\Phi = \Omega \cdot \mathbf{I}_\Phi,$$

for some $\Omega \in End(\mathbf{C}^\mu) \otimes_{\mathcal{O}_S} \Omega_S^1(log\ D)$.

Proof As for the proof of 1, we remark the following equality that yields from Proposition 2.4,

$$\vec{v}_j \int_{t(\gamma)} \phi(x,u) F_1(x,s)^{-1} \cdots F_k(x,s)^{-1} dx = \int_{t(\gamma)} \vec{F}^{-1} d(\phi(x,u) \sum_{p=1}^{n} (-1)^{p-1} h_{j,p}(x,s) dx_1 \overset{p}{\overset{\vee}{\cdots}} dx_n) +$$

$$+ \int_{t(\gamma)} \vec{F}^{-1} \phi(x,u) (\sum_{q=1}^{k} \sum_{r=1}^{k} a_{j,q}^{(r)} F_r F_q^{-1}) dx + \int_{t(\gamma)} \vec{F}^{-1} \sum_{q=1}^{k} F_q^{-1} b_{j,q}(x,u,t,\vec{F}) dx$$

$$= \int_{t(\gamma)} \vec{F}^{-1} d(\phi(x,u) \sum_{p=1}^{n} (-1)^{p-1} h_{j,p}(x,s) dx_1 \overset{p}{\overset{\vee}{\cdots}} dx_n) + \int_{t(\gamma)} \vec{F}^{-1} \phi(x,u) (\sum_{r=1}^{k} a_{j,r}^{(r)}(x,s)) dx,$$

which evidently belongs to $\mathcal{H}^{(-1)}$. The last equality can be explained by the vanishing of the integral

$$\int_{t(\gamma)} F_1^{-1} \cdots 1 \overset{r}{\vee} \cdot F_q^{-2} \overset{q}{\vee} \cdot F_k^{-1} \phi(x,u)(a_{j,q}^r) dx = 0,$$

because of the lack of the residue along $F_r(x,s) = 0$ and

$$\int_{t(\gamma)} \vec{F}^{-1} F_{q_1} F_{q_2} F_q^{-1} \phi(x,u)(b_{j,q}^0(x,s)) dx = 0,$$

in view of the lack of at least one of residues either along $F_{q_1} = 0$ or along $F_{q_2} = 0$. These equalities are derived from the property of the Leray's tube $t(\gamma)$ which needs codimension k residue to give rise to a non-zero integral.

2. Let us rewrite the relations obtained in 1. into the form,

$$dI_{\phi_q}^{(-1)} = \sum_{r=1}^{\mu} \omega_{q,r} I_{\phi_r}^{(-1)},$$

for some $\omega_{q,r} \in \Omega_S^1(-D)$ meromorphic 1-forms with poles along D. These $\omega_{q,r}$ satisfy the following relations,

$$\vec{v}_j(I_{\phi_q}^{(-1)}) = \langle \vec{v}_j, dI_{\phi_q}^{(-1)} \rangle = \langle \vec{v}_j, \sum_{r=1}^{\mu} \omega_{q,r} I_{\phi_r}^{(-1)} \rangle \ \ 1 \le j, q \le \mu.$$

If $\langle \vec{v}_j, \omega_{q,r} \rangle \in \mathcal{O}_S$ for all $\vec{v}_j \in Der_S(log\ D)$ $1 \le j \le \mu$ then $\omega_{q,r} \in \Omega_S^1(log\ D)$ in view of the Theorem 2.3. **Q.E.D.**

Let us introduce a filtration as follows $\mathcal{H}^{(\lambda)} = \bigoplus_{\lambda_1+\cdots+\lambda_k=\lambda} \mathcal{H}^{(\vec{\lambda})}$. For this rough filtration we have the following generalisation of the Griffiths' transversality theorem ([6] Theorem 3.1).

Corollary 2.8 *For every $\vec{v} \in Der_S(log\ D)$, we have the following inclusion relation*

$$\vec{v}(\mathcal{H}^{(\lambda)}) \hookrightarrow \mathcal{H}^{(\lambda)}.$$

Proof For $\partial_{s_j} I_\Phi \in \mathcal{H}^{(-k-1)}$ and $\vec{v}_\ell \in Der_S(log\ D)$ we have

$$\vec{v}_\ell(\partial_{s_j} I_\Phi) = [\vec{v}_\ell, \partial_{s_j}] I_\Phi + \partial_{s_j} \vec{v}_\ell(I_\Phi)$$

$$= [\vec{v}_\ell, \partial_{s_j}] I_\Phi + \partial_{s_j}(B_\ell(s) I_\Phi) = [\vec{v}_\ell, \partial_{s_j}] I_\Phi + (\partial_{s_j} B_\ell(s)) I_\Phi + B_\ell(s)(\partial_{s_j} I_\Phi).$$

As the commutator $[\vec{v}_\ell, \partial_{s_j}]$ is a first order operator, the term above $[\vec{v}_\ell, \partial_{s_j}] I_\Phi$ belongs to $\mathcal{H}^{(-k-1)}$. The term $\partial_{s_j} B_\ell(s) I_\Phi \in \mathcal{H}^{(-k)}$ again belongs to $\mathcal{H}^{(-k-1)}$. Thus we see $\vec{v}_\ell(\partial_{s_j} I_\Phi) \in \mathcal{H}^{(-k-1)}$. In an inductive way, for any $\lambda \le -k$ we prove the statement.

Q.E.D.

3 Multiplication table and the logarithmic vector fields

We consider a miniversal deformation of a mapping $\vec{f}^{(0)}(x)$ which can be written down in the following special form for $s = (u, t)$,

$$\vec{F}(x,s) = \vec{f}^{(0)}(x) + \sum_{\ell=1}^{\tau} t_\ell \vec{e}_\ell(x) + u\vec{e}_0(x) = \begin{pmatrix} F_1(x,t) - u \\ F_2(x,t) \\ \vdots \\ F_k(x,t) \end{pmatrix}, \tag{3.1}$$

for

$$\{\vec{e}_0(x), \cdots, \vec{e}_\tau(x)\} \in Q_f,$$

where $\vec{e}_0(x) = {}^t(-1, 0, \cdots, 0)$. One may consult [9] (6.7) to see that $\vec{F}(x,s)$ really gives a miniversal deformation of $\vec{f}^{(0)}(x)$ by virtue of the definitions (2.3), (2.5). Let us fix a basis $\{\phi_0(x), \cdots, \phi_\tau(x)\}$ of the space $\Phi := \frac{\mathcal{O}_X}{I_{C_0} + \mathcal{O}_X\langle f_1(x)-u, f_2(x), \cdots, f_k(x)\rangle}$. We remark here that the basis of Φ can be represented by functions on x as we can erase the variable u by the relation $f_1(x) = u$ in Φ.It turns out that we can regard $\{\phi_0(x), \cdots, \phi_\tau(x)\}$as a free basis of the $\mathcal{O}_S$ module,

$$\Phi(s) = \frac{\mathcal{O}_{\tilde{X}\times S}}{\mathcal{O}_{\tilde{X}\times S}\langle F_2(x,t), \cdots, F_k(x,t)\rangle + I_{C_0}(t)}.$$

Under this situation, we introduce holomorphic functions $\tau^\ell_{i,j}(s) \in \mathcal{O}_S$ in the following way.

$$\phi_i(x)\vec{e}_j(x) = \sum_{\ell=0}^{\tau} \tau^\ell_{i,j}(s)\vec{e}_\ell(x) \;\; mod(\mathcal{O}_{\tilde{X}\times S}\langle \frac{\partial \vec{F}(x,s)}{\partial x_1}, \cdots, \frac{\partial \vec{F}(x,s)}{\partial x_n}\rangle). \tag{3.2}$$

The functions $\tau^\ell_{i,j}(s) \in \mathcal{O}_S$ exist due to the versality of the deformation $\vec{F}(x,s)$. We denote by

$$T_j(s) = \left(\tau^\ell_{i,j}(s)\right)_{0\leq j,\ell\leq\tau}, \tag{3.3}$$

a $\mu\times\mu$ matrix which is called the matrix of **multiplication table**. We denote the discriminant associated to this deformation by $D \subset S$.

Further on we will make use of the abbreviation $mod(d_x\vec{F}(x,s))$ instead of making use of the expression $mod(\mathcal{O}_{\tilde{X}\times S}\langle \frac{\partial \vec{F}(x,s)}{\partial x_1}, \cdots, \frac{\partial \vec{F}(x,s)}{\partial x_n}\rangle)$.

After Proposition 2.4 the vector field $\vec{v}_1$ constructed in Lemma 2.6 has its lifting $\hat{\vec{v}}_1 \in Der_{\tilde{X}\times S}$. Let us denote by $\check{\vec{v}}_1 = \hat{\vec{v}}_1 - \vec{v}_1 \in \mathcal{O}_{\tilde{X}\times S}\otimes Der_{\tilde{X}}$.

$$\hat{\vec{v}}_1(\vec{F}(x,s))\cdot\phi_i(x) = \check{\vec{v}}_1(\vec{f}^{(0)}(x))\cdot\phi_i(x) + \sum_{\ell=0}^{\tau}\vec{v}_1(s_\ell)\vec{e}_\ell(x)\phi_i(x) + \sum_{\ell=0}^{\tau}s_\ell(\check{\vec{v}}_1 e_\ell(x))\phi_i(x)$$

$$\equiv \sum_{\ell=0}^{\tau}\vec{v}_1(s_\ell)\vec{e}_\ell(x)\phi_i(x) \quad mod(d_x\vec{F}(x,s)).$$

Lemma 3.1 *There exists a vector valued function $M(x,\vec{F}(x,s)) \in (\mathcal{O}_{\tilde{X}\times\mathbf{C}^k})^k$ such that*

$$\hat{\vec{v}}_1(s)(\vec{F}(x,s)) = M(x,\vec{F}(x,s)) \quad mod(d_x\vec{F}(x,s)),$$

with

$$M(x,\vec{F}(x,s)) = M^0\cdot\vec{F}(x,s) + M^1(x,\vec{F}(x,s)),$$

where $M^0 \in GL(k,\mathbf{C})$: a non-degenerate matrix and $M^1(x,\vec{F}(x,s)) \in (\mathcal{O}_{\tilde{X}}\otimes m_S^2)^k$. Especially the first row of $M^0 = (1,0,\cdots,0)$.

Proof First of all we remember a theorem due to [5] §1.1, [13] Propositition 2.3.2 which states that the Krull dimension of the ring of holomorphic functions on the critical set $Cr(\vec{F})$ is equal to $\mu-1$ and this ring is a Cohen- Macaulay ring. Let us denote by $L = {}_nC_k$. We have $(k+L)$tuple of $k\times k-$ minors $j_{k+1}(x,s)\cdots j_{k+L}(x,s)$ of the matrix $(\frac{\partial}{\partial x_1}\vec{F}(x,s), \cdots, \frac{\partial}{\partial x_n}\vec{F}(x,s))$ such that

$$Cr(\vec{F}) = V(< F_1(x,s), \cdots, F_k(x,s), j_{k+1}(x,s), \cdots, j_{k+L}(x,s) >).$$

The lemma 2.6 yields that the lifting $\hat{\vec{v}}_1$ of the vector field $\vec{v}_1$ satisfies the relations,

$$< F_1(x,s),\cdots,F_k(x,s),j_{k+1}(x,s),\cdots,j_{k+L}(x,s) >$$

$$=< \hat{\vec{v}}_1(F_1(x,s)),\cdots,\hat{\vec{v}}_1(F_k(x,s)),\hat{\vec{v}}_1(j_{k+1}(x,s)),\cdots,\hat{\vec{v}}_1(j_{k+L}(x,s)) > .$$

As it has been seen above Proposition 2.4, the vector $\hat{\vec{v}}_1$ prop2 is tangent to $Cr(\vec{F})$. If the above equality does not hold, it would entail the relation

$$D = \{s \in S; \Delta(s) = 0\} \underset{\neq}{\subset} \pi(V(< \hat{\vec{v}}_1(F_1(x,s)),\cdots,\hat{\vec{v}}_1(F_k(x,s)),\hat{\vec{v}}_1(j_{k+1}(x,s)),\cdots,\hat{\vec{v}}_1(j_{k+L}(x,s)) >)),$$

after elimination theoretical consideration. This yields

$$\hat{\vec{v}}_1(F_q(x,s)) = \sum_{\ell=1}^{k} C_q^\ell F_\ell(x,s) + m_q(x,\vec{F}) + \sum_{\ell=k+1}^{k+L} C_q^\ell j_\ell(x,s), 1 \leq q \leq k,$$

$$\hat{\vec{v}}_1(j_p(x,s)) = \sum_{\ell=k+1}^{k+L} C_p^\ell j_\ell(x,s), k+1 \leq p \leq k+L,$$

for $m_q(x,\vec{F}) \in \mathcal{O}_{\tilde{X}} \otimes m_S^2$, $1 \leq q \leq k$ and some constants $C_q^\ell, 1 \leq \ell \leq k$. First we see that the expression $\hat{\vec{v}}_1(j_p(x,s))$ cannot contain terms of $F_q(x,s)$ like $F_q(0,s)$ in view of the situation that the versality of the deformation makes all linear in x variable terms dependent on some of deformation parameters. Secondly the non-degeneracy of the matrix $M^0 := (C_q^\ell)_{1\leq q,\ell \leq k}$ is necessary so that the above equality among ideals holds.

From this relation and the preparation theorem, we see

$$\hat{\vec{v}}_1(\vec{F}(x,s)) = M^0 \cdot \vec{F}(x,s) + M^1(x,\vec{F}(x,s)) + h_{1,1}(x,s)\frac{\partial \vec{F}(x,s)}{\partial x_1} + \cdots + h_{1,n}(x,s)\frac{\partial \vec{F}(x,s)}{\partial x_n},$$

with $M^1(x,\vec{F}(x,s)) = {}^t(m_1(x,\vec{F}),\cdots,m_k(x,\vec{F}))$.

More precisely we can state that $C_1^1 = 1$, $C_1^\ell = 0, 2 \leq \ell \leq k$. The dependence of some coefficients of $\hat{\vec{v}}_1$ on $F_i(x,t)$ is necessary so that $C_1^\ell \neq 0$ for some $2 \leq \ell \leq k$. But this is impossible because if not it would mean that some of the coefficients of $\hat{\vec{v}}_1$ contains factor $F_2(x,s),\cdots,F_k(x,s)$ that contradicts the construction of $\hat{\vec{v}}_1$ in Proposition 2.4. This can be seen from the fact that the expressions $\frac{\partial F_1(x,s)}{\partial x_1}$, $\cdots$, $\frac{\partial F_1(x,s)}{\partial x_n}$, $\frac{\partial F_1(x,s)}{\partial s_1}$, $\cdots$, $\frac{\partial F_1(x,s)}{\partial s_\mu}$ do not contain the deformation parameters present in the polynomials $F_2(x,s),\cdots,F_k(x,s)$. **Q.E.D.**

Lemma 3.2 *A basis of logarithmic vector fields $\vec{v}_0,\cdots,\vec{v}_\tau \in Der_S(log\ D)$ can be produced from the functions $\sigma_i^\ell(s)$ defined as follows,*

$$M(x,\vec{F}(x,s)) \cdot \phi_i(x) = \sum_{\ell=0}^{\tau} \sigma_i^\ell(s)\vec{e}_\ell + \check{\vec{v}}_i(\vec{F}(x,s))$$

$$\equiv \sum_{\ell=0}^{\tau} \sigma_i^\ell(s)\vec{e}_\ell \ \ mod(d_x\vec{F}(x,s)),$$

where the vector valued fucntion $M(x,\vec{F}(x,s))$ denotes the one defined in the Lemma 3.1 and $\check{\vec{v}}_j = \sum_{p=1}^n h_{j,p}(x,s)\frac{\partial}{\partial x_p}$ is a certain vector field with holomorphic coefficients.

Proof
We remark the following relation,

$$\hat{\vec{v}}_1(\vec{F}(x,s))\phi_i(x) = \check{\vec{v}}_1(\vec{f}^{(0)}(x))\phi_i(x) + \sum_{j=1}^{\mu} \vec{v}_1(s_j)\vec{e}_j(x)\phi_i(x) + \sum_{j=1}^{\mu} s_j\check{\vec{v}}_1(\vec{e}_j(x))\phi_i(x)$$

$$\equiv \sum_{j=0}^{\tau} \vec{v}_1(s_j)\vec{e}_j(x)\phi_i(x) \;\; mod(d_x\vec{F}(x,s)).$$

The relation (3.2) above entails,

$$M(x,\vec{F}(x,s))\cdot\phi_i(x) \equiv \sum_{\ell=0}^{\tau}\sum_{j=0}^{\tau} \vec{v}_1(s_j)\tau_{i,j}^{\ell}(s)\vec{e}_\ell(x) \;\; mod(d_x\vec{F}(x,s)).$$

As $\phi_i(x)$ can be considered to be a basis of $\mathcal{O}_S$ module $\Phi(s)$ above, vectors $(\sigma_i^0(s),\cdots,\sigma_i^\tau(s))$, $0 \le i \le \tau$ are $\mathcal{O}_S$ linearly independent at each generic point $S \setminus D$. If we put

$$\sigma_i^\ell(s) = \sum_{j=0}^{\tau} \vec{v}_1(s_j)\tau_{i,j}^{\ell}(s),$$

then the vector field $\hat{\vec{v}}_i \in Der_{\tilde{X}\times S}$

$$\hat{\vec{v}}_i = \sum_{\ell=0}^{\tau} \sigma_i^\ell(s)\frac{\partial}{\partial s_\ell} + \phi_i(x)\check{\vec{v}}_1,$$

is tangent to $Cr(\vec{F})$. The only non-trivial relations that may arise between $\hat{\vec{v}}_i$ and $\hat{\vec{v}}_{i'}$ $i \neq i'$ is

$$\phi_i(x)\hat{\vec{v}}_{i'} = \phi_{i'}(x)\hat{\vec{v}}_i.$$

These vectors give rise to the same push down vector field in $Der_S(log\ D)$. Namely,

$$\pi_*(\phi_i(x)\hat{\vec{v}}_{i'}) = \pi_*(\phi_{i'}(x)\hat{\vec{v}}_i) = \sum_{j=0}^{\tau}\sum_{\ell=0}^{\tau} R_{i,i',j}^{\ell}(s)\frac{\partial}{\partial s_\ell},$$

for the coefficients $R_{i,i',j}^{\ell}(s)$ determined by

$$\sum_{j=0}^{\tau} \vec{v}_1(s_j)\phi_i(x)\phi_{i'}(x)\vec{e}_j(x) \equiv \sum_{j=0}^{\tau}\sum_{\ell=0}^{\tau} R_{i,i',j}^{\ell}(s)\vec{e}_\ell(x) mod(d_x\vec{F}(x,s)).$$

This means that $\hat{\vec{v}}_0,\cdots,\hat{\vec{v}}_\tau$ form a free basis of $Der_{\tilde{X}\times S}(Cr(\vec{F}))$ hence $\vec{v}_0,\cdots,\vec{v}_\tau$ that of $Der_S(log\ D)$.
Q.E.D.

This lemma gives us a correspondence between $\phi_i(x) \in \Phi$ and $\vec{v}_i \in Der_S(log\ D)$, therefore it is quite natural to expect that the mixed Hodge structure on Φ would induce that on $Der_S(log\ D)$, and would hence contribute to describe $B_i(s)$ of Theorem 2.7, 1 in a precise manner. A good understanding of this situation is indispensable to characterize the rational monodromy of solutions to the Gauss-Manin system in terms of the mixed Hodge structure on Φ.

In other words, we formulate the following proposition (see [2] Theorems A2, A4, [11] (3.19), [13] (4.5.3) Corollary 2 for $k = 1$ and [9] (6.13), [3] Theorem 3.2 for k general).

Proposition 3.3 *There exist holomorphic functions $w_j(s) \in \mathcal{O}_S$, $0 \le j \le \tau$ such that the components of the matrix*

$$\Sigma(s) := \sum_{j=0}^{\tau} w_j(s) T_j(s), \tag{3.4}$$

give rise to a basis of logarithmic vector fields $\vec{v}_0, \cdots, \vec{v}_\tau \in Der_S(log\ D)$. Namely, if we write $\Sigma(s) = \left(\sigma_i^\ell(s)\right)_{0 \le i, \ell \le \tau}$, then the expression

$$\vec{v}_i = \sum_{\ell=0}^{\tau} \sigma_i^\ell(s) \frac{\partial}{\partial s_\ell}, \tag{3.5}$$

consists a base element of the $\mathcal{O}_S$ module $Der_S(log\ D)$.

Especially in the case of quasihomogeneous singularity $\vec{f}(x,u)$ we have the following simple description of the vector field that can be deduced from Lemma 3.2. To do this, it is enough to remark that the vector field $\vec{v}_1$ is the Euler vector field by definition and $\vec{v}_1(s_r) = \frac{w(s_r)}{w(s_0)}$, where $w(s_j)$ denotes the quasihomoeneous weight of the variable s_j.

Proposition 3.4 *([4] Theorem 2.4) In the case of quasihomogeneous singularity* (2.1), *the basis* (3.5) *of $Der_S(log\ D)$ can be calculated by*

$$\sigma_i^\ell(s) = \sum_{j=0}^{\tau} w(s_j) s_j \tau_{i,j}^\ell(s).$$

Furthermore, the vector valued function $M(x, \vec{F}(x,s))$ of Lemma 3.1 has the expression,

$$M(x, \vec{F}(x,s)) = M^0 \cdot \vec{F}(x,s) = diag\ (w(f_1), \cdots, w(f_k)) \cdot \vec{F}(x,s).$$

4 Multiplication table and the topology of real hypersurfaces

In this section we continue to consider the situation where $\mu = \tau + 1$ for $k = 1$ in (2.5). We associate to the versal deformation of the hypersurface singularity

$$F(x,s) = f(x) + \sum_{i=0}^{\tau} s_i e_i(x), \tag{4.1}$$

the following matrix $\Sigma(s) = (\sigma_i^\ell(s))_{0 \le i, \ell \le \tau}$ after the model (3.2),

$$F(x,s) e_i(x) = \sum_{\ell=0}^{\tau} \sigma_i^\ell(s) e_\ell(x) \ \ mod(d_x F(x,s)). \tag{4.2}$$

$$e_i(x) e_j(x) \equiv \sum_{\ell=0}^{\tau} \tau_{i,j}^\ell(t) e_\ell(x) \ \ mod(d_x F(x,s)). \tag{4.3}$$

Further on we make use of the convention $e_0(x) = 1$ and $s = (s_0, t)$. We will denote the deformation parameter space $t \in T = (\mathbf{C}^\tau, 0)$.

We recall the Milnor ring for $k = 1$ whose analogy has been introduced in (2.5),

$$Q_F := \frac{\mathcal{O}_{\tilde{X}\times S}}{\mathcal{O}_{\tilde{X}\times S}\langle \frac{\partial F(x,s)}{\partial x_1}, \cdots, \frac{\partial F(x,s)}{\partial x_n}\rangle}.$$

We introduce the Bezoutian matrix $B^F(s)$ whose (i,j) element is defined by the trace of the multiplication action $F(x,s)e_i(x)e_j(x)\cdot$ on the Milnor ring Q_F,

$$F(x,s)e_i(x)e_j(x) \equiv (\sum_{c=0}^{\tau} \sigma_i^c(s)e_c(x))e_j(x)$$

$$\equiv \sum_{c=0}^{\tau} \sigma_i^c(s)(\sum_{r=0}^{\tau} \tau_{c,j}^r(t)e_r(x)) mod(d_x F(x,s)).$$

For the sake of simplicity we will use the following notation,

$$\tau^r(t) = (\tau^r_{c,b}(t))_{0\leq c,b\leq\tau}. \tag{4.4}$$

To clarify the structure of the Bezoutian matrix $B^F(s)$ we introduce a matrix

$$T(t) = \left(\sum_{r=0}^{\tau} \zeta_r(t)\tau^r(t)\right), \tag{4.5}$$

with the notation

$$\zeta_r(t) = tr(e_r(x)\cdot) = \sum_{\ell=0}^{\tau} \tau_{r,\ell}^{\ell}(t). \tag{4.6}$$

The (i,j) element of the matrix $T(t)$ (4.5) equals to $tr(e_i(x)e_j(x)\cdot)$ on the Milnor ring Q_F. It is possible to show that $\{t \in T; det(T(t)) = 0\}$ coincides with the bifurcation set of $F(x,s)$ outside the Maxwell set. Thus we get the Bezoutian matrix

$$B^F(s) = \Sigma(s) \cdot T(t). \tag{4.7}$$

Following statement is a simple application of Morse theory to the multiplication table see [14] Theorem 2.1. From here on we assume that $|s|$ is small enough and denote by $\tilde{X} = \{x \in \mathbf{C}^n; |x| \leq \delta\}$ a closed ball such that all critical points of $F(x,s)$ are located inside $\tilde{X}$.

Proposition 4.1 *sign* $\Sigma(s) \cdot T(t)$ ={ *number of real critical points with respet to the variables* x *in* $F(x,s) > 0$, $x \in \tilde{X} \cap \mathbf{R}^n$} -{ *number of real critical points with respet to the variables* x *in* $F(x,s) < 0$, $x \in \tilde{X} \cap \mathbf{R}^n$}. *Here sign(A) denotes the signature of a symmetric matrix A i.e. the difference between the number of positive and negative eigenvalues.*

Let us denote by $h(x,t)$ the determinant of the Hessian

$$h(x,t) := det\langle \frac{\partial^2 F(x,s)}{\partial x_i \partial x_j}\rangle_{1\leq i,j\leq n}.$$

We associate the following μ holomorphic functions $h_0(t), \cdots, h_\tau(t) \in \mathcal{O}_S$ to the function $h(x,t)$,

$$h(x,t) \equiv \sum_{\ell=0}^{\tau} h_\ell(t)e_\ell(x) \ \ mod(d_x F(x,s)). \tag{4.8}$$

Further by means of (4.7) we introduce the matrix

$$B^H(t) := \sum_{\ell=0}^{\tau} \eta^\ell(t)\tau^\ell(t). \tag{4.9}$$

where

$$\begin{pmatrix} \eta^0(t) \\ \vdots \\ \eta^\tau(t) \end{pmatrix} = T(t) \cdot \begin{pmatrix} h_0(t) \\ \vdots \\ h_\tau(t) \end{pmatrix}.$$

We consider the matrix $B^{HF}(s) = (\cdot)_{0\leq a,b\leq \tau}$ whose $(a,b)-$element is defined by the trace of the following expression on the Milnor ring Q_F,

$$\begin{aligned} h(x,t)F(x,s)e_a(x)e_b(x) &\equiv (\sum_{\ell=0}^{\tau} h_\ell(t)e_\ell(x))(\sum_{c=0}^{\tau}\sigma_a^c(s)\sum_{m=0}^{\tau}\tau_{c,b}^m(t)e_m(x)) \\ &\equiv \sum_{\ell=0}^{\tau}\sum_{c=0}^{\tau}\sum_{m=0}^{\tau} h_\ell(t)\sigma_a^c(s)\tau_{c,b}^m(t)e_\ell(x)e_m(x) \\ &\equiv \sum_{\ell=0}^{\tau}\sum_{c=0}^{\tau}\sum_{m=0}^{\tau} h_\ell(t)\sigma_a^c(s)\tau_{c,b}^m(t)\sum_{r=0}^{\tau}\tau_{\ell,m}^r(t)e_r(x) mod(d_xF(x,s)). \end{aligned} \tag{4.10}$$

If we take the trace of this, we get

$$\equiv \sum_{c=0}^{\tau}\sigma_a^c(s)\sum_{m=0}^{\tau}\sum_{\ell=0}^{\tau} h_\ell(s)\tau_{c,b}^m(t)(\sum_{r=0}^{\tau}\tau_{\ell,m}^r(t)\zeta_r(t)).$$

After (4.8) and (4.9) this matrix has the following expression,

$$B^{HF}(s) = \Sigma(s)\cdot B^H(t). \tag{4.11}$$

We consider the following closures of semi-algebraic sets,

$$W_{=0} := \{x \in \tilde{X} \cap \mathbf{R}^n; F(x,s) = 0\},$$

$$W_{\geq 0} := \{x \in \tilde{X} \cap \mathbf{R}^n; F(x,s) \geq 0\}, W_{\leq 0} := \{x \in \tilde{X} \cap \mathbf{R}^n; F(x,s) \leq 0\}.$$

Theorem 4.2 *The following expression of the Euler characteristics for W_* holds,*

$$\chi(W_{\geq 0}) - \chi(W_{=0}) = \frac{sign(B^H(t)) + sign(B^{HF}(s))}{2}.$$

$$\chi(W_{\leq 0}) - \chi(W_{=0}) = (-1)^n\frac{sign(B^H(t)) - sign(B^{HF}(s))}{2}.$$

Proof

After Szafraniec [14], or simply applying Morse theory to the real fibres of $F(x,s)$, we have the following equalities,

$$\sum_{x\in \text{critical points of} F(x,s)} (sgn\ h(x,t))$$

$$= sign\langle tr(h(x,t)e_i(x)\cdot e_j(x)\cdot)\rangle_{1\le i,j\le n} = \sum_{x\in \text{critical pointspoints of} F(x,s)} (-1)^{\lambda(x)}.$$

Here we denoted by $tr(h(x,t)e_i(x)\cdot e_j(x)\cdot)$ the trace of a matrix defined by the multiplication by the element $h(x,t)e_i(x)\cdot e_j(x)$ considered $mod(d_xF(x,s))$ for the basis $e_i(x), 1\le i\le \mu$.

$$\sum_{x\in \text{critical points of} F(x,s)} (sgn\ h(x,t))(sgn\ F(x,s))$$

$$= sign\langle tr(h(x,t)F(x,s)e_i(x)\cdot e_j(x)\cdot)\rangle_{1\le i,j\le n} = \sum_{x\in \text{critical points of} F(x,s)} (-1)^{\lambda(x)}(sgn\ F(x,s)).$$

Here we denoted by $tr(h(x,t)F(x,s)e_i(x)\cdot e_j(x)\cdot)$ the trace of a matrix defined by the multiplication by the element $h(x,t)F(x,s)e_i(x)\cdot e_j(x)$ considered $mod(d_xF(x,s))$ for the basis $e_i(x), 1\le i\le \mu$. Here $\lambda(x)$ is the Morse index of the function $F(x,s)$ at x and $sgn\ h(x,t) = (-1)^{\lambda(x)}$. **Q.E.D.**

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