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NAMBU-LIE GROUPS ENDOWED WITH MULTIPLICATIVE TENSORS OF TOP ORDER

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ABSTRACT. A multiplicative (Nambu-Poisson) tensor of top order on a Lie group is characterized. As an application, we determine multiplicative structures on 3-dimensional Lie groups.

1. INTRODUCTION

A Nambu-Lie group is defined as a natural generalization of a Poisson Lie group. In fact, if η is a *multiplicative* k-vector field on a Lie group G which satisfies *fundamental identity*, then a pair (G,η) is called a *Nambu-Lie group.* If $k=2$, then (G,η) is especially called a Poisson Lie group [1],[3]. A Nambu-Lie group was studied by J.Grabowski and G.Marmo [2] and I.Vaisman [5]. In [2], they proved that there are no Nambu-Lie structures of order $k\geq 3$ on simple Lie groups. I.Vaisman [5] gave an alternative definition of multiplicativity by defining the k-bracket of 1-forms on G. In this paper, we characterize the properties of multiplicative Nambu-Poisson tensors of top order ($i.e.$, $n=k$). Note that the word "Nambu-Poisson" is void in this case. As an application of these characterizations, we determine multiplicative (Nambu-Lie) structures defined on 3-dimensional Lie groups.

2. NAMBU-LIE GROUPS

Let G be an n-dimensional connected Lie group with the Lie algebra $\mathfrak{g}$. We denote by $\Gamma(\Lambda^{k}TG)$ the set of k-vector fields (or contravariant tensor fields of order k) on G. Let $\mathcal{F}$ be the set of C^{∞}-functions on G. Each element η of $\Gamma(\Lambda^{k}TG)$ defines a k-bracket of functions $f_i\in\mathcal{F}$ as follows.

$$\{f_1,\ldots,f_k\}=\eta(df_1,\ldots,df_k).$$

Since this k-bracket satisfies Leibniz rule, we can define a vector field $X_{f_1,\ldots,f_{k-1}}$ by

$$X_{f_1,\ldots,f_{k-1}}(g)=\{f_1,\ldots,f_{k-1},g\},\ \ g\in\mathcal{F}.$$

Definition 2.1. *An element* η *of* $\Gamma(\Lambda^k TG)$, $k \geq 3$, *is called a Nambu-Poisson tensor of order* k *if* η *satisfies*

$$\mathcal{L}_{X_{f_1,\dots,f_{k-1}}}\eta = 0$$

for any $f_1, \dots, f_{k-1} \in \mathcal{F}$.

Note that if $k = n$, every η is a Nambu-Poisson tensor [4].

Definition 2.2. *An element* η *of* $\Gamma(\Lambda^k TG)$ *is said to be multiplicative if* η *satisfies*

$$\eta_{gh} = L_{g_*}\eta_h + R_{h_*}\eta_g$$

for any $g, h \in G$, *where* L_g *and* R_g *denote, respectively, the left and the right translations. Let* G *be a Lie group endowed with a multiplicative Nambu-Poisson tensor* η. *Then a pair* (G, η) *is called a Nambu-Lie group.*

For an element $\Lambda \in \Lambda^k\mathfrak{g}$, we define vector fields $\bar{\Lambda}$ and $\tilde{\Lambda}$ by

$$\bar{\Lambda}_g = L_{g_*}\Lambda, \quad \tilde{\Lambda}_g = R_{g_*}\Lambda, \text{ for all } g \in G.$$

Then it is clear that $\bar{\Lambda}$ (resp. $\tilde{\Lambda}$) is a left (resp. right) invariant vector field on G. Let us recall the following, which was proved by J-H Lu [3].

Proposition 2.1. *Let* G *be a compact (or semisimple) Lie group. Then for every multiplicative* k*-vector field* $\eta \in \Gamma(\Lambda^k TG)$, *there exists an element* $\Lambda \in \Lambda^k\mathfrak{g}$ *such that*

$$\eta_g = \bar{\Lambda}_g - \tilde{\Lambda}_g$$

for all $g \in G$.

Using the above proposition, we show the following theorem.

Theorem 2.2. *Let* (G, η) *be an* n*-dimensional compact or semisimple Nambu-Lie group, and let* η *be of top order. Then* $\eta = 0$.

Proof. By Proposition 2.1, there exists an element Λ of $\Lambda^n\mathfrak{g}$ such that $\eta = \bar{\Lambda} - \tilde{\Lambda}$. For all $g, h \in G$,

$$Ad_g\bar{\Lambda}_h = R_{g^{-1}_*}L_{g_*}\bar{\Lambda}_h = R_{g^{-1}_*}\bar{\Lambda}_{gh}.$$

On the other hand, since G is a unimodular Lie group, we have

$$Ad_g\bar{\Lambda}_h = (\det Ad_g)\bar{\Lambda}_{ghg^{-1}} = \bar{\Lambda}_{ghg^{-1}}.$$

Hence we obtain that $R_{g^{-1}_*}\bar{\Lambda}_{gh} = \bar{\Lambda}_{ghg^{-1}}$. This means that a left invariant vector field $\bar{\Lambda}$ is also a right invariant vector field. *i.e.*, $R_{h_*}\bar{\Lambda}_g = \bar{\Lambda}_{gh}$.This equation induces

$$\begin{aligned} R_{h_*}\bar{\Lambda}_g &= R_{h_*}L_{g_*}\Lambda = L_{g_*}R_{h_*}\Lambda \\ &= \bar{\Lambda}_{gh} = L_{g_*}L_{h_*}\Lambda. \end{aligned}$$

Thus we have $R_{h_*}\Lambda = L_{h_*}\Lambda$ for all $h \in G$, and this means $\eta = \bar{\Lambda} - \tilde{\Lambda} = 0$. □

Let η be a Nambu-Poisson tensor of order k on G. Then η defines a bundle mapping

$$\sharp_\eta : \underbrace{T^*G \times \cdots \times T^*G}_{k-1 \text{ times}} \longrightarrow TG$$

given by

$$< \beta, \sharp_\eta(\alpha_1, ..., \alpha_{k-1}) >= \eta(\alpha_1, ..., \alpha_{k-1}, \beta),$$

where all the arguments are covectors.

For such a tensor η, I.Vaisman [5] defined a k-bracket of 1-forms by

$$\{\alpha_1, ..., \alpha_k\} = d(\eta(\alpha_1, ..., \alpha_k)) + \sum_{j=1}^{k} (-1)^{k+j} i(\sharp_\eta(\alpha_1, ..., \widehat{\alpha_j}, ..., \alpha_k)) d\alpha_j,$$

where α_j $(j = 1, ..., k)$ are 1-forms on G.

The following theorem proved by I.Vaisman [5] gives one of the characterizations of Nambu-Lie groups.

Theorem 2.3. *If G is a connected Lie group endowed with a Nambu-Poisson tensor field η which vanishes at the unit e of G, then (G, η) is a Nambu-Lie group if and only if the k-bracket of any k left (right) invariant 1-forms of G is a left (right) invariant 1-form.*

Using Theorem 2.3, we characterize a multiplicative tensor η of top order. Let $\mathfrak{g}$ be a Lie algebra of G with a basis $X_1, ..., X_n$. We also denote the extended left invariant vector fields induced from X_i by the same letter. Since η is of top order, η has an expression $\eta = fX_1 \wedge \cdots \wedge X_n$ for some $f \in \mathcal{F}$. Let ω_i $(i = 1, ..., n)$ be left invariant 1-forms dual to X_i. Under these notations we prove

Theorem 2.4. *Let $\eta = fX_1 \wedge \cdots \wedge X_n$, $f \in \mathcal{F}$ be a tensor of top order on G. (Recall that such a tensor is always a Nambu-Poisson tensor.) Then η is multiplicative if and only if $f(e) = 0$ and*

$$X_i f + \Big(\sum_{k=1}^{n} C_{ik}^k\Big) f = q_i, \quad i = 1, ..., n,$$

where $\{C_{ij}^k\}$ are structure constants of $\mathfrak{g}$ with respect to the basis $X_1, ..., X_n$, and q_i $(i = 1, ..., n)$ are some constants.

Proof. By Theorem 2.3, we know that η is multiplicative if and only if $\eta_e = 0$ and

$$\{\omega_1, ..., \omega_n\} = d(\eta(\omega_1, ..., \omega_n)) + \sum_{k=1}^{n} (-1)^{n+k} i(\sharp_\eta(\omega_1, ..., \widehat{\omega_k}, ..., \omega_n)) d\omega_k$$

$$= df + f \sum_{k=1}^{n} i(X_k) d\omega_k = df + f\Big(\sum_{\alpha,k=1}^{n} C_{\alpha k}^k \omega_\alpha\Big)$$

is a left invariant 1-form. Since $< X_i, \{\omega_1, ..., \omega_n\} >$ is constant for any X_i, we have

$$< X_i, \{\omega_1, ..., \omega_n\} >= X_i f + \Big(\sum_{k=1}^{n} C_{ik}^k\Big) f = q_i, \quad i = 1, ..., n.$$

□

3. Examples

In this section, as an application of Theorem 2.4, we calculate Nambu-Lie group structures (*i.e.*, multiplicative Nambu-Poisson tensors) of order 3 on 3-dimensional simply connected Lie groups. Since such tensors are of top degree, we have only to see whether they are multiplicative or not.

Throughout this section, we denote by G the simply connected Lie groups corresponding to Lie algebras $\mathfrak{g}$. Linearly independent three left invariant vector fields are denoted by X, Y, Z. Then $\eta \in \Gamma(\Lambda^3 TG)$ is written as $\eta = fX \wedge Y \wedge Z$, $f \in C^\infty(G)$. It is well-known that there are 9 types of 3-dimensional Lie algebras. If $\mathfrak{g}$ is not a simple Lie algebra, its corresponding simply connected Lie group has global coordinates x, y, z. Hence a function f can be considered to be defined on $\mathbb{R}^3(x, y, z)$.

Type 1. $[\mathfrak{g}, \mathfrak{g}] = 0$. Namely $\mathfrak{g}$ is an abelian Lie algebra. The corresponding Lie group G is given by

$$G = \left\{ \left. \begin{pmatrix} e^x & 0 & 0 \\ 0 & e^y & 0 \\ 0 & 0 & e^z \end{pmatrix} \right| \ x, y, z \in \mathbb{R} \right\}.$$

Using these coordinates x, y, z, left invariant vector fields are written as $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$, $Z = \frac{\partial}{\partial z}$. By Theorem 2.4, a function $f(x, y, z)$ must satisfy $f(0, 0, 0) = 0$, and $\frac{\partial f}{\partial x} = a$, $\frac{\partial f}{\partial y} = b$, $\frac{\partial f}{\partial z} = c$, where a, b, c are some constants. Hence $f = ax + by + cz$, and

$$\eta = (ax + by + cz)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

gives a Nambu-Lie group structure on G.

By the similar method, we can get the results for other types.

Type 2. $\dim[\mathfrak{g}, \mathfrak{g}] = 1$. There are 2 cases as follows.

Case (1). $\mathfrak{g} =$ Heisenberg Lie algebra. $\mathfrak{g}$ is characterized by the condition $[\mathfrak{g}, \mathfrak{g}] \subset$ 1-dimensional center. The corresponding Lie group G is given by

$$G = \left\{ \left. \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right| \ x, y, z \in \mathbb{R} \right\}.$$

A Nambu-Lie group structure on G is given by

$$\eta = (ax + by)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

Case (2). A Lie algebra $\mathfrak{g}$ endowed with a property $[\mathfrak{g},\mathfrak{g}] \not\subset$ the center of $\mathfrak{g}$. The corresponding Lie group G is given by

$$G = \left\{ \begin{pmatrix} e^{y+z} & 0 & xe^{y} \\ 0 & e^{y} & 0 \\ 0 & 0 & e^{y} \end{pmatrix} \middle| \ x, y, z \in \mathbb{R} \right\}.$$

$$\eta = \{ax + c(e^{z} - 1)\} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

gives a Nambu-Lie group structure on G.

Type 3. $\dim[\mathfrak{g},\mathfrak{g}] = 2$. $\mathfrak{g}^{(2)} = 0$. There are 4 cases as follows.

Case (1). Left invariant vector fields X, Y, Z satisfy $[X,Y] = 0$, $[X,Z] = -X$, $[Y,Z] = -X - Y$. The corresponding Lie group G is given by

$$G = \left\{ \begin{pmatrix} e^{-z} & ze^{-z} & xe^{-2z} \\ 0 & e^{-z} & ye^{-2z} \\ 0 & 0 & e^{-2z} \end{pmatrix} \middle| \ x, y, z \in \mathbb{R} \right\}.$$

We know that

$$\eta = c(e^{2x} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

gives a Nambu-Lie group structure on G.

Case(2). Left invariant vector fields X, Y, Z satisfy $[X,Y] = 0$, $[X,Z] = -X$, $[Y,Z] = -Y$. The corresponding Lie group G is given by

$$G = \left\{ \begin{pmatrix} e^{-z} & 0 & xe^{-2z} \\ 0 & e^{-z} & ye^{-2z} \\ 0 & 0 & e^{-2z} \end{pmatrix} \middle| \ x, y, z \in \mathbb{R} \right\}.$$

We have

$$\eta = c(e^{2x} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

Case (3). Let $\mathfrak{g}$ be a Lie algebra endowed with the following bracket relations. $[X,Y] = 0$, $[X,Z] = -X$, $[Y,Z] = -qY$, $(q \neq 0,1)$. The corresponding Lie group G is given by

$$G = \left\{ \begin{pmatrix} e^{-qz} & 0 & xe^{-(q+1)z} \\ 0 & e^{-z} & ye^{-(q+1)z} \\ 0 & 0 & e^{-(q+1)z} \end{pmatrix} \middle| \ x, y, z \in \mathbb{R} \right\}.$$

We have

$$\eta = c(e^{(q+1)x} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

Case (4). Let $\mathfrak{g}$ be a Lie algebra endowed with the following bracket relations. $[X,Y] = 0$, $[X,Z] = -Y$, $[Y,Z] = X - qY$, $(q^2 < 4)$. The corresponding Lie group G has rather complicated expression. Put $k = q/2,\ p = \sqrt{1-k^2} = \sqrt{4-q^2}/2$. Then G is given by

$$G = \left\{ \begin{pmatrix} \frac{1}{p}e^{-kz}(-k\sin(pz) + p\cos(pz)) & -\frac{1}{p}e^{-kz}\sin(pz) & xe^{-2kz} \\ \frac{1}{p}e^{-kz}\sin(pz) & \frac{1}{p}e^{-kz}(p\cos(pz) + k\sin(pz)) & ye^{-2kz} \\ 0 & 0 & e^{-2kz} \end{pmatrix} \middle| \ x, y, z \in \mathbb{R} \right\}.$$

Then

$$\eta = \begin{cases} \frac{c}{q}(e^{qx} - 1)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, & q \neq 0 \\ cx\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, & q = 0 \end{cases}$$

gives a Nambu-Lie group structure on G.

Type 4. $\dim[\mathfrak{g}, \mathfrak{g}] = 3$. It is well-known that such Lie algebras are simple, and there are 2 cases. The corresponding simply connected Lie groups are $G_1 = SU(2)$ and $G_2 = \widetilde{SL(2,\mathbb{R})}$, where $\widetilde{SL(2,\mathbb{R})}/\mathbb{Z} \cong SL(2,\mathbb{R})$. Since G_1 is compact, and G_2 is semisimple, we have $\eta = 0$ by Theorem 2.2.

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