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THE FIXED-POINT HOMOMORPHISM IN EQUIVARIANT SURGERY

MASATSUGU NAGATA

RIMS, Kyoto University

永田雅嗣（京都大学数理解析研究所）

SECTION 1. INTRODUCTION: THE EQUIVARIANT SURGERY EXACT SEQUENCE

Let G be a finite group. The classification of G-manifolds can be approached through the equivariant surgery exact sequence. In the category of locally linear PL-G-manifolds with a certain stability condition ("the gap hypothesis"), a surgery exact sequence was set up by I. Madsen and M. Rothenberg in [MR 2], when the group G is of odd order. One of its central feature is equivariant transversality, which holds only in those circumstances.

Let X be a (locally linear PL) G-manifold with boundary. The main target we wish to investigate is expressed, in this context, as the "structure set" $\widetilde{\mathcal{S}}_G(X,\partial)$, which is the set of equivalence classes of G-simple homotopy equivalences $h: M \to X$ with ∂h a PL-homeomorphism, where two such objects are equivalent when they are connected (in a commutative diagram) with a PL-G-homeomorphism of the domain M.

When one wishes to analyze the surgery exact sequence, one needs to compute the set $\widetilde{\mathcal{N}}_G(X)$ of G-normal cobordism classes of G-normal maps. By virtue of G-transversality, this set is interpreted in terms of bundle theories, and therefore is classified by a G-space F/PL. (See [MR 2, §5].)

Madsen and Rothenberg set up the equivariant surgery exact sequence and identified $\widetilde{\mathcal{N}}_G(X)$ as a term in the sequence, in a suitable category of G-spaces when G is a group of odd order. Here we cite their main results:

The strong gap condition. [MR 2, Theorem 5.11] *If G is a group of odd order and X is a G-oriented PL-G-manifold which satisfies the gap conditions*

$$10 < 2\dim X^H < \dim X^K \qquad \text{for} \quad K \subset H, X^H \neq X^K,$$

then $\widetilde{\mathcal{N}}_G(X/\partial X)$ is in one-to-one correspondence with normal cobordism classes of restricted G-normal maps over X, as defined in [MR 2, 5.9].

The equivariant surgery exact sequence. [MR 2, Theorem 5.12] *If G, X are as above and we assume that X^H is simply-connected for all H, then there is an exact sequence*

$$\to \widetilde{\mathcal{S}}_G(D^1 \times X, \partial) \to \widetilde{\mathcal{N}}_G(D^1 \times X, \partial) \to \mathcal{L}_{1+m} \to \widetilde{\mathcal{S}}_G(X, \partial) \to \widetilde{\mathcal{N}}_G(X/\partial X) \to \mathcal{L}_m(G)$$

where

$$\mathcal{L}_m(G) = \oplus_{(H)} L_{m(H)}(N_G H/H)$$

with $m(H) = \dim X^H$, and the sum is over the conjugacy classes of subgroups of G.

Long time ago ([N 6]) the author have worked on the explicit structure of the terms in the exact sequence, and, in particular, analyzed the equivariant homotopy type of the classifying space F/PL. In this paper, we try to construct an example for particular groups G to illustrate what kind of obstructions lie in determining those homotopy type information.

Madsen and Rothenberg ([MR 2]) had identified the terms of the exact sequence in geometric and homotopy theoretic methods, and the author ([N 6]) had modified their methods to interpret the terms in a homotopy theoretic way.

Two of the terms in the equivariant surgery exact sequence, $\widetilde{\mathcal{N}}_G(X/\partial X)$ and $\mathcal{L}_m(G)$, are defined using homotopy-theoretic and algebraic methods, respectively. Therefore they naturally inherit a Mackey functor structure over the system of subgroups of G. However, the remaining term, the structure set $\widetilde{\mathcal{S}}_G(X, \partial)$, is concerned with homeomorphisms, and so it does not provide a straightforward way to construct a functorial (Mackey) structure with respect to the system of subgroups of G.

Ranicki ([R 1,2]) has identified the structure set term in the equivariant surgery exact sequence with an "algebraically defined structure set," in his terminology. He used categorical constructions to identify the surgery exact sequence itself using algebraically constructed objects, thus making it possible to apply various categorical techniques. Making use of his methods, it is possible to interpret the equivariant structure set $\widetilde{\mathcal{S}}_G(X, \partial)$ in a categorical manner. However, that approach puts one in a stabilization situation, and thus requires a very strong stability hypotheses.

In a series of papers [N 1, 2, 3] we used geometric methods, rather than algebraic, to directly construct a Mackey structure within the terms of the equivariant surgery exact sequence, in the case where the manifolds X are very special ones. So, at least in those situations, the Mackey functor structure is realized in the equivariant surgery exact sequence, without going through the stable homotopy category, thus giving the result to the structure set of the manifold itself.

In this paper, we investigate an explicit example of groups, that is, non-abelian metacyclic groups, for which the equivariant classifying space F/PL was not quite determined in ([N 6]), to see if those methods can be expanded to more general set of groups. If we can determine the structure of F/PL more precisely in those cases, then we could expect to obtain clearer understanding of the Mackey structure in the equivariant surgery exact sequence.

SECTION 2. DEFINITION: THE MACKEY FUNCTOR STRUCTURE

The Mackey functor structure over the system of subgroups of the finite group G is defined as follows. For an $\mathbb{R}G$-module V, let $\mathrm{Iso}(V)$ be the set of isotropy subgroups of the G-module V.

Let $\mathcal{M}$ be an abelian group valued bifunctor over the category $\mathrm{Iso}(V)$, and for the morphisms in $\mathrm{Iso}(V)$, that is, inclusions of subgroups $H < K$, we use the notation $\mathrm{Res}_K^H : \mathcal{M}(K) \to \mathcal{M}(H)$ and $\mathrm{Ind}_H^K : \mathcal{M}(H) \to \mathcal{M}(K)$ for the corresponding morphisms. Also we suppose there is a conjugation morphism $c_g : \mathcal{M}(H) \to \mathcal{M}(H^g)$ for any H and and $g \in G$.

The system $\mathcal{M}, \mathrm{Res}_K^H, \mathrm{Ind}_H^K, c_g$ is called a Mackey functor if the following conditions are satisfied for all $H < K$ in $\mathrm{Iso}(V)$:

$$\begin{aligned}
&c_g = \mathrm{id}_{\mathcal{N}(H)} \quad \text{if} \quad g \in H; && c_{g_1 \circ g_2} = c_{g_1} \circ c_{g_2} \\
&\mathrm{Ind}_{H^G}^{K^g} \circ c_g = c_g \circ \mathrm{Ind}_H^K, && \mathrm{Res}_{K^g}^{H^g} \circ c_g = c_g \circ \mathrm{Res}_K^H \\
&\mathrm{Res}_G^H \circ \mathrm{Ind}_K^G = \sum_{H \backslash G / K} \mathrm{Ind}_{h \cap K^g}^H \circ c_g \circ \mathrm{Res}_K^{k \cap H^{g^{-1}}}
\end{aligned}$$

Let $A(G : V)$ be the Grothendieck group of finite G-sets X such that $\mathrm{Iso}(X) \subset \mathrm{Iso}(V)$. Then a Mackey functor $\mathcal{M}$ over $\mathrm{Iso}(V)$ becomes a natural $A(G : V)$-module, and thus traditional algebraic calculations are applicable to compute such terms. See [MS] for example.

SECTION 3. THE FIXED-POINT HOMOMORPHISM FOR NONABELIAN GROUP ACTIONS

Let us consider the metacyclic group $G = G_{21} = \mathbb{Z}/7 \rtimes_{\alpha} \mathbb{Z}/3$:

$$1 \longrightarrow H = \mathbb{Z}/7 \longrightarrow G \longrightarrow \mathbb{Z}/3 \longrightarrow 1$$

Here $\alpha : \mathbb{Z}/3 \to \operatorname{Aut}\mathbb{Z}/7$ is defined by multiplication by 2. The system $\underline{\underline{RO}}$ of real representation rings is well-known. We fix notation as follows. Let A be a subgroup of order 3. All such are conjugate to each other.

Here the system $\underline{\underline{RO}}$ consists of

$$\begin{aligned} RO(e) &= \mathbb{Z} \ni 1 \\ RO(H) &= \mathbb{Z}^4 \ni 1, z_1, z_2, z_4 \\ RO(A) &= \mathbb{Z}^2 \ni 1, w \\ RO(G) &= \mathbb{Z}^3 \ni 1, w, P \end{aligned}$$

where

$$\begin{aligned} &\operatorname{Res}^H_e(1) = 1, \operatorname{Res}^H_e(z_i) = 2, \\ &\operatorname{Res}^A_e(1) = 1, \operatorname{Res}^A_e(w) = 2, \\ &\operatorname{Res}^G_H(1) = 1, \operatorname{Res}^G_H(w) = 2, \operatorname{Res}^G_H(P) = z_1 + z_2 + z_4, \\ &\operatorname{Res}^G_A(1) = 1, \operatorname{Res}^G_A(w) = w, \operatorname{Res}^G_A(P) = 2 + 2w. \end{aligned}$$

Note that Res^G_H is not surjective but is onto the WH-invariant submodule of $RO(H)$, and therefore we cannot have a decomposition for this system.

We remark that any metacyclic group has a similar system $\underline{\underline{RO}}$.

In ([N 6]), we determined the term $\widetilde{\mathcal{N}}_G(X)$ of the equivariant surgery exact sequence, that is, the set of equivariant normal maps, localized at 2. More precisely, we have

$$\begin{aligned} \widetilde{\mathcal{N}}_G(X)_{(2)} &= [x, F/PL]^G \\ &= [X^*, E^{\underline{\underline{s}}}]_{\mathcal{O}_G} \times \bigoplus_{i \geq 6} H^i_G\left(X; L_i(e)^{\underline{\underline{s}}}\right) \times \bigoplus_{i \geq 2} H^i_G\left(X; \underline{\underline{\hat{\mathcal{L}}_i}}\right). \end{aligned}$$

where

$$\hat{\mathcal{L}}_i(H) = \bigoplus_{(\Gamma) \subset H} \widetilde{L}_i\left(N_H\Gamma/\Gamma\right)$$

is the system (that is, the Mackey functor structure, in the notation of [E]) of the L-group term in the equivariant surgery exact sequence.

Thus we express $\widetilde{\mathcal{N}}_G(X)_{(2)}$ as the product of Bredon cohomology groups and a certain group of homotopy classes of maps between systems, which in turn can be calculated by a natural spectral sequence.

Together with Madsen-Rothenberg's description of $\widetilde{\mathcal{N}}_G(X)$ localized away from 2 as a product of equivariant K-theories, this gives us an algorithm of calculation of the group $\widetilde{\mathcal{N}}_G(X)$.

We now consider the non-injectivity of the fixed-point homomorphism of:

$$(*)\qquad \bigoplus \operatorname{Res}^G_H : H^m_G\left(X; \underline{\underline{M}}\right) \longrightarrow \bigoplus_{(\Gamma)} H^m\left(X^\Gamma \underline{\underline{M}}(G/\Gamma)\right)$$

with $\underline{\underline{M}} = \underline{\underline{\pi}}_n(F/PL)$. This would in turn detect the equivariant k-invariant of F/PL, as investigated in ([N 6]). Non-triviality of the k-invariant would imply the existence of some new information hiding in the Mackey structure of the terms of the equivariant surgery exact sequence that we are interested in.

Assumption. We assume that the homomorphism (*) is injective on the group

$$H^{i+1}_G\left(F/PL\langle i-2\rangle; \underline{\underline{\pi}}(F/PL)\right)$$

in which the i-th equivariant k-invariant of F/PL lies, for $i < n$.

Under this assumption, the k-invariants in dimension less than n are all detected by the nonequivariant k-invariants, and therefore produce a map

$$F/PL \longrightarrow \mathcal{E} \times \prod_{i=2}^{n-1} \mathcal{K}\left(\hat{\underline{\underline{\mathcal{L}}}}_i, i\right)$$

which is an $(n-1)$-equivalence.

In particular, we identify the $(n-1)$-st Postnikov component of F/PL as

$$X = F/PL\langle n-1\rangle = \mathcal{E}_0 \times \mathcal{K}\left(\hat{\underline{\underline{\mathcal{L}}}}_2, 2\right) \times \mathcal{K}\left(\hat{\underline{\underline{\mathcal{L}}}}_4, 4\right) \times \prod_{i=6}^{n-1} \mathcal{K}\left(\underline{\underline{\mathcal{L}}}_i, i\right),$$

which we denote by X throughout this section.

The next k-invariant lies in the group

$$H^{n+1}_G\left(X; \underline{\underline{\pi}}_n(F/PL)\right) \qquad \text{with } \underline{\underline{\pi}}_n(F/PL) = \underline{\underline{\mathcal{L}}}_n.$$

Proposition. *For the group $G = G_{21}$ and X as above, the homomorphism*

$$\bigoplus \operatorname{Res}^G_\Gamma : H^{n+1}_G\left(X; \underline{\underline{\mathcal{L}}}_n\right) \longrightarrow \bigoplus_{(\Gamma)} H^{n+1}\left(X^\Gamma; \mathcal{L}_n(\Gamma)\right)$$

is not injective for some chjoice of n.

Our tool of computation will be the Bredon spectral sequence ([Bre, I.10.4]):

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{C}_G}\left(\underline{\underline{H}}_q(X), \underline{\underline{M}}\right) \implies H^{p+q}_G\left(X; \underline{\underline{M}}\right),$$

where $\underline{\underline{H}}_q(X)$ is the system $G/\Gamma \mapsto H_q(X^\Gamma)$ and $\mathcal{C}_G$ is the category of systems (contravariant functors on $\mathcal{O}_G$). All homology is understood to be with $\mathbb{Z}_{(2)}$-coefficients.

The proof of the Proposition will occupy the rest of this section.

Lemma. *For the group $G = G_{21}$, the homomorphism*

$$\bigoplus \mathrm{Res}_\Gamma^G : H_G^k\left(\mathcal{K}(\underline{\underline{RO}}, m); \underline{\underline{RO}}\right) \longrightarrow \bigoplus_{(\Gamma)} H^k\left(K(RO(\Gamma), m); RO(\Gamma)\right)$$

is not injective for some k with $m + 4 \leq k < 2m$.

Proof. Let $Y = \mathcal{K}\left(\underline{\underline{RO}}, m\right)$ and $\underline{\underline{M}} = \underline{\underline{RO}}$. Consider the Bredon spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{C}_G}\left(\underline{\underline{H}}_q(Y), \underline{\underline{M}}\right) \implies H_G^{p+q}\left(Y; \underline{\underline{M}}\right).$$

Since $RO(\Gamma)$ is a free abelian group, Y^Γ is a product of $K(\mathbb{Z}, m)$'s.

We construct a projective resolution of $\underline{\underline{H}}_q(Y)$ in the category $\mathcal{C}_G$ of systems. Bredon [Bre] pointed out that $\mathcal{C}_G$ has enough projectives and a projective resolution can be condtructed using the projective objects F_S:

$$F_S(G/\Gamma) = \mathbb{Z}[S^\Gamma]$$

for finite G-sets S.

In the stable range $m \leq q < 2m$, generators of $H_q(K(\mathbb{Z}, m); \mathbb{Z})$ are explicitly written down by H. Cartan in [C, 11.6., Théorème 2]. Also in the stable range Künneth theorem implies that generators of $H_q\left(Y^\Gamma; \mathbb{Z}_{(2)}\right)$ are just images of Cartan's elements. More precisely,

$$\begin{aligned}
H_m\left(Y^\Gamma\right) &\cong RO(\Gamma)_{(2)}, \\
H_{m+1}\left(Y^\Gamma\right) &= 0, \\
H_{m+2}\left(Y^\Gamma\right) &\cong RO(\Gamma) \otimes \mathbb{Z}/2, \\
H_{m+3}\left(Y^\Gamma\right) &= 0, \\
H_{m+4}\left(Y^\Gamma\right) &\cong RO(\Gamma) \otimes \mathbb{Z}/2, \qquad \text{etc.}
\end{aligned}$$

If we let F and $F_{(q)}$ respectively denote a projective resolution of $\underline{\underline{RO}}$ in $\mathcal{C}_G$, and of $\underline{\underline{RO}} \otimes \mathbb{Z}/2$ in $\mathcal{C}_G$ with shifted dimension starting from q, respectively, then a projective resolution of $\underline{\underline{H}}_q(Y)$ can be obtained by F or sum of $F_{(q)}$'s, one for each Cartan generator in dimension q, as long as we consider matters below dimension $2m$.

Now $\underline{\underline{RO}}$ being the system as in (5.2), its projective resolution F can be given as follows:

$$\begin{cases}
F^0 &= \left(F_{G/G}\right)^3 \oplus F_{G/H}, \\
F^1 &= F_{G/H} \oplus F_{G/A}, \\
F^t &= F_{G/H} \oplus F_{G/e} \qquad (t \geq 2),
\end{cases}$$

where

$$\begin{aligned}
&F_{G/G}(G/-) = \mathbb{Z},\\
&F_{G/H}(G/e) = F_{G/H}(G/H) = \mathbb{Z}^3, F_{G/H}(G/A) = F_{G/H}(G/G) = 0,\\
&F_{G/A}(G/e) = \mathbb{Z}\oplus\mathbb{Z}^6, F_{G/A}(G/A) = \mathbb{Z}, F_{G/A}(G/H) = F_{G/A}(G/G) = 0\\
&F_{G/G}(G/e) = \mathbb{Z}^{21}, F_{G/G}(G/H) = F_{G/G}(G/A) = F_{G/G}(G/G) = 0.
\end{aligned}$$

where the nontrivial maps are the identity maps, except the $\mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}^6$, which is the inclusion onto the first component.

The maps are given as follows:

$$\begin{aligned}
\phi^0 : F^0 \longrightarrow \underline{\underline{RO}} :&(F_{G/G})^3(G/G) \ni a_1, a_2, a_3 \mapsto 1, w, P\\
&F_{G/H}(G/H) \ni b_1, b_2, b_3 \mapsto z_1, z_2, z_3\\
\phi^1 : F^1 \longrightarrow F^0 :&F_{G/H}(G/H) \ni c_1, c_2, c_3 \mapsto a_2 - 2a_1, a_3 - b_1 - b_2 - b_3, 0\\
&F_{G/A}(G/A) \ni d \mapsto a_3 - 2a_1 - 2a_2\\
&F_{G/A}(G/e) \ni d_2, \dots, d_7 \mapsto b_1 - 2a_1, b_2 - 2a_1, b_3 - 2a_1, 0, 0, 0\\
\phi^2 : F^2 \longrightarrow F^1 :&F_{G/H}(G/H) \ni e_1, e_2, e_3 \mapsto 0, 0, c_3\\
&F_{G/e}(G/e) \ni f_1, \dots, f_{21} \mapsto c_2 - d + d_2 + d_3 + d_4 - 2c_1, d_5, d_6, d_7, 0, \dots, 0\\
\phi^{2s-1} : F^{2s-1} \longrightarrow F^{2s-2} :&F_{G/H}(G/H) \ni e_1, e_2, e_3 \mapsto e_1, e_2, 0\\
&F_{G/e}(G/e) \ni f_1, \dots, f_{21} \mapsto 0, 0, 0, 0, f_5, \dots, f_{21}\\
\phi^{2s} : F^{2s} \longrightarrow F^{2s-1} :&F_{G/H}(G/H) \ni e_1, e_2, e_3 \mapsto 0, 0, e_3\\
&F_{G/e}(G/e) \ni f_1, \dots, f_{21} \mapsto f_1, f_2, f_3, f_4, 0, \dots, 0,
\end{aligned}$$

where $s \geq 2$.

Next we consider the system $\underline{\underline{RO}} \otimes \mathbb{Z}/2$. It is

$$\begin{aligned}
\underline{\underline{RO}} \otimes \mathbb{Z}/2 &= \left(\mathbb{Z}/2 \oplus \underline{\underline{R}}^-\right) \otimes \mathbb{Z}/2\\
&= \underline{\underline{\mathbb{Z}/2}} \oplus \underline{\underline{w}} \oplus \underline{\underline{P}},
\end{aligned}$$

where

$$\begin{aligned}
&\underline{\underline{\mathbb{Z}/2}}(G/-) = \mathbb{Z}/2;\\
&\underline{\underline{w}}(G/e) = \underline{\underline{w}}(G/H) = 0,\\
&\underline{\underline{w}}(G/A) = \underline{\underline{w}}(G/G) = \mathbb{Z}/2,\\
&\underline{\underline{P}}(G/e) = \underline{\underline{P}}(G/A) = 0,\\
&\underline{\underline{P}}(G/H) = \mathbb{Z}/2^3, \underline{\underline{P}}(G/G) = \mathbb{Z}/2,
\end{aligned}$$

where the nontrivial maps are the identity maps, except the $\mathbb{Z}/2 \longrightarrow \mathbb{Z}/2^3$, which is the diagonal map.

Therefore its projective resolution $F_{(q)}$ can be given as follows:

$$F_{(q)} = F_{(\mathbb{Z}/2)} \oplus F_{(w)} \oplus F_{(P)}$$

with dimension shifted, where

$$\begin{aligned}
F^0_{(\mathbb{Z}/2)} &= F^1_{(\mathbb{Z}/2)} = F_{G/G}, \\
F^t_{(\mathbb{Z}/2)} &= 0 \qquad (t \geq 2); \\
F^0_{(w)} &= F_{G/G}, \\
F^1_{(w)} &= F_{G/G} \oplus F_{G/H}, \\
F^2_{(w)} &= F_{G/H}, \\
F^t_{(w)} &= 0 \qquad (t \geq 3); \\
F^0_{(P)} &= F_{G/G} \oplus F_{G/H}, \\
F^1_{(P)} &= F_{G/G} \oplus (F_{G/H})^2 \oplus F_{G/A}, \\
F^2_{(P)} &= F_{G/e}, \\
F^t_{(P)} &= 0 \qquad (t \geq 4),
\end{aligned}$$

where the morphisms are easily computed by the explicit description of the maps ϕ^i in the above.

Now, a direct computation shows that

$$\begin{aligned}
E_2^{p,q} &= \operatorname{Ext}^p_{\mathcal{C}_G}\left(\underline{\underline{H}}_q(Y), \underline{\underline{M}}\right) \\
&= \begin{cases} H^p\left(\operatorname{Hom}_{\mathcal{C}_g}(F, \underline{\underline{M}})\right) & \text{if } q = m \\ \left\{H^p\left(\operatorname{Hom}_{\mathcal{C}_G}\left(F_{(\mathbb{Z}/2)} \oplus F_{(w)} \oplus F_{(P)}, \underline{\underline{M}}\right)\right)\right\}^{A(q,m)} & \text{if } m < q < 2m, \end{cases}
\end{aligned}$$

where $A(q, m)$ is the number of Cartan generators on $H_q(K(\mathbb{Z}, m); \mathbb{Z})$, and

$$H^p\left(\operatorname{Hom}_{\mathcal{C}_g}(F, \underline{\underline{M}})\right) = \begin{cases} \mathbb{Z}^{10} & \text{if } p = 0 \\ \mathbb{Z}^2 & \text{if } p = 1 \\ 0 & \text{if } p \geq 2, \end{cases}$$

$$H^p\left(\operatorname{Hom}_{\mathcal{C}_g}(F_{(\mathbb{Z}/2)}, \underline{\underline{M}})\right) = \begin{cases} 0 & \text{if } p = 0 \\ (\mathbb{Z}/2)^3 & \text{if } p = 1 \\ 0 & \text{if } p \geq 2, \end{cases}$$

$$H^p\left(\mathrm{Hom}_{\mathcal{C}_g}(F_{(w)}, \underline{\underline{M}})\right) = \begin{cases} 0 & \text{if } p = 0 \\ \mathbb{Z}/2 & \text{if } p = 1 \\ (\mathbb{Z}/2)^2 = \mathbb{Z}^3/\Delta + 2\mathbb{Z}^3 & \text{if } p = 2 \\ 0 & \text{if } p \geq 3, \end{cases}$$

$$H^p\left(\mathrm{Hom}_{\mathcal{C}_g}(F_{(P)}, \underline{\underline{M}})\right) = \begin{cases} 0 & \text{if } p = 0 \\ (\mathbb{Z}/2)^3 & \text{if } p = 1 \\ 0 & \text{if } p \geq 2. \end{cases}$$

The unique elements of homological degree 2 in $H^2\left(\mathrm{Hom}_{\mathcal{C}}\left(F_{(w)}, \underline{\underline{M}}\right)\right)$ are produced by the relation

$$\phi^2_{(w)}(c_1) = a - 2b_1 \in F_{G/H}(G/H)$$

in $F_{(w)}$, and the map

$$\mathrm{Res}^G_H(P) = z_1 + z_2 + z_4 \in RO(H)$$

in $\underline{\underline{M}} = \underline{\underline{RO}}$. Both of them refl;ect the fact that Res^G_H is not surjective in the system.

Let us turn to the image of the map $\oplus \mathrm{Res}^G_H$. Given any $\mathcal{C}_G$-resolution F_* of $\underline{\underline{H}}_q(Y)$, if we restrict it to the values of G/Γ, it forms a module resolution $F_*(G/\Gamma)$ of the module $\underline{\underline{H}}_q(Y) = H_q\left(Y^\Gamma\right)$. Also this correspondence gives a cochain map

$$\mathrm{Hom}_{\mathcal{C}_G}\left(F_*, \underline{\underline{M}}\right) \longrightarrow \mathrm{Hom}\left(F_*(G/\Gamma), \underline{\underline{M}}(G/\Gamma)\right)$$

and hence a map of spectral sequences

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{C}}\left(\underline{\underline{H}}_q(Y), \underline{\underline{M}}\right) \longrightarrow {}'E_2^{p,q} = \mathrm{Ext}^p_{\mathbb{Z}}\left(\underline{\underline{H}}_q\left(Y^\Gamma\right), \underline{\underline{M}}(G/\Gamma)\right).$$

The right hand side forms the usual universal coefficient spectral sequence for the space Y^Γ, and hence collapses since

$$H_q\left(Y^\Gamma\right) = \begin{cases} \mathbb{Z}^t & \text{if } q = m \\ (\mathbb{Z}/2)^s & \text{if } q > m. \end{cases}$$

Now that we know

$$\begin{aligned} E_2^{p,q} &= 0 \qquad \text{if } p \geq 3, \\ E_2^{0,q} &= 0 \qquad \text{if } q \geq m+1, \\ E_2^{2,q} &= (\mathbb{Z}/2)^{2A(q,m)}, \\ {}'E_2^{p,q} &= 0 \qquad \text{if } p \geq 2, \end{aligned}$$

and the differentials are

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1},$$

we see that there is no room for nontrivial differentisls, so both of the spectral sequences collapse.

The nontrivial term $E^{2,q}$ is in the kernel of the spectral sequence morphism, and hence is a nontrivial kernel in the $E^{2,q}$. But since $E_\infty^{p,q} = 0$ for $p \geq 3$, this kernel lies in the highest (i.e., smallest) filtration term, thus produces a nontrivial kernel of

$$\mathrm{Res}_\Gamma^G : H_G^{p+q}\left(Y; \underline{\underline{M}}\right) \longrightarrow H^{p+q}\left(Y^\Gamma; \underline{\underline{M}}(G/\Gamma)\right).$$

Since the same $E_2^{p,q}$ is in the kernel for any Γ, it produces a nontrivial kernel of

$$\bigoplus \mathrm{Res}_H^G : H_G^{p+q}\left(Y; \underline{\underline{M}}\right) \longrightarrow \bigoplus_{(\Gamma)} H^{p+q}\left(Y^\Gamma; \underline{\underline{M}}(G/\Gamma)\right).$$

This completes the proof of the Lemma.

Remark. $A(q,m) = \frac{1}{2}\,\mathrm{rank}\, E_2^{2,q}$ is non-zero if

$$q - m = 2, 4, 6, 8, 10, 12, 14, 16, 17, \ldots .$$

(See Cartan's formula in [C].)

We also remark that similar proof works for

$$Y = \mathcal{K}\left(\underline{\underline{RO}}, m\right) \quad \text{or} \quad \mathcal{K}\left(\mathbb{Z}/2 \oplus \underline{\underline{R}}^-, m\right),$$
$$\underline{\underline{M}} = \underline{\underline{RO}} \quad \text{or} \quad \mathbb{Z}/2 \oplus \underline{\underline{R}}^-,$$

and an analogue of the Lemma holds.

We return to the proof of the Proposition, where

$$X = \mathcal{E}_0 \times \mathcal{K}\left(\hat{\underline{\underline{\mathcal{L}}}}_2, 2\right) \times \mathcal{K}\left(\hat{\underline{\underline{\mathcal{L}}}}_4, 4\right) \times \prod_{i=6}^{n-1} \mathcal{K}\left(\underline{\underline{\mathcal{L}}}_i, i\right),$$

and the coefficient system is $\underline{\underline{\mathcal{L}}}_n$.

If we take n to be a multiple of 4, we can choose m in such that m is also a multiple of 4, $m + 4 \leq n + 1 < 2m$ and such that

$$A(n-1, m) \neq 0 \qquad \text{for such } m,$$

by the above remark.

Therefore it suffices to show that there is a natural homomorphism

$$P^* : H_G^*\left(Y; \underline{\underline{RO}}\right) \longrightarrow H_G^*\left(X; \underline{\underline{\mathcal{L}}}_n\right)$$

which is injective. This follows from the next Lemma, which implies that Y is a direct factor of X as a G-space:

Lemma. *The system $\underline{\underline{RO}}$ is included in the system $\underline{\underline{\mathcal{L}_n}}$ as a direct summand of system, if $n \equiv 0 \mod 4$.*

Proof. $\underline{\underline{\mathcal{L}_n}}(G/\Gamma) = \mathcal{L}_n(\Gamma) = \bigoplus_{(\Lambda)\subset\Gamma} L_n(N_\Gamma\Lambda/\Lambda)$ includes $L_n(\Gamma/e) = RO(\Gamma)$ as a "top summand". The system structure of $\underline{\underline{\mathcal{L}_n}}$ splits this collection of $RO(\Gamma)$'s as a direct summand of system, because the "top summand" and the complementary summand are both preserved by the structure. Thus the proof of the Proposition is complete.

Finally we remark that the same situation occurs for actions of general non-abelian metacyclic group G of odd order. In the similar way as above, the non-surjectivityof Res_H^G in the system $\underline{\underline{RO}}$ produces a nontrivial kernel of the fixed-point homomorphism inside the Bredon cohomology group.

The result of the Proposition implies that the Bredon cohomology group in which the euqivariant k-invariant of F/PL lies is not detected by the nonequivariant cohomology of the fixed-point setsm for the group $G = G_{21}$, or more generally, by the above remark, of any nonabelian metacyclic group G of odd order.

This fact suggests that there might be an exotic k-invariant of F/PL, in the sense that it is nontrivial but vanishes after one maps it to nonequivariant data. We hope to construct in future work a new geometric invariant which detects these exotic elements.

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KITASHIRAKAWA, SAKYO-KU, KYOTO 606-8502, JAPAN