数理解析研究所講究録
1502 巻 2006 年 41-53

Extra singularities of geometric solutions to Monge-Ampère equations of three variables.

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1 Introduction.

In this survey article, we review recent results on singularities of solutions to Monge-Ampère equations of two independent variables [12], and give the generic classification for Monge-Ampère equations of three independent variables. Then we find the remarkable difference in generic singularities which appear in the case of two variables and three variables. The details will be given in the forthcoming paper.

Solutions to a Monge-Ampère equation

$$\det\left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right)_{1\le i,j\le n} = g\left(x_1, x_2, \ldots, x_n, z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \cdots, \frac{\partial z}{\partial x_n}\right)$$

for a function $z = z(x), x = (x_1, x_2, \ldots, x_n)$, can be treated as a Legendrian submanifold, a *geometric solution*, in the $(x, z, p = \partial z/\partial x)$ space satisfying a condition due to the equation. Then the singularities of a solution are regarded as Legendrian singularities; singularities of a geometric solution via the Legendrian projection $(x, z, p) \mapsto (x, z)$.

The list of generic singularities of Legendrian projections of Legendrian submanifolds (which are not necessarily geometric solutions) consists of the cuspidal edge (A_2-singularity) and the swallowtail (A_3-singularity) in the case of two variables. See Figure 1.

In the case of three variables, the list consists of A_2, A_3, A_4 and D_4-singularities. The A_4-singularity is called the butterfly. The D_4 singularities are the pyramid (elliptic umbilic, D_4^-) and the purse (hyperbolic umbilic, D_4^+) [4]. Figures 2 and 3 illustrate the caustics, the loci of singularities in (x_1, x_2, x_3)-space in each cases A_2, A_3, A_4, D_4^- and D_4^+.

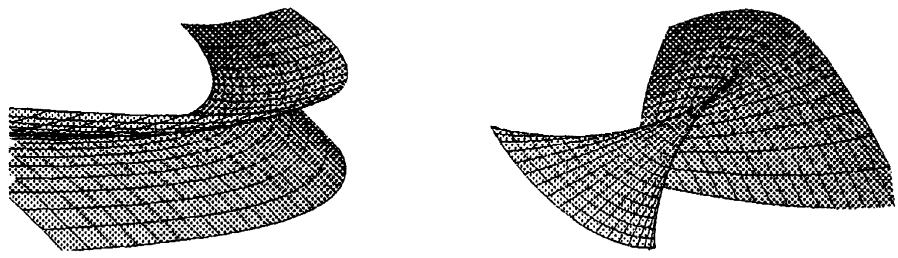

Figure 1: the cuspidal edge (left) and the swallowtail (right)

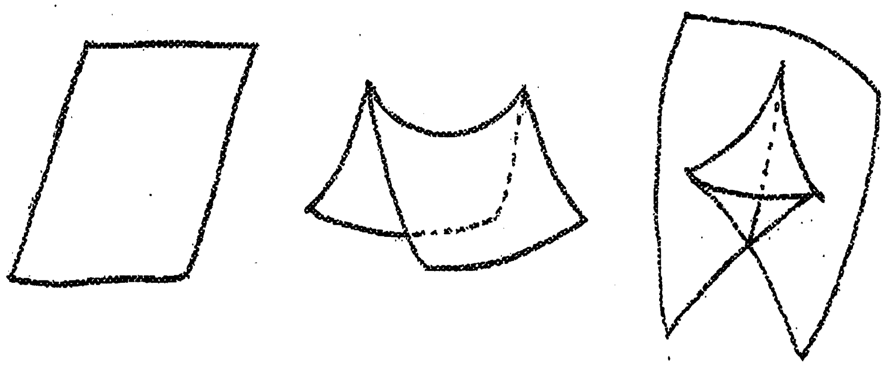

Figure 2: Caustics of A_2, A_3 and A_4-singularities in the three space

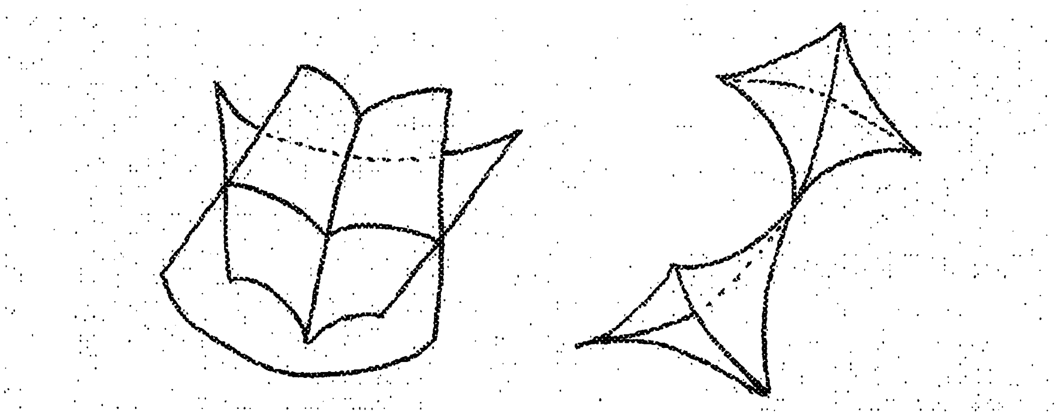

Figure 3: Caustics of D_4^+ and D_4^--singularities in the three space

In [12], we study on the singularities of solutions to the Monge-Ampère equation

$$\det\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} = c,$$

c being a constant, the equation of improper affine spheres, and

$$\det\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} = c\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^2,$$

the equation of surfaces with the constant Gaussian curvature. Then it is shown that generic singularities of solutions to each equation are cuspidal edges and swallowtails as in the case without an equation. Moreover, in the case $c \neq 0$, also the list of generic singularities of dual surfaces turns to be the same. To show the classification results, we used in [12] the criterion of cuspidal edges and swallowtails established in [17].

We clarify our class of Monge-Ampère equations, *Hessian Monge-Ampère equations*, recalling the formulation established in [13]. Then we generalise the classification result in [12] to general Monge-Ampère equations in §3. Moreover, in the case of three variables, we announce that there appear extra singularities in generic solutions to a Monge-Ampère equation, other than A_2, A_3, A_4, D_4-singularities in §4. Moreover in §5 we explain roughly the method of generating families to show the classification results in this paper.

In this paper, all manifolds and mappings are assumed to be of class C^∞ unless otherwise stated.

2 Monge-Ampère equations with Lagrangian pairs.

In [13], we introduce a class of Monge-Ampère equations; Monge-Ampère systems with a Lagrangian pair. Consider $\mathbf{R}^{2n+1}$ with coordinates $(x, z, p) = (x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n)$ and the contact form

$$\theta = dz - p_1 dx_1 - p_2 dx_2 - \cdots - p_n dx_n$$

on $\mathbf{R}^{2n+1}$. The contact distribution $D = \{\theta = 0\} \subset T\mathbf{R}^{2n+1}$ has the decomposition $D = D_1 \oplus D_2$ into the pair of two Lagrangian sub-bundles

$$D_1 = \left\langle \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \ldots, \frac{\partial}{\partial p_n} \right\rangle$$

and

$$D_2 = \left\langle \frac{\partial}{\partial x_1} + p_1 \frac{\partial}{\partial z}, \frac{\partial}{\partial x_2} + p_2 \frac{\partial}{\partial z}, \ldots, \frac{\partial}{\partial x_n} + p_n \frac{\partial}{\partial z} \right\rangle$$

for the symplectic form $d\theta$ on D. We call (D_1, D_2) a *Lagrangian pair.* Since D_1, D_2 are both integrable, we have the Legendrian double fibrations:

$$\begin{array}{ccc} & \mathbf{R}^{2n+1} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbf{R}^{n+1} & & \mathbf{R}^{n+1}, \end{array}$$

where $\pi_1(x, z, p) = (x, z)$ and $\pi_2(x, z, p) = (p,\ x \cdot p - z)$, $x \cdot p = \sum_{i=1}^{n} x_i p_i$, are projections along D_1 and D_2 respectively.

In general, a differential system $\mathcal{M}$ on a contact manifold is called a *Monge-Ampère system* if $\mathcal{M}$ is locally generated by a contact form θ and an n-form ω ([21][22]).

In particular consider an n-form ω on $\mathbf{R}^{2n+1}$ of the form $\omega = \omega_1 - \omega_2$, ω_1, ω_2 satisfying that $u \rfloor \omega_1 = 0$ for any $u \in D_1$, $v \rfloor \omega_2 = 0$ for any $v \in D_2$, $\omega_1 | D_2$ is a volume form on D_2, and that $\omega_2 | D_1$ is a volume form on D_1. Then the differential system generated by θ and ω is called a *Monge-Ampère system with the Lagrangian pair* (D_1, D_2). Then we can take $\omega = \omega_1 - \omega_2$ with

$$\omega_1 = g(x, z, p) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n, \quad \omega_2 = dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n,$$

for a non-vanishing function $g = g(x, z, p)$. Since n and the function g determine the system, we designate it by $\mathcal{M}(n, g)$. Note that we assume g is non-vanishing (on the domain we work on).

An immersed submanifold L^n in $(\mathbf{R}^{2n+1}, D)$ of dimension n is called *Legendrian* if $\theta|_L = 0$ for a contact form θ, namely, if L is an immersed integral submanifold to D. A Legendrian submanifold L in $(\mathbf{R}^{2n+1}, D)$ is called a *geometric solution* to a Monge-Ampère system generated by θ and ω if $(\theta|_L = 0$ and) $\omega|_L = 0$.

A function $z : U \to \mathbf{R}$ on a domain U of $\mathbf{R}^n$ induces a Legendrian submanifold L in $\mathbf{R}^{2n+1}$ by

$$L = \left\{ (x, z, p) = \left(x, z(x), \frac{\partial z}{\partial x} \right) \;\middle|\; x \in U \right\}.$$

Then L is a geometric solution to $\mathcal{M}(n, g)$ if and only if z is a classical solution to the equation

$$\det\left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right)_{1\le i,j\le n} = g\left(x_1, \ldots, x_n, z, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_1}\right)$$

We call this type of equations *Hessian Monge-Amp'ere equations.*

Note that a geometric solution L in $\mathbf{R}^{2n+1}$ gives a multi-valued classical solution if $\pi_1|_L$ is immersive. A singular point of L means a non-immersive point of $\pi_1|_L$.

We denote by Hess(z) the Hessian determinant of $z = z(x_1, x_2, \ldots, x_n)$.

Example 2.1 Consider the equation Hess$(z) = c$, $(c \neq 0)$ for improper affine spheres $z = z(x_1, \ldots, x_n)$ of dimension n. The corresponding Monge-Ampère system $\mathcal{M}(n, c)$ to it is generated by the contact form $\theta = dz - p_1 dx_1 - \cdots - p_n dx_n$ and $\omega = c dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n - dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n$.

Example 2.2 The equation $K = c, (c \neq 0)$ for surfaces of constant Gaussian curvature is described by the Monge-Ampère system $\mathcal{M}(2, c(1 + p_1^2 + p_2^2)^2)$ generated by the contact form $\theta = dz - p_1 dx_1 - p_2 dx_2$ and

$$\omega = c(1 + p_1^2 + p_2^2)^2 dx_1 \wedge dx_2 - dp_1 \wedge dp_2.$$

By Jörgens, Calabi and Pogorelov's theorems, a global convex solution $z : \mathbf{R}^n \to \mathbf{R}$ to the equation Hess$(z) = c$ $(c > 0)$ is necessarily a quadratic polynomial function. By Hilbert's theorem, we see that there does not exist any complete surface satisfying $K = c$ $(c < 0)$. Also we see, by Liebmann's theorem, any complete surface with $K = c$ $(c > 0)$ is a sphere. Therefore it is indispensable to study singularities of solutions to Monge-Ampère equations. Then generic classification of singularities of geometric solutions to the corresponding Monge-Ampère systems provides one of higher perspective beyond intuitive and analytic approaches to the solutions to Monge-Ampère equations. Moreover we classify singularities of the original solution $z = z(x_1, x_2, \ldots, x_n)$ as well as its Legendre transformation $\tilde{z} = \sum_{i=1}^n p_i x_i - z = \sum_{i=1}^n \frac{\partial z}{\partial x_i} x_i - z$.

The geometric foundation on Monge-Ampère equations is given, for instance, in [21][22][18][6][15][5]. For related geometric studies on singularities can be seen in [9][20][17][14]. For a related analytic study on Monge-Ampère equations can be seen, for instance, in [10].

The fundamental observation we will use in particular is the following:

Lemma 2.3 *Let $L \subset \mathbf{R}^{2n+1}$ be a geometric solution to a Monge-Ampère system $\mathcal{M}(n,g)$ for a non-vanishing function g. Then $\ell \in L$ is a singular point of $\pi_1|_L$ if and only if $\ell \in L$ is a singular point of $\pi_2|_L$.*

Proof: Since $\theta|_L = 0$, we see $\ell \in L$ is a singular point of $\pi_1|_L$ if and only if $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n|_L = 0$ at ℓ. Similarly, since $\theta = -(d(\sum_{i=1}^n p_i x_i - z) - \sum_{i=1}^n x_i dp_i) = 0$ on L, we see $\ell \in L$ is a singular point of $\pi_2|_L$ if and only if $dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n|_L = 0$ at ℓ. Now

$$\omega = g(x,z,p)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n - dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n = 0$$

on L, and $g(\ell) \neq 0$. Thus we see $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n|_L = 0$ at ℓ if and only if $dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n|_L = 0$ at ℓ. □

Also the following is fundamental:

Lemma 2.4 *$\pi_{1*}|\mathrm{Ker}(\pi_2|_L)_*$ is injective. Similarly $\pi_{2*}|\mathrm{Ker}(\pi_1|_L)_*$ is injective.*

Proof: Since $(\pi_1, \pi_2) : \mathbf{R}^{2n+1} \to \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ is an embedding, the restriction $(\pi_1|_L, \pi_2|_L) : L \to \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ is an immersion. Therefore Lemma 2.4 follows easily. □

3 Monge-Ampère equations of two variables.

The result in [12] is generalised to the following result:

Theorem 3.1 *Let $g(x_1, x_2, z, p_1, p_2)$ be a non-vanishing analytic function on a domain of $\mathbf{R}^5$. Then, for generic geometric solutions to the Monge-Ampère system $\mathcal{M}(2,g)$ corresponding to the equation*

$$\det\left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right)_{1 \le i,j \le 2} = g\left(x_1, x_2, z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}\right),$$

the pair of π_1-Legendrian singularity and π_2-Legendrian singularity at any point is given exactly by the list:

$$(A_1, A_1), (A_2, A_2), (A_2, A_3), (A_3, A_2).$$

All four cases actually appear in a geometric solution to $\mathcal{M}(2,g)$ and they are stable under small perturbations among geometric solutions to $\mathcal{M}(2,g)$.

For a generic Legendrian submanifold L^2 in $\mathbf{R}^5$, $\theta|_L = 0$, we have six cases:

$$(A_1, A_1), (A_1, A_2), (A_1, A_3), (A_2, A_1), (A_2, A_2), (A_3, A_1).$$

By Theorem 3.1, for a generic L^2 in $\mathbf{R}^5$ with $\theta|_L = 0, \omega|_L = 0$, just the cases $(A_1, A_1), (A_2, A_2)$ are realised as generic singularities of a Monge-Ampère equation, and moreover two cases $(A_2, A_3), (A_3, A_2)$ occur generically as singularities of a Monge-Ampère equation, while they are not generic as singularities of Legendrian immersions via the Legendrian double fibration. The equation provides the essential restriction via Lemma 2.3.

Similarly as in [12], Theorem 3.1 can be proved by using the criterion of [17]. Also the method of generating families can be applied; in the next section, we show the outline of the method, in the case of three variables. The method is applied equally to the case of two variables. We assume g is analytic in Theorem 3.1 and in Theorem 4.1 below. This is because we use the theorem of Cauchy-Kovalevskaya to guarantee the solvability of an initial value problem.

4 Monge-Ampère equations of three variables.

As is mentioned already in Introduction, it is known that the generic Legendrian singularities of three dimension are $A_1, A_2, A_3, A_4, D_4^+, D_4^-$ [4]. However we easily see that the generic singularities of geometric solutions to a Monge-Ampère system with a Lagrangian pair of three variables never have the same list. Regarding with the symmetry between π_1 and π_2, suppose they have the same list, and suppose $\pi_2|_L$ is of type D_4 at $\ell \in L$ for a generic L via both π_1 and π_2. Then $\dim \mathrm{Ker}(\pi_2|_L)_* = 2$. Then, by Lemma 2.4, we have $\pi_1|_L$ is of rank 2 so it must be of type A_k. However, by Lemma 2.3, the singular loci of $\pi_1|_L$ and $\pi_2|_L$ coincide. The singular locus of an A_k-singularity is non-singular itself, On the other hand, the singular locus of a D_4-singularity is a cone, which has a singularity. These lead a contradiction.

In the case of three variables, in fact we get the list

$$A_1, A_2, A_3, A_4, D_4^+, D_4^-, A_3(+,-), A_3(-,-)$$

of generic singularities of geometric solutions.

The singularities of type $A_3(+,-)$ ("cuspidal cone") and $A_3(-,-)$ ("cone-cone") appear also as instantaneous singularities (of codimension one) in wave front evolutions [1][26]. The pictures of the caustics (the singular loci in the (x_1, x_2, x_3)-space) corresponding to $A_3(+,-)$ and $A_3(-,-)$-singularities are given in Figure 4. See also [2][3].

More exactly we have:

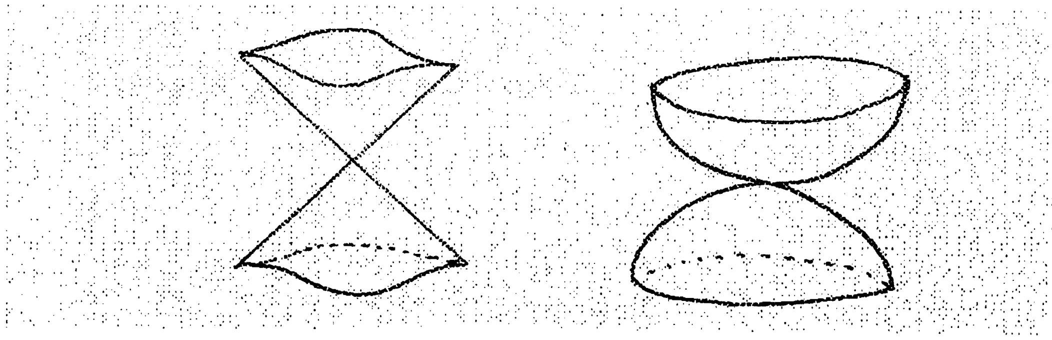

Figure 4: Caustics of $A_3(+,-)$ and $A_3(-,-)$ in the three space

Theorem 4.1 *Let $g(x_1, x_2, x_3, z, p_1, p_2, p_3)$ be a non-vanishing analytic function on a domain of $\mathbf{R}^7$. Then, for generic geometric solutions to the Monge-Ampère system $\mathcal{M}(3, g)$ corresponding to the equation*

$$\det\left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq 3} = g\left(x_1, x_2, x_3, z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \frac{\partial z}{\partial x_3}\right),$$

the pair of π_1-Legendrian singularity and π_2-Legendrian singularity at any point is given exactly by the list:

$$(A_1, A_1),\ (A_2, A_2),\ (A_2, A_3),\ (A_2, A_4),\ (A_3, A_2),\ (A_3, A_3),\ (A_4, A_2),$$
$$(A_3(+,-), D_4^+),\ (A_3(-,-), D_4^-),\ (D_4^+, A_3(+,-)),\ \text{and}\ (D_4^-, A_3(-,-)).$$

All eleven cases actually appear in a geometric solution to $\mathcal{M}(3, g)$ and they are stable under small perturbations among geometric solutions to $\mathcal{M}(3, g)$.

The stratifications of L by singularities of double Legendrian fibrations are illustrated as Figure 5.

Note that, by Theorem 4.1, each of these singularities appears as a *generic* and *stable* singularity of a Monge-Ampère equation. Also note that another singularity $A_3(+,+)$ ("the birth of flying saucer" [2][3]) does not appear generically in solutions of a Monge-Ampère equation.

5 Solutions to generalised Chynoweth-Sewell equations.

Here we exhibit a typical consideration of the proof of Theorem 4.1.

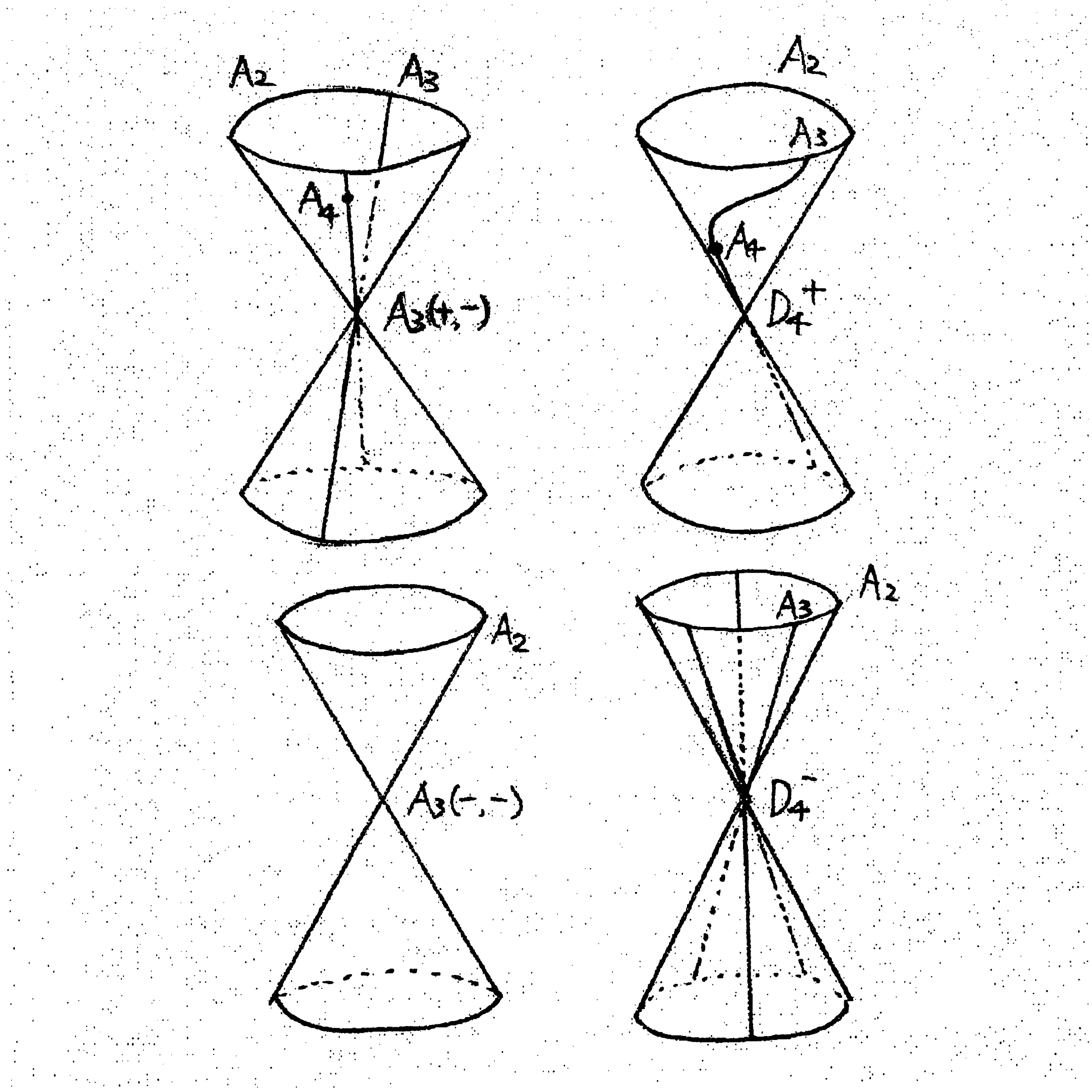

Figure 5: Stratifications by double Legendrian fibrations of a geometric solution.

Let $L^3 \subset \mathbf{R}^7$ be a geometric solution to $\mathrm{Hess}(z) = c$, $(c \neq 0)$. Suppose $\pi_1|_L$ is of rank 2 and $\pi_2|_L$ is of rank 1 at a point ℓ on L. Then we can set

$$L : x_1 = u,\ x_2 = v,\ x_3 = -\frac{\partial h}{\partial w},\ z = h - \frac{\partial h}{\partial w}w,\ p_1 = \frac{\partial h}{\partial u},\ p_2 = \frac{\partial h}{\partial v},\ p_3 = w,$$

for a parameter (u, v, w) centred at ℓ and a generating function $h = h(u, v, w)$. Then the analysis on singularities of solutions to the equation $\mathrm{Hess}(z) = c$ is reduced that of the equation

$$c\frac{\partial^2 h}{\partial w^2} + \begin{vmatrix} \dfrac{\partial^2 h}{\partial u^2} & \dfrac{\partial^2 h}{\partial u \partial v} \\ \dfrac{\partial^2 h}{\partial v \partial u} & \dfrac{\partial^2 h}{\partial v^2} \end{vmatrix} = 0, \cdots\cdots (\mathrm{CS})$$

for $h = h(u, v, w)$. The equation (CS) is called a *Chynoweth-Sewell equation* [5] and appears in meteorology [8].

In general, for the equation

$$\mathrm{Hess}(z) = g(x, z, p),$$

we reduce our classification problem to the analysis of classical solutions to

$$\Gamma(u, v, w)\frac{\partial^2 h}{\partial w^2} + \begin{vmatrix} \dfrac{\partial^2 h}{\partial u^2} & \dfrac{\partial^2 h}{\partial u \partial v} \\ \dfrac{\partial^2 h}{\partial v \partial u} & \dfrac{\partial^2 h}{\partial v^2} \end{vmatrix} = 0, \cdots\cdots (\mathrm{GCS})$$

a generalised Chynoweth-Sewell equation, for a non-vanishing function Γ, by setting

$$\Gamma(u, v, w) = g(u, v, \frac{\partial h}{\partial w}, w\frac{\partial h}{\partial w} - h, -\frac{\partial h}{\partial u}, -\frac{\partial h}{\partial v}, w).$$

The generating family for the projection π_1 of L is given by

$$F(w; x_1, x_2, x_3, z) = z - x_3 w + h(x_1, x_2, w).$$

This means that L is given by

$$L = \left\{ (x_1, x_2, x_3, z, p_1, p_2, p_3) \;\middle|\; F = 0, \frac{\partial F}{\partial w} = 0, p_i = \frac{\partial F}{\partial x_i} \text{ for some } w \right\}.$$

On the other hand, the generating family for the projection π_2 of L is given by

$$G(u, v; p_1, p_2, p_3, \tilde{z}) = \tilde{z} - p_1 u - p_2 v - h(u, v, p_3).$$

This means that L is given by

$$L=\left\{(x_1,x_2,x_3,\tilde{z},p_1,p_2,p_3)\,\middle|\, G=\frac{\partial G}{\partial u}=\frac{\partial G}{\partial v}=0, x_i=\frac{\partial G}{\partial p_i}\text{ for some }(u,v)\right\}.$$

Note that $\tilde{z}=x_1p_1+x_2p_2+x_3p_3-z$.

Solving the initial value problem of (GCS), we get the general form of h and thus F and G.

The initial value problem for $h(u,v,w)$ of (GCS) is solved for given

$$\varphi(u,v)=h(u,v,0),\ \psi(u,v)=\frac{\partial h}{\partial w}(u,v,0).$$

We see $F(w;0,0,0,0)=h(0,0,w)$, and

$$\begin{aligned}\frac{\partial F}{\partial z}(w;0,0,0,0) &= 1, & \frac{\partial F}{\partial x_1}(w;0,0,0,0) &= \frac{\partial h}{\partial u}(0,0,w),\\ \frac{\partial F}{\partial x_2}(w;0,0,0,0) &= \frac{\partial h}{\partial v}(0,0,w), & \frac{\partial F}{\partial x_3}(w;0,0,0,0) &= w.\end{aligned}$$

Suppose

$$\frac{\partial^2 h}{\partial w^2}(0,0,0)=0,\frac{\partial^3 h}{\partial w^3}(0,0,0)=0,\frac{\partial^4 h}{\partial w^4}(0,0,0)\neq 0.$$

Then F is a versal unfolding of $F(w;0,0,0,0)$ if and only if

$$1,\frac{\partial h}{\partial u}(0,0,w),\frac{\partial h}{\partial v}(0,0,w),w$$

form a generator of the quotient vector space

$$Q=\frac{\mathbf{R}[[w]]}{\langle F(w;0,0,0,0),\frac{\partial F}{\partial w}(w;0,0,0,0)\rangle_{\mathbf{R}[[w]]}}.$$

See [4]. This condition is equivalent to that

$$\frac{\partial^3 h}{\partial w^2\partial u}(0,0,0)\neq 0,\ \text{or}\ \frac{\partial^3 h}{\partial w^2\partial v}(0,0,0)\neq 0.$$

Recall that $\pi_2|_L$ is given by

$$(\tilde{z},p_1,p_2.p_3)=(\tilde{z}(u,v),\frac{\partial h}{\partial u},\frac{\partial h}{\partial v},w),$$

with $d\tilde{z}=x_1dp_1+x_2dp_2+x_3dp_3$. Since $\pi_2|_L$ is of rank 1 at $\ell\in L$, we have

$$\frac{\partial^2 h}{\partial u^2}(0,0,0)=0,\frac{\partial^2 h}{\partial u\partial v}(0,0,0)=0,\frac{\partial^2 h}{\partial v^2}(0,0,0)=0.$$

By the equation (GCS) and that $\Gamma(0,0,0) \neq 0$, we see

$$\frac{\partial^3 h}{\partial w^2 \partial u}(0,0,0) = 0, \frac{\partial^3 h}{\partial w^2 \partial v}(0,0,0) = 0.$$

Thus we see the singularity of $\pi_1|_L$ at L is of corank 1 but never of A_k-type. In fact we get the extra singularities $A_3(+,-)$ and $A_3(-,-)$.

Example 5.1 Let consider the equation $\mathrm{Hess}(z) = 1$ of three variables. Then

$$\begin{aligned} h(u,v,w) &= \frac{1}{6}u^3 + \frac{1}{2}uv^2 + uvw + \frac{1}{2}v^2 w - \frac{1}{2}(u^2 - v^2)w^2 - \frac{1}{6}(u - 2v)w^3 \\ &\quad + \frac{1}{12}w^4 + \frac{1}{20}w^5 + \frac{1}{30}w^6 \end{aligned}$$

give a geometric solution $L^3 \subset \mathbf{R}^7$ with $\pi_1|_L$ is of type $A_3(+,-)$ and $\pi_2|_L$ is of type D_4^+ at $0 \in \mathbf{R}^7$.

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