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Symbolic calculus of pseudo-differential operators and curvature of manifolds

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Abstract

The method of construction of the fundamental solution for a heat equations as pseudo-differential operators with parameter time variable is discussed, which is applicable to calculate traces of operators. This gives extensions of a local version of both Gauss-Bonnet-Chern Theorem and Riemann-Roch Theorem. Moreover a characterization of complex manifolds which hold a local version of Riemann-Roch Theorem is obtained.

1 Introduction

In this paper we give, by means of symbolic calculus of pseudo-differential operators, both an extension theorem of a local version of Gauss-Bonnet-Chern theorem given in C.Iwasaki[10] and that of a local version of Riemann-Roch theorem given in C.Iwasaki[11]. We give also a characterization of complex manifolds where a local version of Riemann-Roch theorem holds. For more precise discussion see C.Iwasaki[12] and C.Iwasaki[13] .

Let M be a Riemannian manifold of dimension n without boundary. The Gauss-Bonnet-Chern theorem is stated as follows:

$$\sum_{p=0}^{n}(-1)^{p}\dim H_{p}(M)=\int_{M}C_{n}(x,M)dv,$$

where H_p is the set of harmonic p-forms, $C_n(x,M)dv$ is the Euler form if n is even and $C_n(x,M)dv = 0$ if n is odd. Its analytical proof based on the following formula

$$\sum_{p=0}^{n}(-1)^{p}\dim H_{p}(M)=\int_{M}\sum_{p=0}^{n}(-1)^{p}\,\mathrm{tr}e_{p}(t,x,x)dv,$$

where $e_p(t,x,y)$ denotes the kernel of the fundamental solution $E_p(t)$ of Cauchy problem for the heat equation of Δ_p on differental p-forms $\Gamma(\wedge^{p}T^{*}(M))$;

$$E_{p}(t)\varphi(x)=\int_{M}e_{p}(t,x,y)\varphi(y)dv_{y},\quad \varphi\in\Gamma(\wedge^{p}T^{*}(M))$$

satisfies

$$\begin{aligned}(\frac{d}{dt}+\Delta_{p})E_{p}(t)&=0\qquad \text{in } (0,T)\times M,\\ E_{p}(0)&=I\qquad \text{in } M.\end{aligned}$$

So, we may call a local version of Gauss-Bonnet-Chern theorem holds, if we have

$$\sum_{p=0}^{n}(-1)^p \operatorname{tre}_p(t,x,x) = C_n(x,M) + 0(\sqrt{t}) \tag{1.1}$$

as t tends to 0.

The author has prooved (1.1) in [10], using both algebraic theorem on linear spaces stated in H.l.Cycon,R.G.Froese,W.Kirsch and B.Simon[3] and the method of construction of the fundamental solution by technique of pseudodifferential operators of new weights on symbols. In this paper, a genaralization of a local version of Gauss-Bonnet-Chern theorem is obtained. Before stating our theorems, we introduce notations.

We denote $\mathcal{I}$ the set of index

$$\mathcal{I} = \{I = (i_1, i_2, \cdots, i_r) : 0 \le r \le n, 1 \le i_1 < \cdots < i_\ell \le n\},$$

and

$$\binom{a}{b} = 0 \text{ if } a < b, \text{ or } b < 0, \quad \binom{0}{0} = 1.$$

Fix an integer ℓ such that $0 \le \ell \le n$ in the rest of this paper.

Set the following constants $\{f_p\}_{p=0,1,\cdots,n}$ of the form with arbitrary constants $\{k_j\}_{j=\ell+1,\cdots,n}$

$$f_p = \binom{n-p}{n-\ell} + \sum_{j=\max\{p,\ell+1\}}^{n} k_j \binom{n-p}{n-j} \quad (0 \le p \le n). \tag{1.2}$$

Theorem 1.1 (Main Theorem I) *Let M be a Riemannian manifold without boundary of dimenssion n and let $E_p(t)$ be the fundamental solution on $\Gamma(\wedge^p T^*(M))$. Suppose f_p are of the form (1.2). Then we have*

$$\sum_{p=0}^{n}(-1)^p f_p \operatorname{tre}_p(t,x,x) = C_\ell(x) t^{-\frac{n}{2}+\frac{\ell}{2}} + 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}) \qquad \text{as } t \to 0,$$

where $C_\ell(x)$ and is given as follow;
(1) If ℓ is odd, $C_\ell(x) = 0$
(2)If ℓ is even ($\ell = 2m$), $C_\ell(x) = \sum_{I \in \mathcal{I}, \sharp(I)=\ell} C_I(x)$, for $I = (i_1, i_2, \cdots, i_\ell) \in \mathcal{I}$

$$C_I(x) = (\frac{1}{2\sqrt{\pi}})^n \frac{1}{m!} (\frac{1}{2})^m \sum_{\pi,\sigma \in S_\ell} sign(\pi) sign(\sigma) \times R_{i_{\pi(1)} i_{\pi(2)} i_{\sigma(1)} i_{\sigma(2)}} \cdots \quad \cdots R_{i_{\pi(\ell-1)} i_{\pi(\ell)} i_{\sigma(\ell-1)} i_{\sigma(\ell)}}.$$

Remark 1.2 *Assume $\ell = n$. Then $f_p = 1$ of (1.2) for all p. Theorem 1.1 is a local version of Gauss-Bonnet-Chern theorem.*

Remark 1.3 *Assume $k_j = 0$ for all j. Then $f_p = \binom{n-p}{n-\ell}$ $(0 \le p \le \ell)$, $f_p = 0 (\ell + 1 \le p \le n)$. So Theorem 1.1 coinside with the result in P.Günther and R.Schimming[4].*

Now consider the similar problem for Dolbeault complex on a Kaehler manifold M, that is, a generalization of a local version of Reimann-Roch theorem. Let $e_p(t,x,y)$ denotes the kernel of the fundamental solution $E_p(t)$ of Cauchy problem $\Gamma(\wedge^p T^{*(0,1)}(M))$;

$$E_p(t)\varphi(x) = \int_M e_p(t,x,y)\varphi(y)dv_y, \quad \varphi \in \Gamma(\wedge^p T^{*(0,1)}(M))$$

satisfies

$$\begin{aligned} (\frac{d}{dt} + \mathrm{L}_p)E_p(t) &= 0 \quad \text{in } (0,T)\times M, \\ E_p(0) &= I \quad \text{in } M, \end{aligned}$$

where $L_p = \bar{\partial}_p^* \bar{\partial}_p + \bar{\partial}_{p-1}\bar{\partial}_{p-1}^*$.

The author in [11] have given a proof of a local version of Riemann-Roch theorem, constructing the fundamental solution according to the method of symbolic calculus for a degenerate parabolic operator in C.Iwasaki and N.Iwasaki[9]. There are several papers about a local version of Riemann-Roch theorem. T.Kotake[15] proved this formula for manifolds of dimension 1. V.K.Patodi[17] has proved for Kaehler manifolds of any dimension. P.B.Gilkey[8] also has shown, using invariant theory. E.Getzler[6] treated this problem by different approach. We obtain an extension of this problem as follows:

Theorem 1.4 (Main Theorem II) *Let M be a compact Kaehler manifold whose complex dimension is n, and let $E_p(t)$ be the fundamental solution on $A^{0,p}(M) = \Gamma(\wedge^p T^{*(0.1)}(M))$. Suppose f_p are of the form (1.2). Then we have*

$$\sum_{p=0}^{n}(-1)^p f_p \operatorname{tr} e_p(t,x,x)dv = \left(\frac{1}{2\pi i}\right)^n C_\ell^D(x)t^{-n+\ell} + 0(t^{-n+\ell+1}) \qquad \text{as } t \to 0,$$

where $D_\ell^D(x)$ are defined as follows:
$C_\ell^D(x) = \sum_{I\in\mathcal{I},\sharp(I)=\ell} C_I^D(x)$, *where for* $I = (i_1, i_2, \cdots, i_\ell) \in \mathcal{I}$

$$C_I^D(x) = \left[\det(\frac{\Omega}{e^\Omega - Id})\right]_{2\ell} \wedge dv^{I^c}.$$

Here Ω is a matrix whose (j,k) element is 2-form defined as

$$(\Omega)_{jk} = \sum_{a,b=1}^{n} R_{\bar{k}ja\bar{b}}\omega^a \wedge \bar{\omega}^b$$

and

$$dv^{I^c} = \bar{\omega}^{j_1}\wedge\omega^{j_1}\wedge\bar{\omega}^{j_2}\wedge\omega^{j_2}\cdots\wedge\bar{\omega}^{j_{n-\ell}}\wedge\omega^{j_{n-\ell}},$$

where $I^c = (j_1, j_2, \cdots, j_{n-\ell}) \in \mathcal{I}$ such that $I \cup I^c = \{1, 2, \cdots, n\}$.

Remark 1.5 *Assume $\ell = n$. Then $f_p = 1$ of (1.2) for all p. In this case Theorem 1.4 is a local version of Riemann-Roch theorem.*

It is known that a local version of Riemann-Roch theorem does not hold on complex manifolds by P.B.Gilkey[7]. A characterization of complex manifolds where a local version of Reimann-Roch theorem holds is given by the Kaelher form Φ of complex manifolds as follows.

Theorem 1.6 (Main Theorem III) *(1) If n is even and $\partial\bar{\partial}\Phi \neq 0$, then we have*

$$\sum_{p=0}^{n}(-1)^p \operatorname{tre}_p(t,x,x)dv_x = (2\pi)^{-n}(-1)^{\frac{n}{2}}\frac{(i\partial\bar{\partial}\Phi)^{\frac{n}{2}}}{(\frac{n}{2})!}t^{-\frac{n}{2}} + O(t^{-\frac{n}{2}+\frac{1}{2}}).$$

(2) If $\partial\bar{\partial}\Phi = 0$, then we have

$$\sum_{p=0}^{n}(-1)^p \operatorname{tre}_p(t,x,x)dv_x = (\frac{1}{2\pi i})^n\Big[\sqrt{\det(\frac{\frac{\Lambda}{2}}{\sinh(\frac{\Lambda}{2})})}\, e^{-\frac{1}{2}tr\Omega^{S}}\Big]_{2n} + 0(t),$$

where Λ is a $2n \times 2n$ real anti-symmetric matrix whose (p,q) element is 2-form(See (5.2) for the precise definition).

Our point is that one can prove the above theorems by only calculating the main term of the symbol of the fundamental solution, introducing a new weight of symbols of pseudodifferential operators.

The plan of this papaer is following. In section 2 an algebraic theorem, which is the key of the proof, is stated . The sketch of proof is given in section 3, section 4 and section 5.

2 Algebraic properties for the calculation of the trace

Let V be a vector space of dimension n with an inner product and let $\wedge^p(V)$ be its anti-symmetric p tensors. Set $\wedge^*(V) = \sum_{p=0}^{n}\wedge^p(V)$. Let $\{v_1,\cdots,v_n\}$ be an orthonormal basis for V. Let a_i^* be a linear transformation on $\wedge^*(V)$ defined by $a_i^*v = v_i \wedge v$ and let a_i be the adjoint operator of a_i^* on $\wedge^*(V)$.

Definition 2.1 *Set $\mathcal{A} = \{(\mu_1,\cdots,\mu_k) : 1 \leq k \leq 2n, 1 \leq \mu_1 < \cdots < \mu_k \leq 2n\}$, $\gamma_{2k-1} = a_k + a_k^*$, $\gamma_{2k} = i^{-1}(a_k - a_k^*)$ for $k \in \{1,2,\cdots,n\}$, $\gamma_A = i^{\frac{k(k-1)}{2}}\gamma_{\mu_1}\cdots\gamma_{\mu_k}$ for $A = (\mu_1,\cdots,\mu_k) \in \mathcal{A}$ and $\gamma_\phi = 1$.*

We have

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \quad 1 \leq \mu,\nu \leq 2n$$

and

$$\gamma_A^2 = 1 \text{ for any } A \in \mathcal{A}.$$

The following propositions are shown in [3] under the above assumptions.

Proposition 2.2 *We have the following equality for transformation on $\wedge^*(V)$.*

$$\operatorname{tr}(\gamma_A) = \begin{cases} 0, & \text{if } A \neq \phi; \\ 2^n, & \text{if } A = \phi. \end{cases}$$

Corollary 2.3 *For any $A, B \in \mathcal{A}$*

$$\operatorname{tr}(\gamma_A\gamma_B) = \begin{cases} 0, & \text{if } A \neq B; \\ 2^n, & \text{if } A = B. \end{cases}$$

Definition 2.4 *Set* $\beta_\phi = 1, \beta_j = i\gamma_{2j-1}\gamma_{2j}$ *for* $1 \le j \le n$ *and* $\beta_I = \beta_{i_1}\cdots\beta_{i_k}$ *for* $I = (i_1, \cdots, i_k) \in \mathcal{I}$. $\Gamma_0 = 1$, $\Gamma_k = \sum_{I\in\mathcal{I},\sharp(I)=k}\beta_I$.

It holds that for $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}$

$$\beta_I = \gamma_{\tilde{I}},$$

where $\tilde{I} = (2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \cdots, 2i_k - 1, 2i_k) \in \mathcal{A}$. It is clear that

$$a_k^* a_k = \frac{1}{2}(1 + \beta_k), \quad \beta_j\beta_k = \beta_k\beta_j, \quad \beta_j^2 = 1$$

by the properties of γ_j.

Proposition 2.5 *We have for any* $I = (i_1, \cdots, i_k) \in \mathcal{I}$ *the following assertions;*
(1)If $p < k$

$$\mathrm{tr}[\beta_I a_{j_1} a_{j_2}\cdots a_{j_p} a_{h_1}^* a_{h_2}^* \cdots a_{h_p}^*] = 0.$$

(2)Suppose $p = k$ *and* $\{j_1, j_2, \cdots, j_k\} \ne \{i_1, i_2, \cdots, i_k\}$ *or* $\{h_1, h_2, \cdots, h_k\} \ne \{i_1, i_2, \cdots, i_k\}$. *Then we have*

$$\mathrm{tr}[\beta_I a_{j_1} a_{j_2}\cdots a_{j_p} a_{h_1}^* a_{h_2}^* \cdots a_{h_p}^*] = 0.$$

(3) Let π, σ *be elements of the permutation group of degree* k. *Then we have*

$$\mathrm{tr}[\beta_I a_{i_{\pi(1)}}^* a_{i_{\sigma(1)}} a_{i_{\pi(2)}}^* a_{i_{\sigma(2)}} \cdots a_{i_{\pi(k)}}^* a_{i_{\sigma(k)}}] = 2^{n-k} sign(\pi) sign(\sigma).$$

Let Ψ_p be the projection of $\wedge^*(V)$ on $\wedge^p(V)$. The following Proposition is the key algebraic argument of the proof of this section.

Proposition 2.6 *For any* p $(0 \le p \le n)$ *we have the following equation*

$$\Psi_p = \sum_{q=0}^{n} \mathcal{M}_{pq}\Gamma_q,$$

where

$$\mathcal{M}_{pq} = \sum_{p,q\le j\le n} (-1)^{p+j} 2^{-j} \binom{j}{p}\binom{n-q}{n-j}.$$

Note that a $(n+1)\times(n+1)$ matrix $\mathcal{M} = (\mathcal{M}_{pq})_{0\le p,q\le n}$ is regular because

$$(\mathcal{M}^{-1})_{pq} = \sum_{0\le j\le p,q} (-1)^{p+j} 2^j \binom{q}{j}\binom{n-j}{n-p}.$$

Then we have

Theorem 2.7 *Let* α_p $(\ell + 1 \le p \le n)$ *be constants. The equation*

$$\sum_{q=0}^{n} f_q \Psi_q = (-1)^\ell 2^{\ell-n}\Gamma_\ell + \sum_{p=\ell+1}^{n} \alpha_p \Gamma_p$$

has solution as follows:

$$f_p = (-1)^p\left\{\binom{n-p}{n-\ell} + \sum_{j=\max(\ell+1,p)}^{n} k_j\binom{n-p}{n-j}\right\} \text{ for any p}$$

with constants k_j $(\ell+1 \leq j \leq n)$ *defined by*

$$k_j = (-1)^j 2^{n-j}\left\{2^{\ell-n}(-1)^\ell\binom{j}{\ell} + \sum_{p=\ell+1}^{j}\binom{j}{p}\alpha_p\right\}.$$

Especially
(1) If $\ell = n$, *then*

$$\sum_{q=0}^{n} f_q\Psi_q = (-1)^n\Gamma_n$$

holds if and only if

$$f_p = (-1)^p \text{ for any } p.$$

(2) If $\alpha_p = (-1)^p 2^{\ell-n}\binom{p}{\ell}$ $(\ell+1 \leq p \leq n)$, *we have* f_p *of the following form*

$$f_p = \begin{cases}(-1)^p\binom{n-p}{n-\ell}, & (0 \leq p \leq \ell);\\ 0, & (\ell+1 \leq p \leq n).\end{cases}$$

(3) If $\alpha_p = (-1)^\ell 2^{\ell-n}\binom{p}{\ell}$ $(\ell+1 \leq p \leq n)$, *we have* f_p *of the following form*

$$f_p = \begin{cases}0, & (0 \leq p \leq n-\ell-1);\\ (-1)^{n-\ell+p}\binom{p}{n-\ell}; & (n-\ell \leq p \leq n).\end{cases}$$

3 The proof of Main Theorem I (Riemannian manifolds)

Let M be a smooth Riemannian manifold of dimension n with a Riemannian metric g. Let $X_1, X_2, \cdots, X_n$ be a local orthonormal frame of $T(M)$ in a lokal path U. And let $\omega^1, \omega^2, \cdots, \omega^n$ be its dual. The differential d and its dual ϑ acting on $\Gamma(\wedge^p T^*(M))$ are written as follows,using the Levi-Civita connection ∇ (See Appendix A of S.Murakami[16]):

$$d = \sum_{j=1}^{n} e(\omega^j)\nabla_{X_j}, \qquad \vartheta = -\sum_{j=1}^{n} \imath(X_j)\nabla_{X_j},$$

where we use the following notations.

Notations.

$$e(\omega^j)\omega = \omega^j \wedge \omega, \ \imath(X_j)\omega(Y_1, \cdots, Y_{p-1}) = \omega(X_j, Y_1, \cdots, Y_{p-1}).$$

Let $R(X,Y)$ be the curvature transformation, that is

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Set

$$R(X_i, X_j)X_k = \sum_{\ell=1}^{n} R_{\ell kij}X_\ell \ \ 1 \leq i, j, k, \leq n.$$

The Laplacian $\Delta = d\vartheta + \vartheta d$ on $\sum_{p=0}^{n} \Gamma(\wedge^p T^*(M))$ has the following Weitzenböck's formula:

$$\Delta = -\{\sum_{j=1}^{n} \nabla_{X_j} \nabla_{X_j} - \sum_{j=1}^{n} \nabla_{(\nabla_{X_j} X_j)} + \sum_{i,j=1}^{n} e(\omega) \imath(X_j) R(X_i, X_j)\}. \tag{3.1}$$

We use the following notations in the rest of this section.

$$a_j^* = e(\omega *), \qquad a_k = \iota(X_k).$$

The fundamental solution $E(t)$ has a expansion, due to [10].

$$E(t) \sim \sum_{j=0} u_j(t, x, D),$$

where $u_j(t, x, D)$ are pseudodifferential operators with parameter t.

The following statement is obtained in p.255 of [10]. The kernel of pseudo-differential operator with symbol $u_0(t, x, \xi)$ is obtained as

$$\begin{aligned}\tilde{u}_0(t, x, x) &= (2\pi)^{-n} \int_{\mathbf{R}^n} u_0(t, x, \xi) d\xi \\ &= (\frac{1}{2\sqrt{\pi t}})^n \sqrt{\det g} e^{-tR} \Big(1 + 0(\sqrt{t})\Big),\end{aligned}$$

where

$$R = \sum_{i,j,k,q=1}^{n} R_{qkij} a_i^* a_j a_k^* a_q.$$

We shall calculate

$$\text{tr } (\beta_I \tilde{u}_0(t, x, x)) dx = (\frac{1}{2\sqrt{\pi t}})^n \text{tr } (\beta_I e^{-tR}) dv (1 + 0(\sqrt{t})), \tag{3.2}$$

for $I \in \mathcal{I}$, $\sharp(I) = r$.

Using

$$e^{-tR} = \sum_{k=0}^{\infty} \{\frac{(-1)^k}{k!} R^k t^k\},$$

and by Proposition 2.5 we have

$$\text{tr } (\beta_I e^{-tR}) = \begin{cases} \text{tr}(\beta_I \frac{(-1)^m}{m!} R^m) t^m + 0(t^{m+1}), & \text{if } r = 2m\ ; \\ 0(t^{\frac{r+1}{2}}), & \text{if } r \text{ is odd.} \end{cases} \tag{3.3}$$

We have the following proposition.

Proposition 3.1 *For* $I = (i_1, i_2, \cdots, i_r) \in \mathcal{I}$ $(r = 2m)$

$$\begin{aligned}\text{tr}(\beta_I (-1)^m R^m) = 2^{n-r-m} \sum_{\pi, \sigma \in S_r} sign(\pi) sign(\sigma) \\ \times R_{i_{\pi(1)} i_{\pi(2)} i_{\sigma(1)} i_{\sigma(2)}} \cdots R_{i_{\pi(r-1)} i_{\pi(r)} i_{\sigma(r-1)} i_{\sigma(r)}}.\end{aligned} \tag{3.4}$$

By (3.2),(3.3) and Proposition 3.1 we have

$$\text{(3.5)} \qquad \mathrm{tr}\ (\beta_I \tilde{u}_0(t,x,x))dx = \begin{cases} 2^{n-r} t^{-\frac{n}{2}+\frac{r}{2}} C_I(x)dv + 0(t^{-\frac{n}{2}+\frac{r}{2}+1}), & \text{if } r = 2m \text{ ;} \\ 0(t^{-\frac{n}{2}+\frac{r}{2}+\frac{1}{2}}), & \text{if } r \text{ is odd} \end{cases}$$

with $C_I(x)$ defined in Defifnition 1.1. Similarly we have

$$\text{(3.6)} \qquad \mathrm{tr}\ (\beta_I \tilde{u}_j(t,x,x))dx = 0(t^{-\frac{n}{2}+\frac{r}{2}+\frac{j}{2}}).$$

By Theorem 2.7 we obtain

$$\begin{aligned} \mathrm{tr}\ (\sum_{p=0}^{n} f_p e_p(t,x,x)) &= (-1)^{\ell} 2^{\ell-n} \sum_{I \in \mathcal{I}, \sharp(I)=\ell} \mathrm{tr}\ (\beta_I e(t,x,x)) \\ &\quad + \sum_{p=\ell+1}^{n} \alpha_p \mathrm{tr}\ (\Gamma_p e(t,x,x)). \end{aligned}$$

By (3.5) and (3.6) we have

$$\mathrm{tr}\ (\Gamma_p e(t,x,x)) = 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}).$$

Applying (3.5) (3.6), we have

$$\mathrm{tr}\ (\sum_{p=0}^{n} f_p e_p(t,x,x)) = \begin{cases} C_I(x) t^{-\frac{n}{2}+\frac{\ell}{2}} + 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}), & \text{if } \ell \text{ is even ;} \\ 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}), & \text{if } \ell \text{ is odd.} \end{cases}$$

4 The proof of Main Theorem II (Kaehler manifolds)

Let M be a compact Kaehler manifold whose complex dimension is n with a hermitian metric g. Set $Z_1, Z_2, \cdots, Z_n$ be a local orthonormal frame of $T^{1,0}(M)$ in a local patch of chart U. And let $\omega^1, \omega^2, \cdots, \omega^n$ be its dual. The differential $\bar{\partial}$ and its dual $\bar{\partial}^*$ acting on $A^{0,p}(M)$ are given as follows, using the Levi-Civita connection ∇:

$$\bar{\partial} = \sum_{j=1}^{n} e(\bar{\omega}^j) \nabla_{\bar{Z}_j}, \bar{\partial}^* = -\sum_{j=1}^{n} \imath(\bar{Z}_j) \nabla_{Z_j},$$

where we use the following notations.

Notations.

$$Z_{\bar{j}} = \bar{Z}_j, \quad \omega^{\bar{j}} = \bar{\omega}^j \quad (j = 1, \cdots, n),$$
$$e(\omega^\alpha)\omega = \omega^\alpha \wedge \omega, \quad \imath(Z_\alpha)\omega(Y_1, \cdots, Y_{p-1}) = \omega(Z_\alpha, Y_1, \cdots, Y_{p-1}),$$
$$(\alpha \in \Lambda = \{1, \cdots, n, \bar{1}, \cdots, \bar{n}\}).$$

Let $R(Z_\alpha, \bar{Z}_\beta)$ be the curvature transformation;

$$R(Z_\alpha, Z_\beta) = [\nabla_{Z_\alpha}, \nabla_{Z_\beta}] - \nabla_{[Z_\alpha, Z_\beta]} \quad (\alpha, \beta \in \Lambda).$$

The curvarute transformations satisfy

$$R(Z_i, Z_j) = 0, \quad R(\bar{Z}_i, \bar{Z}_j) = 0,$$

because of M is a Kaehler manifold. Set

$$R(Z_i, \bar{Z}_j)Z_\beta = \sum_{\gamma\in\Lambda} R_{\bar{\gamma}\beta i\bar{j}} Z_\gamma \quad (\beta \in \Lambda).$$

The Laplacian $L = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$ on $A^{0,*}(M) = \sum_{p=0}^n A^{0,p}(M)$ has the following Bochner-Kodaira formula:

$$L = -\frac{1}{2}\{\sum_{j=1}^{n}(\nabla_{Z_j}\nabla_{\bar{Z}_j} + \nabla_{\bar{Z}_j}\nabla_{Z_j}) - \sum_{j=1}^{n}\nabla_{(\nabla_{Z_j}\bar{Z}_j + \nabla_{\bar{Z}_j}Z_j)} - \sum_{j=1}^{n} R(Z_j, \bar{Z}_j)\}.$$

We use the following notations in the rest of this section.

$$e(\bar{\omega}^j) = a_j^*, \iota(\bar{Z}_k) = a_k.$$

The fundamental solution has a expansion, due to [11]

$$E(t) \sim \sum_{j=0} u_j(t, x, D),$$

where $u_j(t, x, D)$ are pseudodifferential operators with parameter t and the main part of their symbols $u_0(t, x, \xi)$ is represented of the pricise form. The kernel $\tilde{u}_0(t, x, x)$ of pseudodifferential operator with symbol $u_0(t, x, \xi)$ is obtained as in p.90 [11]

$$\tilde{u}_0(t, x, x) = (2\pi t)^{-n} \det(\frac{t\mathcal{M}_0}{\exp(t\mathcal{M}_0) - Id})\sqrt{\det g}.$$

Note that

$$e(t, x, x)dv = \tilde{u}_0(t, x, x)dx(1 + 0(t)) \tag{4.1}$$

We shall calculate

$$\text{tr } (\beta_I \tilde{u}_0(t, x, x))dx = (2\pi t)^{-n} \text{tr } \Big(\beta_I \det(\frac{t\mathcal{M}_0}{\exp(t\mathcal{M}_0) - Id})\Big)dv, \tag{4.2}$$

for $I \in \mathcal{I}$, $\sharp(I) = r$, where $(\mathcal{M}_0)_{jk} = R(Z_j, \bar{Z}_k) = -\sum_{p,q=1}^n R_{p\bar{q}j\bar{k}} a_q^* a_p$
$= \sum_{p,q=1}^n R_{j\bar{k}\bar{q}p} a_q^* a_p$.

Set

$$\det(\frac{t\mathcal{M}_0}{\exp(t\mathcal{M}_0) - Id}) = \sum_{j=0}^n A_j t^j.$$

Then we have by proposition 2.5

$$\text{tr } \Big(\beta_I \det(\frac{t\mathcal{M}_0}{\exp(t\mathcal{M}_0) - Id})\Big) = \text{tr } (\beta_I A_r)t^r + 0(t^{r+1}). \tag{4.3}$$

Set Ω a matrix whose (j, k) element is 2-form defined by

$$(\Omega)_{jk} = -\sum_{p,q=1}^n R_{j\bar{k}\bar{q}p}\bar{\omega}^q \wedge \omega^p = \sum_{p,q=1}^n R_{\bar{k}jp\bar{q}}\omega^p \wedge \bar{\omega}^q.$$

Then we have

Proposition 4.1

$$\text{tr }(\beta_I A_r)dv = (-1)^r(\frac{1}{i})^n 2^{n-r}\Big[\det(\frac{\Omega}{\exp\Omega - Id})\Big]_{2r} \wedge dv^{I^c}. \tag{4.4}$$

From (4.2),(4.3) and Proposition 4.1 the following equation holds.

$$\begin{aligned}&\text{tr }(\beta_I \tilde{u}_0(t,x,x))dx \\ &= (-1)^r(\frac{1}{2\pi i})^n 2^{n-r} t^{-n+r} \quad \Big[\det(\frac{\Omega}{\exp\Omega - Id})\Big]_{2r} \wedge dv^{I^c} + 0(t^{-n+r+1}).\end{aligned}$$

Now by Theorem 2.7 we obtain

$$\begin{aligned}\text{tr }(\sum_{p=0}^{n} f_p e_p(t,x,x)) &= (-1)^\ell 2^{\ell-n} \sum_{I\in\mathcal{I},\sharp(I)=\ell} \text{tr }(\beta_I e(t,x,x)) \\ &\quad + \sum_{p=\ell+1}^{n} \alpha_p \text{tr }(\Gamma_p e(t,x,x))\end{aligned}$$

with some constants $a_j(\ell+1 \le j \le n)$. Applying (4.5) we have

$$\sum_{p=\ell+1}^{n} \alpha_p \text{tr }(\Gamma_p e(t,x,x))dv = 0(t^{-n+\ell+1}).$$

By (4.1) and (4.5) we obtain

$$\begin{aligned}\text{tr }(\sum_{p=0}^{n} f_p e_p(t,x,x))dv &= (\frac{1}{2\pi i})^n t^{-n+\ell} \sum_{I\in\mathcal{I},\sharp(I)=\ell} \Big[\det(\frac{\Omega}{\exp\Omega - Id})\Big]_{2\ell} \wedge dv^{I^c} \\ &\quad + 0(t^{-n+\ell+1}).\end{aligned}$$

5 The proof of Main Theorem III (Complex Manifolds)

Let M be a complex manifold with a Hermitian metric g. Choose a local chart and a orthnormal system as in the previous section. Using the Levi-Civita connection ∇, we have the following representation for differential d and its adjoint ϑ acting on $A^{p,q}(M)$:

$$d = \sum_{j=1}^{n} e(\omega^j)\nabla_{Z_j} + \sum_{j=1}^{n} e(\bar{\omega}^j)\nabla_{\bar{Z}_j}, \quad \vartheta = -\sum_{j=1}^{n} \imath(Z_j)\nabla_{\bar{Z}_j} - \sum_{j=1}^{n} \imath(\bar{Z}_j)\nabla_{Z_j}.$$

The connection ∇ is the Levi-Civita connection . So both ∇g and the torsion T vanish. But ∇ does not preserve type of vector fields, that is, $\nabla I \neq 0$ for the complex structure I. In this case, generally we have $R(Z_i, Z_j) \neq 0, \quad R(\bar{Z}_i, \bar{Z}_j) \neq 0$. For the representation of our operator L, we introduce connection ∇^S and $\nabla^{\bar{S}}$ and give characterization of these connections.

Definition 5.1
(1) Let $\nabla^{\bar{S}}$ be the Hermitian connection of M, that is, the unique connection which satisfies the following conditions;

$$\nabla^{\bar{S}} g = 0, \quad \nabla^{\bar{S}} I = 0, \quad T^{\bar{S}}(V, \bar{W}) = 0 \quad \Big(V \in T^{(1,0)}(M), \bar{W} \in T^{(0,1)}(M)\Big).$$

Let $\tilde{S}^{\gamma}_{\alpha\beta}$ $(\alpha,\beta,\gamma\in\Lambda)$ *be the following functions of this connection* $\nabla^{\tilde{S}}$;

$$\nabla^{\tilde{S}}_{Z_\alpha} Z_j = \sum_{k=1}^{n} \tilde{S}^k_{\alpha j} Z_k, \quad \nabla^{\tilde{S}}_{Z_\alpha} Z_{\bar{j}} = \sum_{k=1}^{n} \tilde{S}^{\bar{k}}_{\alpha\bar{j}} \bar{Z}_k.$$

(2) Let ∇^S *be the unique connection which satisfies the following conditions;*

$$\nabla^S g = 0, \quad \nabla^S I = 0,$$

and

$$g(\bar{W}, T^S(U,V)) + g(U, T^S(\bar{W},V)) = 0 \quad \text{for } U,V\in T^{(1,0)}(M), \bar{W}\in T^{(0,1)}(M).$$

Proposition 5.2 *We have the following representation*

$$\bar{\partial} = \sum_{r=1}^{n} a^*_{\bar{r}} D_{\bar{r}}, \quad \bar{\partial}^* = -\sum_{r=1}^{n} a_{\bar{r}}(D_r + \sum_{j=1}^{n} c^{\bar{j}}_{\bar{j}r}), \tag{5.1}$$

where

$$D_\alpha = Z_\alpha - \sum_{j,k=1}^{n} c^{\bar{k}}_{\alpha\bar{j}} a^*_{\bar{j}} a_{\bar{k}} - \sum_{j,k=1}^{n} \tilde{S}^k_{\alpha j} a^*_j a_k \quad (\alpha\in\Lambda)$$

with functions $c^{\gamma}_{\alpha\beta}$ $(\alpha,\beta,\gamma\in\Lambda)$ *defined by* $\nabla_{Z_\alpha} Z_\beta = \sum_{\gamma\in\Lambda} c^{\gamma}_{\alpha\beta} Z_\gamma$.

The following proposition holds for the Kaehler form $\Phi(u,v) = g(Iu,v)$.

Proposition 5.3

$$i\partial\bar{\partial}\Phi = \sum_{j,k,\ell,m=1}^{n} \omega_{\bar{\ell}\bar{m}jk} \bar{\omega}^{\ell} \wedge \bar{\omega}^{m} \wedge \omega^{j} \wedge \bar{\omega}^{k},$$

where

$$\omega_{\bar{\ell}\bar{m}jk} = -\frac{1}{2} R_{\bar{\ell}\bar{m}jk} + \frac{1}{2}\sum_{r=1}^{n}\{c^{\bar{r}}_{\bar{m}j} c^{r}_{k\bar{\ell}} + c^{\bar{r}}_{\bar{\ell}k} c^{r}_{j\bar{m}} - c^{\bar{r}}_{\bar{m}k} c^{r}_{j\bar{\ell}} - c^{\bar{r}}_{\bar{\ell}j} c^{r}_{k\bar{m}}\}.$$

We have the following representation formula for L on complex manifolds instead of Bochner-Kodaira formula, using (5.1).

Theorem 5.4 *It holds on* $A^{0,*}(M) = \sum_{q=0}^{n} A^{0,q}(M)$,

$$\begin{aligned} L = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = & -\frac{1}{2}\{\sum_{j=1}^{n}(\nabla^S_{Z_j}\nabla^S_{\bar{Z}_j} + \nabla^S_{\bar{Z}_j}\nabla^S_{Z_j}) - \nabla^S_{\bar{D}} \\ & + \sum_{j,k,r=1}^{n} e(\bar{\omega}^j)\iota(\bar{Z}_k) g(R^{\tilde{S}}(\bar{Z}_j, Z_k) Z_r, \bar{Z}_r)\} \\ & - \sum_{\ell,m,j,k=1}^{n} \omega_{\bar{\ell}\bar{m}jk} \bar{a}^*_{\ell} \bar{a}^*_{m} \bar{a}_j \bar{a}_k - 2\sum_{r,\ell,k=1}^{n} \omega_{\bar{r}\bar{\ell}rk} \bar{a}^*_{\ell} \bar{a}_k, \end{aligned}$$

where

$$D = \sum_{r=1}^{n}\{\nabla^S_{\bar{Z}_r} Z_r + \nabla^S_{Z_r}\bar{Z}_r + T^S(Z_r, \bar{Z}_r)\}.$$

Remark 5.5 *If M is a Kaehler manifold, then we have the following equations;*

$$\partial\bar{\partial}\Phi = 0,\ \nabla^S = \nabla^{\bar{S}} = \nabla,\ T^S = T^{\bar{S}} = 0.$$

In the case $\partial\bar{\partial}\Phi \neq 0$, we can construct the fundamental solution for $(\frac{\partial}{\partial t} + L)$ on complex manifolds as a pseudo-differntial operator, using the above Theorem 5.4 instead of (3.1). By the similar argument we obtain the assertion(1) of The Main theorem III. But in this cse, the supsertrace has the singularity with respect to t as $t \to 0$. So, in this case we may say that "a local version of Reimann-Roch theorem does not hold."

On the other hand, in the case $\partial\bar{\partial}\Phi = 0$ introducing a curvature transformation $R^M(\bar{Z}_j, Z_k)$ which corresponds a new connection $\nabla^M = 2\nabla - \nabla^S$, we obtain the assertion (2) of the Main Theorem III by (5.1) and the above theorem. Here Λ is defined as follows:

(5.2)
$$(\Lambda)_{pq} = \sum_{\ell,m=1}^{n} g(R^M(\bar{Z}_\ell, Z_m)X_q, X_p)\bar{\omega}^\ell \wedge \omega^m,$$

$$Z_j = \frac{1}{\sqrt{2}}(X_j - iX_{n+j}), \bar{Z}_j = \frac{1}{\sqrt{2}}(X_j + iX_{n+j}).$$

This assertion coincides with the result which is proved in J.M.Bismut[2] by the probabilistic method.

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