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# Multiple Euler factors

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## 1 Introduction

When we discuss zeros of the Riemann zeta function $\zeta(s)$, the Euler product expression and the functional equation play fundamental roles. That is, the region $\mathrm{Re}(s)>1$ is zero-free because the Euler product converges absolutely in $\mathrm{Re}(s)>1$, and from the functional equation the set of zeros of $\zeta(s)$ in $\mathrm{Re}(s)<0$ coincides with $\{-2n : n\in \mathbb{Z}_{>0}\}$. But in general it is difficult to analyze zeros of $\zeta(s)$ in $0<\mathrm{Re}(s)<1$.

In 1992 Kurokawa [K] introduced the *absolute tensor product* to break this difficulty. Roughly speaking, it constructs the new zeta function $(Z_{1}\otimes\cdots\otimes Z_{r})(s)$ from ordinary zeta functions $Z_{1}(s), \ldots, Z_{r}(s)$ such that zeros and poles of $(Z_{1}\otimes\cdots\otimes Z_{r})(s)$ are located at $\rho_{1}+\cdots+\rho_{r}$, where $Z_{j}(\rho_{j})=0$ or $\infty$. If $Z_{1}(s), \ldots, Z_{r}(s)$ have Euler product expressions and functional equations, we expect that $(Z_{1}\otimes\cdots\otimes Z_{r})(s)$ also has (the generalization of) Euler product expressions and functional equations. If these expectations are true, new informations about zeros and poles of $Z_{j}(s)$ may be obtained by the same manner as $\zeta(s)$. But in general, from our present knowledge we need enormous calculations to obtain the Euler product expressions of $(Z_{1}\otimes\cdots\otimes Z_{r})(s)$ (See [KK2] for the case of $((\zeta\otimes\zeta)(s))$. Then, it is also expected that there are somewhat relations between the Euler factor of $(Z_{1}\otimes\cdots\otimes Z_{r})(s)$ and the absolute tensor product of Euler factors of $Z_{j}(s)$. In this survey we mainly treat the latter and we call it the *multiple Euler factor*. From the

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recent investigations [K, KK1, A1, KW, A3, A4] we know that the multiple Euler factors are essentially expressed by the exponential of the polylogarithm. From [KK2, A2] we also know that the multiple Euler factors have functional equations.

The purpose of this survey is to explain the construction and expectations of the absolute tensor products and to introduce the recent developments about the multiple Euler factors. For the background of this theory we refer to an excellent survey [M] by Manin.

This survey is organized as follows. In Section 2 we recall the construction of the absolute tensor product. In Section 3 we explain its expectations and their reasons. In Section 4 we introduce the recent results for the multiple Euler factors.

## 2 Construction of the absolute tensor product

**Definition 2.1** (regularized product). Let $m : \mathbb{C} \to \mathbb{Z}$ express the order of zeros of a meromorphic function. Put

$$\zeta_m(w, s) := \sum_{\rho \in \mathbb{C}} \frac{m(\rho)}{(s-\rho)^w}, \tag{2.1}$$

where $-\pi \leq \arg(s-\rho) < \pi$. We assume the following conditions (i) and (ii):

(i) The right hand side of (2.1) converges absolutely for sufficiently large $\mathrm{Re}(w)$.

(ii) $\zeta_m(w,s)$ has a meromorphic continuation to $\mathrm{Re}(w) > -\varepsilon$ for some $\varepsilon > 0$ and $\zeta_m(w,s)$ is holomorphic at $w = 0$.

Then, the *regularized product* is defined by

$$\coprod_{\rho \in \mathbb{C}} ((s-\rho))^{m(\rho)} := \exp\left[ -\frac{\partial}{\partial w}\zeta_m(w,s)\bigg|_{w=0} \right]. \tag{2.2}$$

**Remark 2.2.** Under suitable (not strong) assumptions for $m$, we can prove that the order of zeros of (2.2) at $s = \rho$ equals $m(\rho)$. When $m(\rho) < 0$, at $s = \rho$ (2.2) is a pole with order $|m(\rho)|$. See [JL, Section 2 in Part I] for detail.

**Remark 2.3.** The regularized product was extended by Illies [I] under weaker assumptions. See also [HKW] for needed regularized products.

**Definition 2.4** (absolute tensor product). Let

$$Z_j(s) \cong \prod_{\rho \in \mathbb{C}} ((s-\rho))^{m_j(\rho)} \qquad (j=1,\ldots,r),$$

where $f(s) \cong g(s)$ means that there exists $Q(s) \in \mathbb{C}[s]$ satisfying $g(s) = e^{Q(s)} f(s)$. We assume that $m_j(\rho) = 0$ for sufficiently large $\mathrm{Re}(\rho)$. Then, their ***absolute tensor product*** is defined by

$$(Z_1 \otimes \cdots \otimes Z_r)(s) := \prod_{\rho_1,\ldots,\rho_r \in \mathbb{C}} ((s-\rho_1-\cdots-\rho_r))^{m(\rho_1,\ldots,\rho_r)},$$

where

$$m(\rho_1,\ldots,\rho_r) := m_1(\rho_1)\cdots m_r(\rho_r) \times \begin{cases} 1 & \text{if } \mathrm{Im}(\rho_1),\ldots,\mathrm{Im}(\rho_r) \geq 0, \\ (-1)^{r-1} & \text{if } \mathrm{Im}(\rho_1),\ldots,\mathrm{Im}(\rho_r) < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

**Remark 2.5.** Roughly speaking, from Remark 2.2, zeros and poles of $(Z_1 \otimes \cdots \otimes Z_r)(s)$ are located at $\rho_1 + \cdots + \rho_r$, where $Z_j(\rho_j) = 0$ or $\infty$. Strictly, under suitable assumptions the order of zeros of $(Z_1 \otimes \cdots \otimes Z_r)(s)$ at $s = \rho$ is given by

$$\sum_{\rho_1+\cdots+\rho_r=\rho} m(\rho_1,\ldots,\rho_r).$$

Here, we remark that under the assumption $\rho_1 + \cdots + \rho_r = \rho$, $m(\rho_1,\ldots,\rho_r) = 0$ except for finitely many $(\rho_1,\ldots,\rho_r) \in \mathbb{C}^r$ because of the assumption of $m_j$ and the definition of $m$.

# 3 Expectations of the absolute tensor product

In this section we explain expectations for the absolute tensor product of zeta functions and their reasons. To put it simply, when $Z_1(s),\ldots,Z_r(s)$ are zeta functions, we expect that $(Z_1 \otimes \cdots \otimes Z_r)(s)$ has the properties similar to those of zeta functions. We formalize this below.

**Expectation 3.1.** *Assume that* $Z_1(s), \ldots, Z_r(s)$ *are meromorphic functions of finite order and satisfy the following conditions* (i) *and* (ii)*:*

(i) *(Euler product)*

$$Z_j(s) = \prod_{p \in P_j} H_p^j(N_j(p)^{-s}) \qquad (\mathrm{Re}(s) > \sigma_j),$$

*where* $P_j$ *are at most countable sets of generalized primes,* $H_p^j(T) \in 1 + T\mathbb{C}[[T]]$ *and* $N_j : P_j \longrightarrow \mathbb{R}_{>1}$.

(ii) *(functional equation)*

$$\hat{Z}_j(\sigma_j - s) = \hat{Z}_j(s),$$

*where* $\hat{Z}_j(s) := F_j(s)Z_j(s)$ *and* $F_j(s)$ *is expressed in terms of multiple gamma functions.*

*Then, we expect that*

(1) *(Euler product)*

$$(Z_1 \otimes \cdots \otimes Z_r)(s) \cong \prod_{(p_1,\ldots,p_r) \in P_1 \times \cdots \times P_r} H_{(p_1,\ldots,p_r)}(N_1(p_1)^{-s}, \ldots, N_r(p_r)^{-s}),$$

*where* $H_{(p_1,\ldots,p_r)}(T_1, \ldots, T_r)$ *are* $r$*-variable formal series with constant term* 1 *if* $N_1(p_1), \ldots, N_r(p_r)$ *are distinct and* $H_{(p_1,\ldots,p_r)}$ *have their degenerate forms otherwise.*

(2) *(functional equation)*

$$\hat{Z}(\sigma_1 + \cdots + \sigma_r - s) = \hat{Z}(s)^{(-1)^{r-1}}.$$

*Here* $Z(s) := F(s)(Z_1 \otimes \cdots \otimes Z_r)(s)$ *and* $F(s)$ *is expressed in terms of* $(G_1 \otimes \cdots \otimes G_r)(s)$*, where* $G_j(s) = F_j(s)$ *or* $Z_j(s)$ *and* $(G_1, \ldots, G_r) \neq (Z_1, \ldots, Z_r)$.

(3) *There are somewhat relations between* $H_{(p_1,\ldots,p_r)}$ *and* $H_{p_1}^1 \otimes \cdots \otimes H_{p_r}^r$. *(We call the latter the* multiple Euler factor.*)*

The reason for (2) is that zeros and poles of $(Z_1 \otimes \cdots \otimes Z_r)(s)$ distribute almost symmetrically. But we remark that real zeros and poles of $Z_j(s)$ break this symmetry (see (2.3)).

Next we explain the reason for (1). For this purpose we first recall the relation between Euler products with functional equations and Weil's explicit formula [W] (see also [H]). For simplicity we consider the Riemann zeta function $\zeta(s)$. The Euler product expression for the Riemann zeta function is given by

$$\zeta(s) = \prod_{p:\text{primes}} (1-p^{-s})^{-1} \qquad (\mathrm{Re}(s) > 1) \tag{3.1}$$

and the functional equation is given by $\xi(s) = \xi(1-s)$, where $\xi(s) := s(s-1)\pi^{-(s/2)}\Gamma(\frac{s}{2})\zeta(s)$. On the other hand, Weil's explicit formula is given as follows:

**Proposition 3.2.** [2] *Let $h(t)$ be a holomorphic even function on the strip $|\mathrm{Im}(t)| \leq \frac{1}{2} + \delta$ such that $|h(t)| \ll (1+|t|)^{-1-\delta}$ on the above strip. Then, we have*

$$\begin{aligned}\sum_{\substack{\mathrm{Re}(\gamma)>0,\\ \xi(\frac{1}{2}+i\gamma)=0}} h(\gamma) &= -\frac{1}{2\pi}\sum_{p:\text{primes}}\sum_{n=1}^{\infty}\frac{\log p}{p^{n/2}}\hat{h}\left(\frac{n\log p}{2\pi}\right) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(u)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)du \\ &\quad + \frac{1}{2}h\left(\frac{i}{2}\right) - \frac{\log\pi}{4\pi}\hat{h}(0),\end{aligned} \tag{3.2}$$

*where*

$$\hat{h}(x) := \int_{-\infty}^{\infty} h(u)e(-ux)du \qquad (e(x) := e^{2\pi i x}).$$

For the proof of Proposition 3.2, we consider the integral

$$\frac{1}{2\pi i}\int_{\partial D_T} H(t)\frac{\xi'}{\xi}\left(t+\frac{1}{2}\right)dt, \tag{3.3}$$

where $H(t) := h(t/i)$ and $D_T := \{x+iy : |x| < \frac{1}{2}+\delta, 0 < y < T\}$. We calculate (3.2) by two methods and take the limit $T \to \infty$. First we calculate (3.3) by the residue formula

[2] Usually (3.2) with $h(t) = f(t) + f(-t)$ for suitable $f$ is called Weil's explicit formula.

and $T \to \infty$. Then, we obtain the left hand side of (3.2). Second we calculate (3.3) by the Euler product and the functional equation and take the limit $T \to \infty$. Then, we obtain the right hand side of (3.2). Hence we get Proposition 3.2.

On the other hand, we recover the Euler product expression (3.1) from Proposition 3.2 by specializing

$$h(t) = \frac{1}{(t+(s-\frac{1}{2})i)^2} + \frac{1}{(t-(s-\frac{1}{2})i)^2}$$

for fixed $\mathrm{Re}(s) > 1$.

We return to the reason for (1). For simplicity we consider the case $Z_j(s) = \zeta(s)$ for any $j$. From the above discussion it would be sufficient to obtain the explicit formula for $\zeta^{\otimes r}(s)$ when we want the Euler product for $\zeta^{\otimes r}(s)$. For this purpose we consider

$$\frac{1}{(2\pi i)^r}\int_{\partial D_T}\cdots\int_{\partial D_T} G(t_1+\cdots+t_r)\frac{\xi'}{\xi}\left(t_1+\frac{1}{2}\right)\cdots\frac{\xi'}{\xi}\left(t_r+\frac{1}{2}\right)dt_r\cdots dt_1, \qquad (3.4)$$

where $G(t) := g(t/i)$ and $g(t)$ is a holomorphic function on the strip $|\mathrm{Im}(t)| \le \frac{r}{2}+\delta$ satisfying $g(-t) = (-1)^{r-1}g(t)$ and $|g(t)| \ll (1+|t|)^{-r(1+\delta)}$ on the above strip. Calculating (3.4) by the same manner as (3.3), we would obtain the explicit formula for $\zeta^{\otimes r}(s)$. In particular, from the Euler product for $\zeta(s)$ we expect that a summation through $(p_1, \dots, p_r)$ with prime numbers $p_j$ appears in the explicit formula. We remark that Koyama-Kurokawa [KK2] calculated the Euler product expression for $(\zeta \otimes \zeta)(s)$ by the above method.

The reason for (3) is the same as (1). We remark that the explicit formula for $(H^1_{p_1} \otimes \cdots \otimes H^r_{p_r})(s)$ would be obtained by the same manner as $\zeta^{\otimes r}(s)$ and we expect that the explicit formula for $(Z_1 \otimes \cdots \otimes Z_r)(s)$ is related to that for $(H^1_{p_1} \otimes \cdots \otimes H^r_{p_r})(s)$. But most parts for (3) remain mysterious.

# 4 Multiple Euler factors

In this section we introduce the recent results for multiple Euler factor of the Riemann zeta function. That is, we consider $(\zeta_{p_1} \otimes \cdots \otimes \zeta_{p_r})(s)$, where $\zeta_p(s) := (1-p^{-s})^{-1}$ for prime numbers $p$.

First we introduce the polylogarithm type expression for $(\zeta_{p_1} \otimes \cdots \otimes \zeta_{p_r})(s)$ as follows:

**Theorem 1** ([A4]). *Let $p_1, \ldots, p_k$ be distinct prime numbers and $r_1, \ldots, r_k$ be positive integers. Then, in* $\mathrm{Re}(s) > 0$ *we have*

$$\begin{aligned}&(\zeta_{p_1}^{\otimes r_1} \otimes \cdots \otimes \zeta_{p_k}^{\otimes r_k})(s)\\ \cong\ &\exp\left[\sum_{j=1}^{k}\sum_{n=1}^{\infty}\frac{1}{n}\left[g_{r_j}\left(\frac{1}{2\pi i n}\frac{\partial}{\partial u}\right)\left(\frac{p_j^{-nsu}}{u\prod_{l\neq j}(e(n\frac{\log p_j}{\log p_l}u)-1)^{r_l}}\right)\right]_{u=1}\right],\end{aligned} \tag{4.1}$$

*where*

$$g_N(z) := \begin{cases} \frac{(z-1)(z-2)\cdots(z-(N-1))}{(N-1)!} & \text{if } N \geq 2, \\ 1 & \text{if } N = 1. \end{cases}$$

**Remark 4.1.** Following special cases of Theorem 1 had been proved when we obtained Theorem 1:

(1) when $k = 1$ ([K]),

(2) when $r_1 = \cdots = r_k = 1$ ([KK1, $k = 2$], [A1, $k = 3$], [KW, $k \geq 4$]),

(3) when $k = 2$ ([A3]).

**Remark 4.2.** Theorem 1 is the generalization of the formula $\zeta_p(s) = \exp[\sum_{n=1}^{\infty} n^{-1}p^{-ns}]$ for $\mathrm{Re}(s) > 0$.

**Remark 4.3.** The absolute convergence of (4.1) depends on the consequence of Baker's result [B, Theorem 3.1] about diophantine approximations. That is, for distinct prime numbers $p$ and $q$ there exists $c = c(p,q) > 0$ such that

$$\left|1 - e\left(n\frac{\log q}{\log p}\right)\right| = O(n^c) \quad \text{as } n \to \infty.$$

**Remark 4.4.** When $r_1 = \cdots = r_k = 1$, Theorem 1 says

$$\begin{aligned}(\zeta_{p_1} \otimes \cdots \otimes \zeta_{p_r})(s) &\cong \exp\left[\sum_{k=1}^{r}\sum_{n=1}^{\infty}\left(\frac{p_k^{-ns}}{n\prod_{j\neq k}(e(n\frac{\log p_k}{\log p_j})-1)}\right)\right] \\ &\in 1+(p_1^{-s},\ldots,p_r^{-s})\mathbb{C}[[p_1^{-s},\ldots,p_r^{-s}]].\end{aligned}$$

This implies that Expectation 3.1 (1) holds for $Z_j(s) = \zeta_{p_j}(s)$.

Suppose that $p_1, \ldots, p_r$ are distinct or same prime numbers and we define $\zeta_{p_1,\ldots,p_r}(s)$ by

$$\zeta_{p_1,\ldots,p_r}(s) = \begin{cases} \exp\left[\sum_{k=1}^{r}\frac{1}{(-2\pi i)^{k-1}}g_r^{(k-1)}\left(-\frac{s\log p}{2\pi i}\right)\mathrm{Li}_k(p^{-s})\right] & \text{if } p_1 = \cdots = p_r =: p, \\ \exp\left[\sum_{k=1}^{r}\sum_{n=1}^{\infty}\frac{p_k^{-ns}}{n\prod_{j\neq k}(e(n\frac{\log p_k}{\log p_j})-1)}\right] & \text{if } p_1, \ldots, p_r \text{ are distinct,}\end{cases}$$

where

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty}\frac{x^n}{n^k} \qquad (|x|<1).$$

This is the right hand side of (4.1) when $k = 1$ and when $r_1 = \cdots = r_k = 1$. (We need a little calculation to lead it in the case $p_1 = \cdots = p_r$. See [A3, (5.3)].) From Theorem 1 $\zeta_{p_1,\ldots,p_r}(s)$ is originally defined in $\mathrm{Re}(s) > 0$ and has a meromorphic continuation to all $s \in \mathbb{C}$. We introduce a functional equation for $\zeta_{p_1,\ldots,p_r}(s)$ as follows:

**Theorem 2** ([A2]). *Let $p_1, \ldots, p_r$ be distinct or same prime numbers. Then, we have*

$$\zeta_{p_1,\ldots,p_r}(-s) = \left\{\zeta_{p_1,\ldots,p_r}(s)\left(\prod_{k=1}^{r-1}Z_{r,k}(s,(p_1,\ldots,p_r))\right)\exp\left[2\pi i\zeta_r\left(0,s,\left(\frac{2\pi i}{\log p_1},\cdots\frac{2\pi i}{\log p_r}\right)\right)\right]\right\}^{(-1)^{r-1}},$$

*where*

$$\begin{aligned} Z_{r,k}(s,(p_1,\ldots,p_r)) &:= \prod_{1\le j_1<\cdots<j_k\le r}\zeta_{p_{j_1},\ldots,p_{j_k}}(s), \\ \zeta_r(w,s,(\omega_1,\ldots,\omega_r)) &:= \sum_{n_1,\ldots,n_r\ge 0}(n_1\omega_1+\cdots+n_r\omega_r+s)^{-w}.\end{aligned}$$

**Remark 4.5.** Koyama-Kurokawa [KK2] obtained Theorem 2 in the case $r = 2$.

**Remark 4.6.** The asymmetric term

$$\prod_{k=1}^{r-1} Z_{r,k}(s,(p_1,\dots,p_r))$$

arises from the real pole $s = 0$ of $\zeta_p(s)$. See Expectation 3.1 (2) and its reason.

**Remark 4.7.** $\zeta_r(0,s,(\omega_1,\dots,\omega_r))$ is a polynomial with respect to $s$. In fact, we have

$$\zeta_r(0,s,(\omega_1,\dots,\omega_r)) = \mathrm{Res}_{t=0}\frac{e^{-st}}{t\prod_{k=1}^{r}(1-e^{-\omega_k t})}.$$

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