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Toeplitz Ψ^*-algebras via unitary group representations

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Abstract

As it was pointed out in [12] there are construction methods for spectral invariant Fréchet operator algebras such as Ψ^*- and Ψ_0-algebras in the bounded operators on a Hilbert space having prescribed properties. For the Segal-Bargmann space H and using systems of unbounded closable Toeplitz operators T_f where f is in a certain class $\mathrm{SP}_{\mathrm{Lip}}(\mathbb{C}^n)$ of symbols we show that these algebras contain all Toeplitz operators T_h with $h \in L^\infty(\mathbb{C}^n)$. Let ρ be the Segal-Bargmann representation of the Heisenberg group $\mathbb{H}_n$ in the bounded operators on H. As an application of our results above we characterize a class of smooth Toeplitz operators in the Ψ^*-algebra of smooth elements with respect to ρ.

1 Introduction

Subsequent to the results in [12] it frequently has been remarked that the abstract concept of (locally) spectral invariant Fréchet algebras such as Ψ_0- and Ψ^*-algebras successfully can be applied to the structural analysis of certain algebras of pseudo-differential operators. Applications arise in complex analysis, analytic perturbation theory of Fredholm operators and non-abelian cohomology for analyzing isomorphisms of abelian groups in K-theory. By generalizing a characterization of the Hörmander classes $\Psi^0_{\rho,\delta}$ [1] by commutator conditions (see Theorem 2.1) a construction method for algebras of the above mentioned type with prescribed properties have been given in [12].

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[1] $0 \le \delta \le \rho \le 1$ and $\delta < 1$

Let $H := H^2(\mathbb{C}^n, \mu)$ be the Segal-Bargmann space of Gaussian square integrable entire functions on $\mathbb{C}^n$. We denote by P the orthogonal projection from $L^2(\mathbb{C}^n, \mu)$ onto H and we write M_f for the multiplication with a measurable symbol f. In the initial stage of this paper we consider iterated commutators of closable Toeplitz operators $T_f := PM_f$ on H having symbols in a certain class $\mathrm{SP}_{\mathrm{Lip}}(\mathbb{C}^n)$ of measurable and in general unbounded functions on $\mathbb{C}^n$. For a system $\mathcal{S}_m := [\, T_{f_1}, \cdots, T_{f_m}\,]$ of operators with $f_j \in \mathrm{SP}_{\mathrm{Lip}}(\mathbb{C}^n)$ and in the sense of [12] the Ψ_0-algebra $\Psi_\infty^{\mathcal{S}_m}$ in the bounded operators $\mathcal{L}(H)$ on H can be defined by commutator methods with respect to $\mathcal{S}_m$. We show that $\Psi_\infty^{\mathcal{S}_m}$ contains all Toeplitz operators with bounded measurable symbols. More precisely:

Theorem A *The symbols map $L^\infty(\mathbb{C}^n) \ni h \mapsto T_h \in \Psi_\infty^{\mathcal{S}_m}$ is well-defined and continuous.*

Let $\mathbb{H}_n$ be the Heisenberg group and α be the Segal-Bargmann representation of $\mathbb{H}_n$ in $\mathcal{L}(H)$, c.f. [10]. The map α is well-known to be unitary, irreducible and strongly continuous. In particular, the Ψ^*-algebra $\Psi^\infty(\mathbb{H}_n) \subset \mathcal{L}(H)$ of smooth elements with respect to α arise in a natural way and it can be characterized by commutator methods. We describe a symmetric subspace $\mathcal{S}_s \subset L^\infty(\mathbb{C}^n)$ with the induced topology such that:

Theorem B *The symbols map $\mathcal{S}_s \ni h \mapsto T_h \in \Psi^\infty(\mathbb{H}_n)$ is well-defined and continuous.*

This result can be stated in terms of the algebra construction. Let $\mathcal{A}$ be the algebra of multiplication operators on $V := L^2(\mathbb{C}^n, \mu)$ with bounded measurable symbols. In a natural way α extends to a representation of $\mathbb{H}_n$ into $\mathcal{L}(V)$ and the corresponding operator algebras $\Psi^k(\mathcal{A}, \mathbb{H}_n)$ of C^k-elements in $\mathcal{A}$ form a decreasing scale. Note that $M_f \in \Psi^k(\mathcal{A}, \mathbb{H}_n)$ is related to the smoothness of the symbols $f \in L^\infty(\mathbb{C}^n)$. Clearly, $\mathcal{A}$ projects under P onto the space $\mathcal{A}_P := P\mathcal{A}P$ of Toeplitz operators with bounded symbols. Theorem B states:

$$P\,\Psi^k(\mathcal{A}, \mathbb{H}_n)\,P = P\,\Psi^{k+1}(\mathcal{A}, \mathbb{H}_n)\,P \subset \mathcal{L}(H) \qquad \text{for all } k \in \mathbb{N}.$$

Heuristically, the smoothness of f cannot be recovered by commutator methods from the Toeplitz operator T_f. We want to remark here that these results are related to an observation in [14], [3]. Let $\beta : L^2(\mathbb{R}^n) \to H$ be the Bargmann isometry and f a bounded measurable function on $\mathbb{C}^n$. The assignment $\beta^{-1}\, T_f \beta$ can be shown to be a pseudo-differential operator $W_{\sigma(f)}$ on $L^2(\mathbb{R}^n)$ in its Weyl quantization. By Identifying $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ the Weyl symbol $\sigma(f)$ and f are related via the heat equation on $\mathbb{R}^{2n}$. There is $t_0 > 0$ such that:

$$\sigma(f) = e^{-t_0\Delta} f := \text{ solution of the heat equation with initial data } f \text{ at the time } t_0.$$

Moreover, σ maps the space of continuous functions with compact support into the symbol class $\mathcal{S}_{\rho,\delta}^{-\infty}$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Corresponding to Theorem A and B it can be checked that $f \mapsto \sigma(f)$ is continuous with respect to the $L^\infty(\mathbb{C}^n)$ topology and the usual Fréchet topology on $\mathcal{S}_{\rho,\delta}^{-\infty}$.

In our first section we remind of some basic definitions and results related to the construction of Ψ_0- and Ψ^*-algebras. For Toeplitz operators having symbols of polynomial growth at infinity an invariant subspace $\mathrm{H}_{\exp}(\mathbb{C}^n)$ of H is defined in section 3. Moreover, the existence of bounded extensions for a class of iterated commutators of Toeplitz operators on $H_{\exp}(\mathbb{C}^n)$ and Theorem A are proved. Section 4 contains the proof of Theorem B and finally we have added some examples and applications in section 5.

2 Fréchet operator algebras with prescribed properties

The following definition due to B. Gramsch have been given in [11]:

Definition 2.1 Let $\mathcal{B}$ be a Banach-algebra with unit e and let $\mathcal{F}$ be a continuously embedded Fréchet algebra in $\mathcal{B}$ with $e \in \mathcal{F}$. Then $\mathcal{F}$ is called *Ψ_0-algebra* if it is *locally spectral invariant* in $\mathcal{B}$, i.e. there is $\varepsilon > 0$ with

$$\{ a \in \mathcal{F} : \| e - a \|_{\mathcal{B}} < \varepsilon \} \subset \mathcal{F}^{-1}.$$

Moreover, one defines:

- If $\mathcal{B}$ is a C^*-algebra and $\mathcal{F}$ is a symmetric Ψ_0-algebra in $\mathcal{B}$, then $\mathcal{F}$ is called *Ψ^*-algebra*. ($\mathcal{F}$ automatically is *spectral invariant*, i.e. $\mathcal{F} \cap \mathcal{B}^{-1} = \mathcal{F}^{-1}$).
- If the topology of $\mathcal{F}$ is generated by a system $[\, q_j : j \in \mathbb{N}\,]$ of sub-multiplicative semi-norms with $q_j(e) = 1$ for $j \in \mathbb{N}$, then $\mathcal{F}$ is called *sub-multiplicative* or *locally m-convex* (E. Michael, 1952) Ψ_0- resp. Ψ^*-algebra.

The concept of Ψ^*- and Ψ_0-algebras allows to treat phenomenas of local structure. As it was observed for algebras of Pseudo-differential operators, C^∞-properties such as pseudo- or micro-locality are preserved by taking closures in the Fréchet topology. Important examples of Ψ^*-algebras are given by the Hörmander classes $\Psi^0_{\rho,\delta}$ [2] of zero order where $\mathcal{B} := \mathcal{L}(L^2(\mathbb{R}^n))$. It is known that $\Psi^0_{\rho,\delta}$ can be described in terms of commutator conditions.

Theorem 2.1 *(R. Beals, '77, [6])*
An operator $B : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is of class $\Psi^0_{\rho,\delta}$ iff for $\alpha, \beta \in \mathbb{N}_0^n$ all iterated commutators:

$$ad\big[- ix\big]^{\alpha} ad\big[i\partial_x \big]^{\beta}(B) : H^{s-\rho|\alpha|+\delta|\beta|} \to H^s \tag{2.1}$$

admit bounded extensions between suitable Sobolev spaces to $L^2(\mathbb{R}^n)$.

On the one hand the spectral invariance of $\Psi^0_{\rho,\delta}$ follows from the commutator characterizations in Theorem 2.1, see [19], [20]. On the other hand, by replacing ix and $i\partial_x$ above with a system of closable and densely defined operators, conditions of the type (2.1) have been used to define (submultiplicative) Ψ_0-algebras in a fairly general situation, see [12]. Below we give the definitions and remind of some basic results.

[2] $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$

2.1 Commutator Methods

Given a topological vector space X we write $L(X)$ (resp. $\mathcal{L}(X)$) for the linear (resp. bounded linear) operators on X.

Definition 2.2 (Iterated commutators)
For a system $\mathcal{S}_m := [A_1, \cdots, A_m]$ where $A_j, B \in L(X)$ we call m the *length* of $\mathcal{S}_m$. We inductively define the *iterated commutators* $\text{ad}[\emptyset](B) := B$ and:

- $\text{ad}[\mathcal{S}_1](B) := [A_1, B] = A_1B - BA_1$,
- $\text{ad}[\mathcal{S}_{j+1}](B) := \text{ad}[A_{j+1}](\text{ad}[\mathcal{S}_j](B))$ for $j = 1, \cdots, m-1$.

In the case of $A = A_j$ where $j = 1, \cdots, m$ we also write:

- $\text{ad}^0[A](B) := B$ and $\text{ad}^m[A](B) := \text{ad}[\mathcal{S}_m](B)$.

With these notations it follows for finite systems $\mathcal{S}_j$ and $\mathcal{S}_k$ in $L(X)$:

$$\text{ad}[\mathcal{S}_j](\text{ad}[\mathcal{S}_k](B)) = \text{ad}[\mathcal{S}_k, \mathcal{S}_j](B).$$

Let H be a Hilbert space and $\mathcal{F} \subset \mathcal{L}(H)$ be a sub-multiplicative Ψ^*-algebra. Assume that the topology of $\mathcal{F}$ is generated by a sequence $(q_j)_{j\in\mathbb{N}}$ of semi-norms and without lost of generality let $q_0 := \|\cdot\|_{\mathcal{L}(H)}$. Given a finite system $\mathcal{V}$ of closed and densely defined operators $A : H \supset \mathcal{D}(A) \to H$ and following [12] we define:

- $\mathcal{I}(A) := \{ a \in \mathcal{F} : a(\mathcal{D}(A)) \subset \mathcal{D}(A) \}$,
- $\mathcal{B}(A) := \{ a \in \mathcal{I}(A) : [A, a]$ extends to an element $\delta_A(a) \in \mathcal{F} \}$.

Inductively, one obtains:

- $\Psi_0^{\mathcal{V}} := \mathcal{F}$, with semi-norms $q_{0,j} := q_j$ for $j \in \mathbb{N}$,
- $\Psi_1^{\mathcal{V}} := \bigcap_{A\in\mathcal{V}} \mathcal{B}(A)$,
- $\Psi_k^{\mathcal{V}} := \{ a \in \Psi_{k-1}^{\mathcal{V}} : \delta_A a \in \Psi_{k-1}^{\mathcal{V}}$ for all $A \in \mathcal{V} \}$ where $k \geq 2$,
- $\Psi_\infty^{\mathcal{V}} := \bigcap_{k\in\mathbb{N}} \Psi_k^{\mathcal{V}}$.

This process leads to a decreasing scale of algebras in $\mathcal{F}$:

$$\mathcal{F} = \Psi_0^{\mathcal{V}} \supset \cdots \Psi_n^{\mathcal{V}} \supset \Psi_{n+1}^{\mathcal{V}} \supset \cdots \supset \Psi_\infty^{\mathcal{V}} := \bigcap_{k\in\mathbb{N}} \Psi_k^{\mathcal{V}}. \tag{2.2}$$

For $n \geq 1$, we inductively define a system $(q_{n,j})_{j\in\mathbb{N}}$ (resp. $(q_{n,j})_{j,n\in\mathbb{N}}$) of norms on $\Psi_n^{\mathcal{V}}$ (resp. on $\Psi_\infty^{\mathcal{V}}$) by:

$$q_{n,j}(a) := q_{n-1,j}(a) + \sum_{A\in\mathcal{V}} q_{n-1,j}(\delta_A a). \tag{2.3}$$

According to [12], $\Psi^{\mathcal{V}}_{\infty}$ is a sub-multiplicative Ψ_0-algebra in $\mathcal{F}$. In the case where each $A \in \mathcal{V}$ is symmetric we replace $\mathcal{B}(A)$ by:

$$\mathcal{B}^*(A) := \{\, a \in \mathcal{B}(A) : a^* \in \mathcal{B}(A) \,\}.$$

Then the algebras $\Psi^{\mathcal{V}}_n$ are symmetric and $\Psi^{\mathcal{V}}_{\infty}$ is a Ψ^*-algebra in $\mathcal{L}(H)$. Let $D \subset H$ be a *core* for $\mathcal{V}$, i.e. the inclusion $D \hookrightarrow \mathcal{D}(A)$ is dense with respect to the graph norm for all $A \in \mathcal{V}$. Then it was shown in [2], [3]:

Proposition 2.1 *Assume that $a \in \mathcal{F}$ and property (E_k) holds for $k \in \mathbb{N} \cup \{\infty\}$:*

(E_k): D is invariant under all $A \in \mathcal{V}$ and $a \in \mathcal{F}$. Moreover, assume that for any system

$$\mathcal{A} \subset \mathcal{S}_k(\,\mathcal{V}\,) := \Big\{ [\, A_1, \cdots, A_j \,] : \text{ where } A_l \in \mathcal{V} \text{ and } 1 \leq l \leq j \leq k \Big\}.$$

$ad[\,\mathcal{A}\,](\,a\,) : H \supset D \longrightarrow H$ has a continuous extensions to $C(\mathcal{A}, a) \in \mathcal{F}$.

Then $a \in \Psi^{\mathcal{V}}_k$ and $C(\mathcal{A}, a)$ is a bounded extension of $ad\,[\,\mathcal{A}\,](a) : H \subset \mathcal{D}(A) \longrightarrow H$ to H for any operator $A \in \mathcal{V}$.

The (locally) spectral invariance of $\mathcal{A} \subset \mathcal{B}$ is preserved under projections $p = p^2 \in \mathcal{A}$. It is readily verified that $\mathcal{A}_p := p\,\mathcal{A}\,p$ is (locally) spectral invariant in $\mathcal{B}_p := p\,\mathcal{B}\,p$. If in addition $\mathcal{B}$ is a C^*-algebra, $\mathcal{A}$ is symmetric in $\mathcal{B}$ and $p = p^*$, then $\mathcal{A}_p$ is symmetric and spectral invariant in $\mathcal{B}_p$.

With (2.2) and an orthogonal projection $p \in \Psi^{\mathcal{V}}_n$, $n \in \mathbb{N} \cup \{\infty\}$ from H onto a closed subspace $H_0 \subset H$ there is a scale of projected algebras in $\mathcal{L}(\,H_0\,)$:

$$\mathcal{L}(\,H_0\,) \supset \mathcal{F}_p = \Psi^{\mathcal{V}}_{0\,p} \supset \cdots \Psi^{\mathcal{V}}_{n-1\,p} \supset \Psi^{\mathcal{V}}_{n\,p}. \tag{2.4}$$

It can be shown that (2.4) arises by commutator methods with a system $\mathcal{V}_p$ of closed operators on H_0 where $\mathcal{D}(\,A_p\,) := p[\,\mathcal{D}(\,A\,)\,]$ and

$$\mathcal{V}_p := \{\, A_p := p\,A\,p : H_0 \supset \mathcal{D}(\,A_p\,) \longrightarrow H_0 : A \in \mathcal{V} \,\}.$$

Defining (2.4) by commutator conditions with respect to $\mathcal{V}_p$ only requires that $p \in \Psi^{\mathcal{V}}_1$. Thus this method gives a natural extension of (2.4) to an infinite scale for $n \in \mathbb{N}$.

There is a corresponding scale of *$\mathcal{V}$-Sobolev spaces* in H:

- $\mathcal{H}^0_{\mathcal{V}} := H$ with the norm $p_0 := \|\cdot\|_H$.
- $\mathcal{H}^1_{\mathcal{V}} := \bigcap_{A \in \mathcal{V}} \mathcal{D}(A)$.
- $\mathcal{H}^k_{\mathcal{V}} := \Big\{\, x \in \mathcal{H}^{k-1}_{\mathcal{V}} : Ax \in \mathcal{H}^{k-1}_{\mathcal{V}} \text{ for all } A \in \mathcal{V} \,\Big\},\quad k \geq 2.$
- $\mathcal{H}^{\infty}_{\mathcal{V}} := \bigcap_{k \in \mathbb{N}} \mathcal{H}^k_{\mathcal{V}}$.

We endow $\mathcal{H}^k_{\mathcal{V}}$ with the norm

$$p_k(x) := p_{k-1}(x) + \sum_{A\in\mathcal{V}} p_{k-1}(Ax), \qquad x \in \mathcal{H}^k_{\mathcal{V}}.$$

Let the topology of $\mathcal{H}^\infty_{\mathcal{V}}$ be defined by the system of norms $(p_k)_{k\in\mathbb{N}_0}$. It can be shown that $(\mathcal{H}^k_{\mathcal{V}}, p_k)$ is a Banach spaces and $(\mathcal{H}^\infty_{\mathcal{V}}, (p_k)_{k\in\mathbb{N}})$ turns into a Fréchet space. Moreover, each $A \in \mathcal{V}$ induces a bounded operator $\tilde{A}_k : \mathcal{H}^k_{\mathcal{V}} \to \mathcal{H}^{k-1}_{\mathcal{V}}$. For $n \in \mathbb{N} \cup \{\infty\}$ it was shown in [12] that all maps

$$\Psi^{\mathcal{V}}_k \times \mathcal{H}^k_{\mathcal{V}} \longrightarrow \mathcal{H}^k_{\mathcal{V}} : (a, x) \mapsto a(x)$$

are bilinear and continuous. The following result on *regularity* was proved in [13]:

Theorem 2.2 *Let $A \in \Psi^{\mathcal{V}}_\infty$ be a Fredholm operator and $u \in H$ with $Au = f \in \mathcal{H}^k_{\mathcal{V}}$ for some $k \in \mathbb{N} \cup \{\infty\}$. Then it follows that $u \in \mathcal{H}^k_{\mathcal{V}}$.*

3 On the Segal-Bargmann Projection

Throughout this paper we write $\langle x, y \rangle := x_1 \bar{y}_1 + \cdots x_n \bar{y}_n$ for the Hermitian inner product on $\mathbb{C}^n$ and $|x| := \sqrt{\langle x, x \rangle}$. For $c > 0$ and the Lebesgue measure v let us denote by μ_c the Gaussian measure on $\mathbb{C}^n$ given by:

$$d\mu_c = c^n \pi^{-n} \exp\left(-c|\cdot|^2\right) dv.$$

With $\mu := \mu_1$ let $H^2(\mathbb{C}^n, \mu)$ be the *Segal-Bargmann space* of μ-square integrable entire functions on $\mathbb{C}^n$. We denote by P the orthogonal projection from $L^2(\mathbb{C}^n, \mu)$ onto $H^2(\mathbb{C}^n, \mu)$. The reproducing kernel K (resp. the normalized kernel k) corresponding to $H^2(\mathbb{C}^n, \mu)$ are known to be:

(a) $K(y, x) := \exp(\langle y, x \rangle)$,

(b) $k_x(y) := K(y, x) \, \| K(\cdot, x) \|^{-1} = \exp(\langle y, x \rangle - \frac{1}{2}|x|^2)$

where $\|\cdot\|$ denotes the $L^2(\mathbb{C}^n, \mu)$-norm. For $z, w \in \mathbb{C}^n$ we write $\tau_w(z) := z + w$ for the shift by w. Consider the space of measurable symbols on $\mathbb{C}^n$ given by:

$$\mathcal{T}(\mathbb{C}^n) := \{ g : g \circ \tau_x \in L^2(\mathbb{C}^n, \mu) \quad \text{for all} \quad x \in \mathbb{C}^n \}.$$

For $g \in \mathcal{T}(\mathbb{C}^n)$ and with the natural domain of definition

$$\mathcal{D}(T_g) := \left\{ f \in H^2(\mathbb{C}^n, \mu) \, : gf \in L^2(\mathbb{C}^n, \mu) \right\} \tag{3.1}$$

the *Toeplitz operator* T_g on $H^2(\mathbb{C}^n, \mu)$ is densely defined by:

$$T_g : \mathcal{D}(T_g) \ni f \mapsto P(fg).$$

If g has *polynomial growth at infinity* we can determine an invariant subspace for T_g:

We inductively define a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_1 := \frac{1}{4}$ and $a_{n+1} := [4 \cdot (1 - a_n)]^{-1}$ for all $n \geq 2$. It can be checked that:

(a) $a_n < \frac{1}{2}, \quad \forall\, n \in \mathbb{N}$,

(b) $(a_n)_{n\in\mathbb{N}}$ is strictly increasing,

(c) $\lim_{n\to\infty} a_n = \frac{1}{2}$.

Let $\mathbb{P}[\,\mathbb{C}^n\,]$ be the space of all *polynomials* on $\mathbb{C}^n$ in the variables $z := (\,z_1, \cdots, z_n\,)$ and $\bar{z} := (\,\bar{z}_1, \cdots, \bar{z}_n\,)$. We write $\mathbb{P}_a[\,\mathbb{C}^n\,]$ for all *analytic polynomials* and set:

$$L_{\exp}(\,\mathbb{C}^n\,) := \Big\{\, f \in L^2(\mathbb{C}^n, \mu) : \exists\, c < \frac{1}{2}, 0 < D \text{ s.t. } |\, f(z)\,| \leq D \exp\big(\,c\,|\,z\,|^2\,\big) \text{ a.e.} \Big\}.$$

Because of $\mathbb{P}[\,\mathbb{C}^n\,] \subset L_{\exp}(\mathbb{C}^n)$ it follows that $L_{\exp}(\mathbb{C}^n)$ is dense in $L^2(\mathbb{C}^n, \mu)$. With the space $\mathcal{H}(\mathbb{C}^n)$ of entire functions on $\mathbb{C}^n$ we define a subspace of $H^2(\mathbb{C}^n, \mu)$ by:

$$H_{\exp}(\,\mathbb{C}^n\,) := \mathcal{H}(\,\mathbb{C}^n\,) \cap L_{\exp}(\,\mathbb{C}^n\,),$$

Consider the symbols having polynomial growth at ∞:

$$\mathrm{Pol}(\,\mathbb{C}^n\,) := \big\{\, f : \exists\, j \in \mathbb{N} \text{ s.t. } |\, f(z)\,|\,\big(\,1 + |\,z\,|^2\,\big)^{-\frac{j}{2}} \in L^\infty(\mathbb{C}^n)\,\big\}.$$

Proposition 3.1 *It holds $P[\,L_{\exp}(\,\mathbb{C}^n\,)\,] \subset H_{\exp}(\,\mathbb{C}^n\,)$ and for f in $Pol(\,\mathbb{C}^n\,)$:*

$$T_f\Big[\,H_{\exp}(\,\mathbb{C}^n\,)\,\Big] \subset H_{\exp}(\,\mathbb{C}^n\,) \subset \mathcal{D}(\,T_f\,) \tag{3.2}$$

Proof: It is obvious that $H_{\exp}(\,\mathbb{C}^n\,) \subset \mathcal{D}(\,T_f\,)$. Because the multiplication by f clearly maps $\mathrm{H}_{\exp}(\mathbb{C}^n)$ into $\mathrm{L}_{\exp}(\mathbb{C}^n)$ it is sufficient to prove the first assertion of Proposition 3.1. For $g \in L_{\exp}(\,\mathbb{C}^n\,)$ there are $c < \frac{1}{2}$ and $D > 0$ such that a.e.:

$$|\,g(\,z\,)\,| \leq D \exp\big(\,c\,|\,z\,|^2\,\big).$$

By (a), (b) and (c) and with $(a_n)_{n\in\mathbb{N}}$ above we can choose $n_0 \in \mathbb{N}$ with $c < a_{n_0} < \frac{1}{2}$. Using the transformation formula and the reproducing property of K we obtain:

$$\begin{aligned}
|\,[\,Pg\,](\,z\,)\,| &\leq \int_{\mathbb{C}^n} |\,g \exp\{\,\langle\, z, \cdot\,\rangle\,\}|\, d\mu \\
&\leq D\,\pi^{-n} \int_{\mathbb{C}^n} \exp\Big\{\,\mathrm{Re}\langle\, z, \cdot\,\rangle - \big[\,1 - a_{n_0}\,\big]\,|\cdot|^2\,\Big\}\, dv \\
&= D\,\big(\,1 - a_{n_0}\,\big)^{-n} \int_{\mathbb{C}^n} \exp\Big\{\,2\mathrm{Re}\,\big\langle\, 2^{-1}\,\big(\,1 - a_{n_0}\,\big)^{-\frac{1}{2}} z, \cdot\,\big\rangle\,\Big\}\, d\mu \\
&= D\,\big(\,1 - a_{n_0}\,\big)^{-n} \exp\Big\{\,\underbrace{\big[\,4\,(\,1 - a_{n_0}\,)\,\big]^{-1}}_{=a_{n_0+1}}\,|\,z\,|^2\,\Big\}.
\end{aligned}$$

From (a) above we conclude that $Pg \in H_{\exp}(\,\mathbb{C}^n\,)$. □

Hence all finite products of Toeplitz operators with symbols in $\mathrm{Pol}(\mathbb{C}^n)$ are well-defined on the dense subspace $H_{\exp}(\mathbb{C}^n)$ of $H^2(\mathbb{C}^n, \mu)$. In particular, all iterated commutators of P and multiplication operators M_f with $f \in \mathrm{Pol}(\mathbb{C}^n)$ can been considered as elements in $\mathrm{L}(\mathrm{L}_{\exp}(\,\mathbb{C}^n\,))$. In fact, they can be written as integral operators and a standard application of the *Schur test* leads to a criterion for the boundedness.

Lemma 3.1 *Let $L : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be a measurable function such that:*

$$|L(x,y)| \leq |F(x-y)| \exp\Big\{Re\langle x,y\rangle\Big\}$$

where $F \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$. Then the integral operator A on $L^2(\mathbb{C}^n, \mu)$ defined by

$$[Af](z) := \int_{\mathbb{C}^n} L(z,\cdot) f\, d\mu$$

is bounded on $L^2(\mathbb{C}^n, \mu)$ with $\| A \| \leq 2^n \| F \|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}$.

Proof: With $p := q = \exp(\frac{1}{2}|\cdot|^2)$ on $\mathbb{C}^n$ it follows that:

$$\begin{aligned}\int_{\mathbb{C}^n} |L(\cdot,y)|\, p\, d\mu &\leq \frac{1}{\pi^n}\int_{\mathbb{C}^n} |F(\cdot-y)| \exp\Big\{ \mathrm{Re}\langle\cdot,y\rangle - \frac{1}{2}|\cdot|^2 \Big\}\, dv\\ &= \frac{1}{\pi^n}\int_{\mathbb{C}^n} |F| \exp\Big\{ \mathrm{Re}\langle\cdot+y,y\rangle - \frac{1}{2}|\cdot+y|^2 \Big\}\, dv\\ &= 2^n p(y) \| F \|_{L^1(\mathbb{C}^n,\mu_{\frac{1}{2}})}.\end{aligned}$$

Similarly, we get $\int |L(x,\cdot)|\, p\, d\mu \leq 2^n p(x) \| F \|_{L^1(\mathbb{C}^n,\mu_{\frac{1}{2}})}$. Applying the *Schur test* we obtain the desired result. □

Consider the subspace $\mathrm{SP}_{\mathrm{Lip}}(\mathbb{C}^n)$ of $\mathrm{Pol}(\mathbb{C}^n)$ defined by:

$$\mathrm{SP}_{\mathrm{Lip}}(\mathbb{C}^n) := \Big\{ f \in \mathrm{Pol}(\mathbb{C}^n) : \exists\, c, D > 0 \text{ s.t. } |f(z) - f(w)| \leq D \exp(c|z-w|) \Big\}.$$

As an application of Lemma (3.1) we can prove:

Proposition 3.2 *Let $m \in \mathbb{N}$ and $\mathcal{S}_m := \{M_{f_1}, \cdots, M_{f_m}\}$ with $f_j \in SP_{Lip}(\mathbb{C}^n)$. Then the commutator $ad[\mathcal{S}_m](P) \in L(L_{\exp}(\mathbb{C}^n))$ has a continuous extension to $L^2(\mathbb{C}^n, \mu)$.*

Proof: It is easy to check that the commutator $\mathrm{ad}[\mathcal{S}_m](P)$ can be written as an integral operator on $L^2(\mathbb{C}^n, \mu)$ with kernel:

$$K_m(z,u) = \exp\big(\langle z,u\rangle\big) \prod_{j=1}^{m} \{ f_j(z) - f_j(u) \}. \tag{3.3}$$

By (3.3) and our assumptions on $f_j \in \mathcal{S}_m$ we can choose $c, D > 0$ such that

$$|K_m(z,u)| \leq D \exp\big(c|z-u| + \mathrm{Re}\langle z,u\rangle\big).$$

Because of $F := D\exp(c|\cdot|) \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$ Lemma 3.1 implies the assertion. □

We remark that by (3.3) the maps $\mathrm{ad}[\mathcal{S}_m](P)$ are invariant under permutations of the system $\mathcal{S}_m$. Now, we can prove the boundedness of a class of iterated commutators.

Corollary 3.1 *Let $g \in L^\infty(\mathbb{C}^n)$ and $\mathcal{S}_m := \{M_{f_1}, \cdots, M_{f_m}\}$ with $f_j \in SP_{Lip}(\mathbb{C}^n)$. Then the commutator*

$$ad\,[\,\mathcal{S}_m\,]\Big(\,[\,P, M_g\,]\,\Big) \in L(\,L_{\exp}(\mathbb{C}^n)\,)$$

has a bounded extensions $A(\,\mathcal{S}_m, g\,)$ to $L^2(\mathbb{C}^n, \mu)$ and (3.4) below is continuous between Banach spaces:

$$L^\infty(\mathbb{C}^n) \ni g \mapsto A(\,\mathcal{S}_m, g\,) \in \mathcal{L}(\,L^2(\mathbb{C}^n, \mu)\,). \tag{3.4}$$

Proof: It can be checked by induction or our remark following Proposition 3.2 that:

$$\text{ad}\,[\,\mathcal{S}_m\,]\Big(\,[\,P, M_g\,]\,\Big) = \Big[\text{ad}\,[\,\mathcal{S}_m\,](\,P\,), M_g\Big] \in \text{L}(\,L_{\exp}(\mathbb{C}^n)\,).$$

Because M_g is bounded and ad $[\,\mathcal{S}_m\,](\,P\,)$ has a bounded extension to $L^2(\mathbb{C}^n, \mu)$ by Proposition 3.2 we conclude the desired result. □

Given a finite set $\mathbf{X} := \{\,X_1, \cdots, X_n\,\} \subset \mathcal{L}(\,L^2(\mathbb{C}^n, \mu)\,)$ we denote by $\mathcal{A}(\mathbf{X})$ the algebra generated by $\mathbf{X}$. Moreover, we write:

$$\mathcal{A}_P(\,\mathbf{X}\,) := P\,\mathcal{A}(\,\mathbf{X}\,)\,P := \{\,PAP : A \in \mathcal{A}(\,\mathbf{X}\,)\,\}.$$

for the corresponding projected algebra in $\mathcal{L}(\,H^2(\mathbb{C}^n, \mu)\,)$. By Proposition 3.1 and for all $m \geq 1$ it follows that the commutator:

$$\text{ad}[\,\mathcal{S}_{m-1}\,](\,[\,P, M_{f_m}\,]\,) = -\text{ad}[\,\mathcal{S}_m\,](\,P\,)$$

can be regarded as bounded operators on $L^2(\mathbb{C}^n, \mu)$.

Proposition 3.3 *Let $g \in L^\infty(\mathbb{C}^n)$ and $\mathcal{T}_m := \{\,T_{f_1}, \cdots, T_{f_m}\,\}$ with $f_j \in SP_{Lip}(\mathbb{C}^n)$. Then*

$$ad[\,\mathcal{T}_m\,]\;(\,T_g\,) \in L\,(\,H_{\exp}\,(\mathbb{C}^n)\,)$$

is well-defined. More precisely, with $\mathcal{S}_m := \{\,M_{f_1}, \cdots, M_{f_m}\,\}$ it holds:

$$ad[\,\mathcal{T}_m\,](\,T_g\,) \in \mathcal{A}_P\Big\{\,ad\,[\,\mathcal{N}\,](\,P\,), M_g :\ with\ \mathcal{N} \subset \mathcal{S}_m\,\Big\} \tag{3.5}$$

and $ad[\,\mathcal{T}_m\,](\,T_g\,)$ has a bounded extension $C(\,\mathcal{T}_m, g\,)$ to $H^2(\mathbb{C}^n, \mu)$. Moreover, the symbols map

$$L^\infty(\mathbb{C}^n\,) \ni g \mapsto C(\,\mathcal{T}_m, g\,) \in \mathcal{L}(\,H^2(\,\mathbb{C}^n, \mu\,)\,) \tag{3.6}$$

is continuous between Banach spaces.

Proof: By Proposition 3.1 the iterated commutators ad$[\,\mathcal{T}_m\,](\,T_g\,)$ are well-defined. It is a straightforward computation that:

$$\text{ad}[\,\mathcal{T}_1\,]\;(\,T_g\,) = P\,\Big[\,[\,P, M_{f_1}\,], [\,P, M_g\,]\,\Big]\,P$$

which proves (3.5) in the case $m = 1$. By induction assume ad $[\mathcal{T}_j](T_g)$ has the form:

$$\mathrm{ad}[\mathcal{T}_j](T_g) = \sum_{l\in\mathcal{I}} P\, A_l\, M_g\, B_l\, P \tag{3.7}$$

where $\mathcal{I}$ is a finite index set, I the identity operator and

$$A_l, B_l \in \mathcal{A}(\mathcal{S}_j) := \mathcal{A}\Big\{ \mathrm{ad}[\mathcal{N}](P), I: \text{ with } \mathcal{N} \subset \mathcal{S}_j \Big\}. \tag{3.8}$$

Then it follows that:

$$\mathrm{ad}[\mathcal{T}_{j+1}](T_g) = \sum_{l\in\mathcal{I}} [T_{f_{j+1}}, P\, A_l\, M_g\, B_l\, P].$$

To prove (3.7) in the case $j+1$ it is sufficient to show for all $l \in \mathcal{I}$ the existence of a finite set $\widetilde{\mathcal{I}} \subset \mathbb{N}$ and operators $C_k, D_k \in \mathcal{A}(\mathcal{S}_{j+1})$ such that

$$[T_{f_{j+1}}, P\, A_l\, M_g\, B_l\, P] = \sum_{k\in\widetilde{\mathcal{I}}} P\, C_k\, M_g\, D_k\, P. \tag{3.9}$$

Note that (3.9) follows from $T_{f_{j+1}}\, P\, A_l\, M_g\, B_l\, P = P\, M_{f_{j+1}}\, P\, A_l\, M_g\, B_l\, P$ and

$$[M_{f_{j+1}}, Q] \in A(\mathcal{S}_{j+1})$$

for $Q \in \{P, A_l, B_l\}$. The continuity of (3.6) is a direct consequence of (3.7). □

As an immediate consequence of Proposition 3.2 we remark:

Lemma 3.2 *Let $f \in SP_{Lip}(\mathbb{C}^n)$ and $\mathcal{D}(T_f)$ as in (3.1). Then the Toeplitz operator T_f is densely defined and closed on $\mathcal{D}(T_f)$.*

Proof: Because of $f \in \mathcal{T}(\mathbb{C}^n)$ it follows that T_f is densely defined. Moreover,

$$M_f = T_f + [M_f, P] : \mathcal{D}(T_f) \subset H^2(\mathbb{C}^n, \mu) \longrightarrow L^2(\mathbb{C}^n, \mu). \tag{3.10}$$

Proposition 3.2 with $j = 1$ shows that the commutator $[M_f, P]$ has a continuous extension to $H^2(\mathbb{C}^n, \mu)$. Choose a sequence $(h_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T_f)$ such that:

(i) $\lim_{n\to\infty} h_n = h \in H^2(\mathbb{C}^n, \mu)$,

(ii) $\lim_{n\to\infty} T_f h_n = g \in H^2(\mathbb{C}^n, \mu)$.

Then we conclude from the continuity of $[M_f, P]$ and (3.10) that

$$fh = \lim_{n\to\infty} fh_n \in L^2(\mathbb{C}^n, \mu)$$

Hence $h \in \mathcal{D}(T_f)$ and $g = \lim_{n\to\infty} P(fh_n) = T_f h$. □

Let $\mathcal{T}_m := \{ T_{f_1}, \cdots, T_{f_m} \}$ be a system of Toeplitz operators where $f_j \in \mathrm{SP}_{\mathrm{Lip}}(\mathbb{C}^n)$ for $j = 1, \cdots, n$. From Lemma 3.2 it follows that the domains $\mathcal{D}(T_{f_j})$ are closed with respect to the graph norm $\|\cdot\|_{\mathrm{gr}} := \|\cdot\| + \|T_{f_j} \cdot\|$. Consider $D_j \subset H^2(\mathbb{C}^n, \mu)$ defined by:

$$D_j := \|\cdot\|_{\mathrm{gr}} - \text{closure of } \mathrm{H}_{\exp}(\mathbb{C}^n) \text{ in } \mathcal{D}(T_{f_j}).$$

If we consider T_{f_j} as a closed operator on D_j we can define a scale of algebras (2.2) by commutator methods with the system $\mathcal{S}_m$. By Lemma 2.1 with $D := \mathrm{H}_{\exp}(\mathbb{C}^n)$ our result in Proposition 3.3 can be formulated as follows:

Theorem 3.1 *The symbol map $L^\infty(\mathbb{C}^n) \ni h \mapsto T_h \in \Psi_\infty^{\mathcal{S}_m}$ is well-defined and continuous.*

Note that an application of Theorem 2.2 in the case of $\mathcal{V} := \mathcal{S}_m$ gives a regularity result for Fredholm Toeplitz operators with bounded symbols.

4 Toeplitz Ψ^*-algebras via the Segal-Bargmann representation

There is a unitary representation of the Heisenberg group $\mathbb{H}_n$ in $\mathcal{L}(L^2(\mathbb{C}, \mu))$. By identifying $\mathbb{H}_n$ with $\mathbb{C}^n \times \mathbb{R}$ the group law is given by, [10]:

$$(z, t) * (w, s) := (z + w, t + s + 2^{-1} \operatorname{Im}\langle w, z \rangle).$$

For $z \in \mathbb{C}^n$ and $f \in L^2(\mathbb{C}^n, \mu)$ we define the operator $W_z f := k_z \cdot f \circ \tau_{-z}$. It follows by an easy calculation:

Lemma 4.1 *$H^2(\mathbb{C}^n, \mu)$ is an invariant subspace for all W_z where $z \in \mathbb{C}^n$. Moreover,*

(1) W_z is unitary with $W_z^* = W_{-z} = W_z^{-1}$,

(2) The commutator $ad[P]\, W_z$ vanishes,

(3) For $z, w \in \mathbb{C}^n : W_z W_w = \exp(i \operatorname{Im}\langle w, z \rangle) W_{z+w}$.

By Lemma 4.1 a unitary representation $\tilde{\rho} : \mathbb{H}_n \to \mathcal{L}(L^2(\mathbb{C}^n, \mu))$ of $\mathbb{H}_n$ is given by:

$$\tilde{\rho}(z, t) := e^{it} W_{\frac{z}{\sqrt{2}}}.$$

Moreover, the restriction of $\tilde{\rho}(z, t)$ to $H^2(\mathbb{C}^n, \mu)$ gives rise to a unitary representation ρ of $\mathbb{H}_n$ in $\mathcal{L}(H^2(\mathbb{C}^n, \mu))$. It is well-known that ρ is irreducible and strongly continuous and it is referred to as *Segal-Bargmann representation*, c.f. [10].

For any $A \in \mathcal{B} := \mathcal{L}(H^2(\mathbb{C}^n, \mu))$ we define the map:

$$\begin{aligned} \Phi_A : \quad & \mathbb{H}_n \longrightarrow \mathcal{B} \\ & (z, t) \mapsto \rho(z, t)\, A\, \rho(z, t)^{-1} = W_{\frac{z}{\sqrt{2}}}\, A\, W_{\frac{-z}{\sqrt{2}}}. \end{aligned} \tag{4.1}$$

In particular, note that for $f \in L^\infty(\mathbb{C}^n)$

$$\Phi_{T_f}(z,t) = T_{f \circ \tau_{-\frac{z}{\sqrt{2}}}}.$$

For $k \in \mathbb{N} \cup \{\infty\}$ we consider the C^k-elements

$$\Psi^k := \{ A \in \mathcal{B} : \Phi_A \in C^k(\mathbb{H}_n, \mathcal{B}) \}$$

defined via ρ. To any $z \in \mathbb{C}^n$ we associate $\varphi_A^z : \mathbb{R} \to \mathcal{B}$ by $\varphi_A^z(s) := W_{sz} A W_{-sz}$. According to (4.1) it follows that:

$$\Psi^k = \bigcap_{z \in \mathbb{C}^n} \Psi^{k,z} \quad \text{where} \quad \Psi^{k,z} := \{ A \in \mathcal{B} : \varphi_A^z \in C^k(\mathbb{R}, \mathcal{B}) \}. \tag{4.2}$$

Here we characterize the C^k-Toeplitz operators (i.e. the Toeplitz operators $T_f \in \Psi^k$) in terms of their symbols. We use a characterization of Ψ^∞ by commutator conditions and apply our results of the previous section.

For all $z \in \mathbb{C}^n$ the map $(W_{sz})_{s \in \mathbb{R}} \subset \mathcal{B}$ defines a strongly continuous unitary group of operators. By V^z we denote its infinitesimal generator with domain of definition:

$$\mathcal{D}(V^z) := \{ h \in H^2(\mathbb{C}^n, \mu) : V^z h := \lim_{s \to 0} s^{-1}(W_{sz} - I)h \ \text{ exists} \ \}.$$

By *Stone's Theorem* iV^z is selfadjoint and associated to $\mathcal{V}^z := [\, iV^z \,]$ there is a scale:

$$\mathcal{B} := \Psi_0^{\mathcal{V}^z} \supset \cdots \Psi_n^{\mathcal{V}^z} \supset \Psi_{n+1}^{\mathcal{V}^z} \supset \cdots \supset \Psi_\infty^{\mathcal{V}^z} := \bigcap_{k \in \mathbb{N}} \Psi_k^{\mathcal{V}^z} \tag{4.3}$$

of algebras in $\mathcal{B}$ defined by commutator methods with $\mathcal{V}^z$ as it was described in (2.2) of section 2.1. In particular, $\Psi_\infty^{\mathcal{V}^z}$ is a Ψ^*-algebra and it is well-known that (4.2) and (4.3) are related as follows, see [16]:

Proposition 4.1 *For $z \in \mathbb{C}^n$ let $\mathcal{V}^z := [\, iV^z \,]$ then:*

(i) $\Psi^{k,z} \subset \Psi_k^{\mathcal{V}^z}$ *for* $k \in \mathbb{N}$,

(ii) $\Psi_{k+1}^{\mathcal{V}^z} \subset \Psi^{k,z}$ *for* $k \in \mathbb{N}_0$ *and* $\Psi^{\infty,z} = \Psi_\infty^{\mathcal{V}^z}$.

Using the fact that convergence in $H^2(\mathbb{C}^n, \mu)$ implies uniformly compact convergence on $\mathbb{C}^n$ we can calculate V^z explicitly. Let $h \in \mathcal{D}(V^z)$ and $w \in \mathbb{C}^n$:

$$[\, V^z h \,](w) = \frac{d}{ds}[\, k_{sz}(w)\, h(w - sz) \,]_{|_{s=0}} = \Big\{ \langle w, z \rangle - \sum_{j=1}^n z_j \frac{\partial}{\partial w_j} \Big\} h(w). \tag{4.4}$$

It easily can be seen that all the monomials $m_\alpha(z) := z^\alpha$ for $\alpha \in \mathbb{N}_0^n$ are contained in the domain $\mathcal{D}(V^z)$. Moreover, from the standard identities $M_{w_j} := T_{w_j}$ and $\frac{\partial}{\partial w_j} := T_{\overline{w_j}}$ it follows that the restriction of V^z to $\mathbb{P}_a[\mathbb{C}^n]$ coincides with an unbounded Toeplitz operator:

$$V^z p := T_{\langle \cdot, z \rangle - \langle z, \cdot \rangle} p = 2i\, T_{\mathrm{Im}\langle \cdot, z \rangle} p, \qquad p \in \mathbb{P}_a[\, \mathbb{C}^n \,].$$

In the following we write:

$$g_z := 2i\,\mathrm{Im}\langle\,\cdot, z\,\rangle$$

for the symbol of the Toeplitz operator appearing above. Consider the space $\mathcal{D}(T_{g_z})$ with the graph norm $\|\cdot\|_{\mathrm{gr}} := \|\cdot\| + \|T_{g_z}\cdot\|$. By Lemma 3.2 it follows that $(\mathcal{D}(T_{g_z}), \|\cdot\|_{\mathrm{gr}})$ is a Banach space containing $\mathbb{P}_a[\,\mathbb{C}^n\,]$ and $H_{\exp}(\mathbb{C}^n)$.

Lemma 4.2 *For all $z \in \mathbb{C}^n$ the embedding $\mathbb{P}_a[\,\mathbb{C}^n\,] \hookrightarrow H_{\exp}(\mathbb{C}^n)$ is dense with respect to the graph norm topology. Moreover,*

$$H_{\exp}(\mathbb{C}^n) \subset \mathcal{D}(\,V^z\,) \cap \mathcal{D}(\,T_{g_z}\,) \tag{4.5}$$

and the restrictions of V^z and T_{g_z} to $H_{\exp}(\mathbb{C}^n)$ coincide.

Proof: For $f \in H_{\exp}(\mathbb{C}^n)$ we can choose $c_1 \in (\,0, \frac{1}{2}\,)$ and $D_1 > 0$ such that:

$$|\,f(w)\,| \leq D_1 \exp\left(\,c_1\,|\,w\,|^2\,\right)$$

for all $z \in \mathbb{C}^n$. Hence, $f \in L^2(\mathbb{C}^n, \mu_r)$ for all $r \in (2c_1, 1)$. Fix c_2, c_3 with $2c_1 < c_2 < c_3 < 1$ and choose $D_2 > 0$ with

$$|\,w\,|^2 \leq D_2 \exp\Big(\,[\,c_3 - c_2\,]\,|\,w\,|^2\Big)$$

for all $w \in \mathbb{C}^n$. Then we obtain for all $p \in \mathbb{P}_a[\,\mathbb{C}^n\,]$:

$$\begin{aligned}\|\,T_{g_z}(f-p)\,\|^2 &\leq \|\,g_z(\,f-p\,)\,\|^2 \\ &\leq 2\,|\,z\,|^2 \int_{\mathbb{C}^n} |\cdot|^2\,|\,f-p\,|^2\,d\mu \\ &\leq 2D_2\,|\,z\,|^2\,r^{-n}\,\|\,f-p\,\|^2_{L^2(\mathbb{C}^n,\mu_r)} < \infty\end{aligned}$$

where $r = 1 - c_3 + c_2 \in (\,2c_1, 1\,)$. Because $\mathbb{P}_a[\,\mathbb{C}^n\,]$ is dense in $L^2(\mathbb{C}^n, \mu_r) \cap \mathcal{H}(\mathbb{C}^n)$ for all $r > 0$ the first assertion follows.

Now, (4.5) immediately can be derived from $T_{g_z}p = V^z p$ for $p \in \mathbb{P}_a[\mathbb{C}^n]$ and the density result above which implies that:

$$H_{\exp}(\,\mathbb{C}^n\,) \subset \mathrm{closure}(\,\mathbb{P}_a\,[\,\mathbb{C}^n\,], \|\cdot\|_{\mathrm{gr}}\,) \subset \mathcal{D}(\,V^z\,) \cap \mathcal{D}(\,T_{g_z}\,).$$

Finally, we apply the continuity of $V^z, T_{g_z} : (\,\mathbb{P}_a[\,\mathbb{C}^n\,], \|\cdot\|_{\mathrm{gr}}\,) \to H^2(\mathbb{C}^n, \mu)$. □

For $z \in \mathbb{C}^n$ we denote by $\widetilde{V}^z$ the infinitesimal generator of $(W_{sz})_{s\in\mathbb{R}}$ considered as strongly continuous group of unitary operators on $L^2(\mathbb{C}^n, \mu)$. Let $\mathcal{D}(\,\widetilde{V}^z\,)$ be its domain of definition, then V^z can be obtained by restricting $\widetilde{V}^z$ to $\mathcal{D}(\,V^z\,)$. For $f \in \mathrm{SP}_{\mathrm{Lip}}(\mathbb{C}^n)$ and $r \in \mathbb{N}$ we write

$$\mathcal{A}_r(\,f\,) := \mathcal{A}(\,[\underbrace{M_f, \cdots, M_f}_{\text{r-times}}]\,) \subset \mathcal{L}(\,L^2(\mathbb{C}^n, \mu)\,)$$

where the algebra on the right hand side was defined in (3.8) of Proposition 3.3.

Lemma 4.3 *The domain $\mathcal{D}(\widetilde{V}^z)$ is invariant under $A \in \mathcal{A}_r(f)$ where f is a linear function on $\mathbb{C}^n$. Moreover, the commutator $[A, \widetilde{V}^z]$ vanishes as an operator on $\mathcal{D}(\widetilde{V}^z)$.*

Proof: It is sufficient to show that for all $j \in \mathbb{N}$ the space $\mathcal{D}(\widetilde{V}^z)$ is invariant under the operators

$$a_j(f) := \mathrm{ad}^j [M_f](P).$$

Note that $\mathrm{L}_{\exp}(\mathbb{C}^n)$ is an invariant under W_z and it holds $W_{-z}M_fW_z = M_{f\circ\tau_z}$. Because W_z commutes with P it follows that:

$$W_{-z}\, a_j(f)\, W_z = \mathrm{ad}^j \left[M_{f\circ\tau_z}\right] \left(P\right) = a_j(f).$$

We have used the linearity of f for the second equality. Hence, the commutator $[A, W_z]$ vanishes for all $A \in \mathcal{A}_r(f)$. Fix $h \in \mathcal{D}(\widetilde{V}^z)$ and $A \in \mathcal{A}_r(f)$, then:

$$\frac{1}{s}\{W_{sz} - I\}\, A\, h = A\,\frac{1}{s}\{W_{sz} - I\}\, h \to A\,\widetilde{V}^z\, h$$

as s tends to 0. It follows that $Ah \in \mathcal{D}(\widetilde{V}^z)$ with $\widetilde{V}^zAh = A\,\widetilde{V}^zh$. □

Remark 4.1 Let W be any subspace of $H := H^2(\mathbb{C}^n, \mu)$ such that $H_{\exp}(\mathbb{C}^n) \subset W$. Consider the operators:

$$\mathcal{O}_W := \left\{ A \in \mathrm{L}(W, H) : H_{\exp}(\mathbb{C}^n) \text{ is an invariant space for } A \right\}.$$

Let $A \in \mathcal{O}_W$ and assume there is $A^* \in \mathcal{O}_W$ with $\langle Af, g \rangle = \langle f, A^*g \rangle$ for all $f, g \in W$. Because of $K(\cdot, \lambda) \in H_{\exp}(\mathbb{C}^n)$ for all $\lambda \in \mathbb{C}^n$ it follows that A can be written as an integral operator with kernel:

$$K_A(z, w) = \overline{A^*K(\cdot, z)(w)}. \tag{4.6}$$

In particular, A completely is determined by the restriction of A^* to $H_{\exp}(\mathbb{C}^n)$. Assume that A has a continuous extensions $\tilde{A}$ from $H_{\exp}(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n, \mu)$. Fix $g \in H^2(\mathbb{C}^n, \mu)$ and a sequence $(g_n)_n \subset H_{\exp}(\mathbb{C}^n)$ with $g = \lim_{n\to\infty} g_n$. Then it follows for $z \in \mathbb{C}^n$:

$$\begin{aligned}[\tilde{A}g](z) &= \lim_{n\to\infty} \langle Ag_n, K(\cdot, z) \rangle \\ &= \lim_{n\to\infty} \langle g_n, A^*K(\cdot, z) \rangle = \langle g, A^*K(\cdot, z) \rangle\end{aligned}$$

and $\tilde{A}$ is given by the same integral formula. In particular, A has a (unique) extension from W to $H^2(\mathbb{C}^n, \mu)$.

Let $h \in L^\infty(\mathbb{C}^n)$ and $f : \mathbb{C}^n \to \mathbb{C}$ be a linear function. We write $C_j(f, h)$ for the continuous extensions of the commutators

$$\mathrm{ad}^j [T_f](T_h) \in \mathrm{L}(H_{\exp}(\mathbb{C}^n))$$

to $H^2(\mathbb{C}^n, \mu)$, (note that $f \in \mathrm{SP}_{\mathrm{Lip}}(\mathbb{C}^n)$ and Proposition 3.3).

Corollary 4.1 *Let $h \in L^\infty(\mathbb{C}^n)$. Assume that $\mathcal{D}(\widetilde{V}^z)$ is invariant under the multiplication operator M_h. Then $\mathcal{D}(V^z)$ is invariant under $C_j(f,h)$ for all $j \in \mathbb{N}$.*

Proof: According to (3.7) there is a finite index set $\mathcal{I}$ and $A_l, B_l \in \mathcal{A}_j(f)$ such that

$$\mathrm{ad}^j[T_f](T_h) = \sum_{l \in \mathcal{I}} P\, A_l\, M_h\, B_l\, P.$$

Due to our assumption on h and by Lemma 4.3 the assertion follows. □

Now, we can proof our main result on the smoothness of Toeplitz operators with respect to the Segal-Bargmann representation ρ of the Heisenberg group:

Theorem 4.1 *Let $h \in \mathcal{S}_s := \mathcal{S} \cap \overline{\mathcal{S}}$ where $\overline{\mathcal{S}} = \{\bar{h} : h \in \mathcal{S})$ and*

$$\mathcal{S} := \left\{ h \in L^\infty(\mathbb{C}^n) : \ s.\ t.\ \mathcal{D}(\widetilde{V}^z) \text{ is invariant under } M_h \text{ for all } z \in \mathbb{C}^n \right\}.$$

Then the symbol map into the Ψ^*-algebra Ψ^∞ given by:

$$\mathcal{S}_s \ni h \mapsto T_h \in \Psi^\infty$$

is well-defined and continuous if $\mathcal{S}_s$ carries the $L^\infty(\mathbb{C}^n)$-topology.

Proof: Using our notation in (4.2) and (4.3) we must show that $T_h \in \Psi^{\infty,z} = \Psi_\infty^{\mathcal{V}^z}$ for all complex directions $z \in \mathbb{C}^n$ and $\mathcal{V}^z := [iV^z]$:

$\mathcal{D}(V^z)$ is invariant under T_q for $q \in \{h, \bar{h}\} \subset \mathcal{S}_s$ and by Lemma 4.2 it follows that the commutators $A_1 := [iV^z, T_q]$ and $[T_{ig_z}, T_q]$ coincide on $\mathrm{H}_{\exp}(\mathbb{C}^n)$. Because iV^z is self-adjoint we can define $A_1^* := [T_{\bar{q}}, iV^z]$ and $W := \mathcal{D}(V^z)$ in Remark 4.1. The operator $[T_{ig_z}, T_q]$ has a bounded extension $C_1(ig_z, q)$ from $\mathrm{H}_{\exp}(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n, \mu)$. We conclude from Remark 4.1 that $C_1(ig_z, q)$ is an extension of A_1 from W to $H^2(\mathbb{C}^n, \mu)$ and $T_q \in \Psi_1^{\mathcal{V}^z}$. By induction we must prove for $j \in \mathbb{N}$:

(1) The domain of definition $\mathcal{D}(V^z)$ is invariant under $C_j(ig_z, q)$,

(2) The commutators $A_{j+1} := [iV^z, C_j(ig_z, q)]$ have the bounded extension $C_{j+1}(ig_z, q)$ from $\mathcal{D}(V^z)$ to $H^2(\mathbb{C}^n, \mu)$.

Assertion (1) is a direct consequence of Corollary 4.1 and (2) can be derived from Remark 4.1 with $A_{j+1}^* := [C_j(ig_z, q)^*, iV^z]$ on $W := \mathcal{D}(V^z)$ [3] and the fact that A_{j+1} has the continuous extension $C_{j+1}(ig_z, q)$ from $H_{\exp}(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n, \mu)$. The continuity of the symbols map follows from (2.3) together with the continuity of (3.6) in Proposition 3.3. □

[3] Note that by Corollary 4.1 and the identity $C_j(ig_z, q)^* = (-1)^j C_j(ig_z, \bar{q})$ the commutator A_{j+1}^* is well-defined on $\mathcal{D}(V^z)$.

5 Examples and Applications

Let $\mathcal{A}$ denote the subalgebra of $\mathcal{L}(\,L^2(\mathbb{C}^n,\mu)\,)$ of all multiplication operators with bounded symbols $h \in L^\infty(\mathbb{C}^n)$. For $z \in \mathbb{C}^n$ and with $\widetilde{\mathcal{V}^z} := [\,i\widetilde{V}^z\,]$ there is a scale of algebras arising by commutator methods:

$$\mathcal{A} \supset \Psi_1^{\widetilde{\mathcal{V}^z}} \supset \cdots \Psi_n^{\widetilde{\mathcal{V}^z}} \supset \Psi_{n+1}^{\widetilde{\mathcal{V}^z}} \supset \cdots \Psi_\infty^{\widetilde{\mathcal{V}^z}} = \bigcap_{n\in\mathbb{N}} \Psi_n^{\widetilde{\mathcal{V}^z}}. \tag{5.1}$$

In general, the inclusions above will be proper. As an immediate consequence of Theorem 4.1 it follows for the projected scale of vector spaces:

$$\mathcal{A}_P \supset \Psi_1^{\widetilde{\mathcal{V}^z}}{}_P = \cdots = \Psi_n^{\widetilde{\mathcal{V}^z}}{}_P = \Psi_{n+1}^{\widetilde{\mathcal{V}^z}}{}_P = \cdots = \Psi_\infty^{\widetilde{\mathcal{V}^z}}{}_P. \tag{5.2}$$

Here $\mathcal{A}_P \subset \mathcal{L}(\,H^2(\mathbb{C}^n,\mu)\,)$ is the space of Toeplitz operators with bounded measurable symbols. By passing from (5.1) to the scale (5.2) the underlying C^k-structure is lost.

We give an example of a class of bounded functions g such that $\mathcal{D}(\widetilde{V}^z)$ is an invariant subspace for M_g and $M_{\bar{g}}$ for all $z \in \mathbb{C}^n$.

Example 5.1 Denote by $C_c^\infty(\mathbb{C}^n)$ the space of compactly supported smooth functions. For $z = (\,z_1,\cdots,z_n\,) \in \mathbb{C}^n$ we write $z_j := x_j + iy_j$ and with $\alpha,\beta \in \mathbb{N}_0^n$:

$$z^{\alpha,\beta} := x^\alpha y^\beta, \qquad \partial^{\alpha,\beta} := \frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}.$$

Fix $h \in \mathcal{D}(\widetilde{V}^z)$ and $z \in \mathbb{C}^n$. For $g \in C_c^\infty(\mathbb{C}^n)$ (real valued) and $s \neq 0$ we write:

$$\frac{1}{s}[\,W_{sz} - I\,]\,M_g h = \frac{1}{s}\,[\,M_{g\circ\tau_{-tz}} - M_g\,]\,W_{sz}h + M_g\,\frac{1}{s}[\,W_{sz} - I\,]h. \tag{5.3}$$

The second term converges in $L^2(\mathbb{C}^n,\mu)$ as $s \to 0$. Consider the smooth and compactly supported function $dg(\,z,\cdot\,) := -\langle\,\mathrm{grad}\,g(\,\cdot\,), z\,\rangle_{\mathbb{R}^{2n}}$. Then:

$$\begin{aligned} C_{s,z} &:= \left\| \frac{1}{s}\left[M_{g\circ\tau_{-sz}} - M_g\right] - M_{dg(\,z,\cdot\,)} \right\| \\ &= \left\| \frac{1}{s}\left[g\circ\tau_{-sz} - g\right] - dg(\,z,\cdot\,) \right\|_\infty \leq \sum_{|\alpha|+|\beta|=2} \frac{|\,s\,|}{(\alpha+\beta)!}\,\|\,\partial^{\alpha,\beta}g\,\|_\infty\,|\,z^{\alpha,\beta}\,|. \end{aligned}$$

Hence $\lim_{s\to 0} C_{s,z} = 0$ and the right hand side of

$$\left\| \frac{1}{s}\left[M_{g\circ\tau_{-sz}} - M_g\right] W_{sz}h - M_{dg(\,z,\cdot)}h \right\| \leq C_{s,z}\,\|\,h\,\| + \|\,dg(\,z,\cdot\,)\,\|_\infty\,\|\,(\,W_{sz} - I\,)h\,\|$$

tends to 0 as $s \to 0$. It follows $gh \in \mathcal{D}(\,V^z\,)$. With our notation of Theorem 4.1 we conclude that $C_c^\infty(\mathbb{C}^n) \subset \mathcal{S}_s$. By the continuity of

$$L^\infty(\,\mathbb{C}^n\,) \subset \mathcal{S}_s \ni h \mapsto T_h \in \Psi^\infty$$

and the fact that $C_c^\infty(\mathbb{C}^n)$ is uniformly dense in the space $C_0(\mathbb{C}^n)$ of all continuous functions vanishing at infinity it follows that $\{\,T_h\,:\,h \in C_0(\mathbb{C}^n)\,\} \subset \Psi^\infty$.

In our second example we construct a compact operator $A \in \mathcal{B} := \mathcal{L}(H^2(\mathbb{C}, \mu))$ which is not contained in $\Psi^{1,z}$ for any $z \in \mathbb{C}$ (with our notation in (4.2)). As a consequence and using Example 5.1 A is not limit point of finite sums of finite products of Toeplitz operators with symbols in $C_0(\mathbb{C})$ and with respect to the Fréchet topology of $\Psi^{\infty,z}$. However, since A is compact it can be approximated by Toeplitz operators with smooth and compactly supported symbols in the topology of $\mathcal{B}$, c.f. [8].

Example 5.2 For $j \in \mathbb{N}_0$ let $P_j \in \mathcal{B}$ be the rank one projection onto $span\{ m_j := z^j \}$. With a sequence $a := (a_n)_{n\in\mathbb{N}}$ tending to zero consider the compact diagonal operator:

$$A := \sum_{j\in\mathbb{N}} a_j P_j \in \mathcal{B}.$$

With $z \in \mathbb{C}$, $|z| = 1$ and $g_z := 2i\,\mathrm{Im}\langle \cdot, z\rangle$ we compute $[T_{g_z}, A]m_j = [V^z, A]m_j$ explicitly for all $j \in \mathbb{N}$. By (4.4) one obtains that:

$$\begin{aligned}[T_{g_z}, A]\, m_j &= a_j\, T_{g_z}\, m_j - A\big[\, \bar{z}\, m_{j+1} - j\, z\, m_{j-1} \,\big] \\ &= a_j \left(\bar{z}\, m_{j+1} - j\, z\, m_{j-1} \right) - \left(a_{j+1}\, \bar{z}\, m_{j+1} - j\, a_{j-1}\, z\, m_{j-1} \right) \\ &= \left(a_j - a_{j+1} \right) \bar{z}\, m_{j+1} - j\, z \left(a_j - a_{j-1} \right) m_{j-1}.\end{aligned}$$

With $e_j := (j!)^{-\frac{1}{2}} z^j$ we have $\langle e_j, e_l \rangle_2 = \delta_{l,j}$ for all $j, l \in \mathbb{N}$. Hence it follows that

$$\| [T_{g_z}, A]\, e_j \|_2^2 = (j + 1)\, |\, a_j - a_{j+1}\, |^2 + j\, |\, a_j - a_{j-1}\, |^2 . \tag{5.4}$$

We choose a such that the right hand side of (5.4) tends to infinity for $j \to \infty$. This can be done by the choice of an oscillating sequence $a_j := (-1)^j j^{-\frac{1}{4}}$. Then it follows

$$(j+1)\, |\, a_j - a_{j+1}\, |^2 = (j+1)\, |\, j^{-\frac{1}{4}} + (j + 1)^{-\frac{1}{4}}\, |^2 \geq \sqrt{j+1}$$

and so the right hand side of (5.4) is unbounded for $j \to \infty$. Hence $[T_{g_z}, A]$ has no bounded extension to $H^2(\mathbb{C}^n, \mu)$ and $A \notin \Psi^{1,z}$ by Proposition 4.1.

Let $\beta : L^2(\mathbb{R}^n) \to H^2(\mathbb{C}^n, \mu)$ denote the *Bargmann isometrie*, c.f. [10]. Our results on Toeplitz operators on $H^2(\mathbb{C}^n, \mu)$ can be used in the analysis of a class of *Gabor-Daubechies windowed localization operators* $L_h := \beta^{-1} T_h \beta$ [4] on $L^2(\mathbb{R}^n)$ where $h \in L^\infty(\mathbb{C}^n)$, c.f. [9]. It was remarked in [14] the operator L_h can be considered as a pseudodifferential operator $W_{\sigma(h)}$ in Weyl quantization with *Weyl symbol* $\sigma(h)$ on $\mathbb{R}^{2n}$. Via the identification of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ the correspondence between h and $\sigma(h)$ can be expressed in terms of the heat equation on $\mathbb{R}^{2n}$. More precisely, $\sigma(h)$ is a solution with initial data h at a fixed time $t_0 > 0$. In the next example we describe how the operators introduced in the previous sections transform under β, c.f. [10].

[4]Here the window is a Hermite function on $\mathbb{R}^n$

Example 5.3 For $u \in L^2(\mathbb{R}^n)$ it is well-known that βu can be expressed by the integral:

$$[\beta u](z) = (2\pi)^{-\frac{n}{4}} \int_{\mathbb{R}^n} u(x) \exp\{ \langle x, z \rangle - \frac{1}{4}|x|^2 - \frac{1}{2}\langle z, \bar{z} \rangle \} dx.$$

Fix $a = p + iq \in \mathbb{C}^n$, then it can be checked that $W_a \in \mathcal{L}(H^2(\mathbb{C}^n, \mu))$ transform as:

$$B_a u := [\beta^{-1} W_a \beta](u) = u(\cdot - 2p) \exp\{ iq(p - \cdot) \}.$$

In particular, in the case $q = 0$ the unitary operator B_a is a usual shift in direction $2p$. For $j = 1, \cdots, n$ it is readily verified that T_{z_j} and $T_{\bar{z}_j}$ transform in the following way:

(i) $\beta^{-1} T_{z_j} \beta = \frac{1}{2} x_j - \partial_{x_j}$,

(ii) $\beta^{-1} T_{\bar{z}_j} \beta = \frac{1}{2} x_j + \partial_{x_j}$

From (i), (ii) and for $\alpha \in \mathbb{N}_0^n$ one obtains the identity:

$$\beta \, \partial_x^\alpha = (-1)^{|\alpha|} T_{i\mathrm{Im} z_1}^{\alpha_1} \cdots T_{i\mathrm{Im} z_n}^{\alpha_n} \beta =: (-1)^{|\alpha|} T_{i\mathrm{Im} z}^{\alpha} \beta.$$

Let $g \in \mathcal{D}(\mathbb{R}^n)$ be a test function and fix $f \in H_{\exp}(\mathbb{C}^n)$. It follows that:

$$\langle \beta^{-1} f, \partial_x^\alpha g \rangle_{L^2(\mathbb{R}^n)} = \langle f, \beta \, \partial_x^\alpha g \rangle = \langle \beta^{-1} T_{i\mathrm{Im} z_1}^{\alpha_1} \cdots T_{i\mathrm{Im} z_n}^{\alpha_n} f, g \rangle_{L^2(\mathbb{R}^n)}.$$

Here we have used the fact that $H_{\exp}(\mathbb{C}^n)$ is invariant under all unbounded Toeplitz operators $T_{i\mathrm{Im} z_j}$ which was proved in Proposition 3.1. It follows that:

$$\mathbf{D} := \beta^{-1}[H_{\exp}(\mathbb{C}^n)] \subset H^\infty(\mathbb{R}^n) = \bigcap_{k \in \mathbb{N}} H^k(\mathbb{R}^n)$$

where $H^s(\mathbb{R}^n)$ denotes the *k-th Sobolev space*. Hence, for $\alpha, \beta \in \mathbb{N}_0^n$ the restriction of (2.1) in Theorem 2.1 to $\mathbf{D}$:

$$\mathrm{ad}[-ix]^\alpha \mathrm{ad}[i\partial_x]^\beta (B) : \mathbf{D} \to \mathbf{D} \tag{5.5}$$

is well-defined for any $B \in L(\mathbf{D})$. With the choice $h \in L^\infty(\mathbb{C}^n)$ and $L_h := \beta^{-1} T_h \beta \in L(\mathbf{D})$ we obtain by conjugating (5.5) with β and using (i), (ii) above:

$$\mathrm{ad}[iT_{2\mathrm{Re} z}]^\alpha \, \mathrm{ad}[T_{\mathrm{Im} z}]^\beta (T_h) : H_{\exp}(\mathbb{C}^n) \to H_{\exp}(\mathbb{C}^n). \tag{5.6}$$

It follows by Proposition 3.3 that the operators in (5.6) have bounded extensions to $H^2(\mathbb{C}^n, \mu))$ and so (5.5) can be extended continuously to $L^2(\mathbb{R}^n)$. Hence we have proved a weaker version of the defining property (2.1) for $\Psi^0_{\rho,\delta}$ in Theorem 2.1.

Since the Gaussian measure μ is invariant under unitary transformations of $\mathbb{C}^n$, there is a natural group representation of U_n in $\mathcal{L}(H^2(\mathbb{C}^n, \mu))$ generating Ψ^*-algebras of smooth elements. As a final example we want to remark:

Example 5.4 Let $A \in \mathbb{R}^{n\times n}$ be self-adjoint and consider the unitary group:

$$\mathbb{R} \ni t \mapsto e^{itA} \in U_n.$$

The group of unitary composition operators $C_t f := f \circ e^{itA}$ on $H^2(\mathbb{C}^n, \mu)$ can be shown to be strongly continuous, cf. [3]. The restriction of the infinitesimal generator L_A of $(C_t)_{t\in\mathbb{R}}$ to $\mathbb{P}_a[\mathbb{C}^n]$ coincides with an (unbounded) Toeplitz operator. More precisely, it was shown in [3] that:

$$L_A p = \left[T_{\langle Az,z\rangle} - n \cdot \text{trace}(A) \right] p, \qquad p \in \mathbb{P}_a[\mathbb{C}^n].$$

Hence, in general the symbol of L_A regarded as a Toeplitz operator is a polynomial of degree 2, which is not globally lipschitz continuous on $\mathbb{C}^n$. Proposition 3.3 cannot be applied in this situation and the smoothness of a Toeplitz operator T_f with bounded symbols f with respect to $(C_t)_t$ requires further assumption on the symbol f. For a more detailed calculation we refer to [3].

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References

[1] V. BARGMANN, *On a Hilbert space of analytic functions and an associated integral transform,* Comm. Pure Appl. Math. 14, 187-214, (1961).

[2] W. BAUER, *Local Properties of the Segal-Bargmann projection and Ψ_0-algebras*, submitted for the Journal of the ISAAC conference, Catania (2005).

[3] W. BAUER, *Toeplitz operators on finite and infinite dimensional spaces with associated Ψ^*-Fréchet algebras*, Ph. D. thesis, Shaker Verlag Aachen, (2006).

[4] W. BAUER, *Hilbert-Schmidt Hankel operators on the Segal-Bargmann space,* Proc. Amer. Math. Soc. 132, pp. 2989-2998, (2004).

[5] W. BAUER, *Mean oscillation and Hankel operators on the Segal-Bargmann space,* Int. Equ. Oper. Theory 52, 1-15, (2005).

[6] R. BEALS , *Characterization of pseudodifferential operators and applications*, Duke Math. J. 44, 45-57, (1977), ibid. 46 (1979), p. 215.

[7] C.A. BERGER, L.A. COBURN, *Toeplitz operators on the Segal Bargmann space,* Trans. Amer. Math. Soc. 301, 813-829, (1994).

[8] C.A. BERGER, L.A. COBURN, *Heat flow and Berezin-Toeplitz estimates,* Amer. J. Math., 116 No. 3, 563-590, (1994).

[9] L.A. COBURN, *The Bargmann isometry and Gabor-Daubechies wavelet localization operators. Systems, approximation, singular integral operators, and related topics* (Bordeaux, 2000), 169-178, Oper. Theory Adv. Appl., 129, Birkhäuser, Basel, (2001).

[10] G.B. FOLLAND, *Harmonic analysis in phase space,* Annals of Mathematics Studies, Princeton University Press 122, (1989).

[11] B. GRAMSCH, *Relative Inversion in der Störungstheorie von Operatoren und Ψ-Algebren,* Math. Ann. 269, 27-71, (1984).

[12] B. GRAMSCH, J. UEBERBERG, K. WAGNER, *Spectral invariance and submultiplicativity for Fréchet algebras with applications to pseudo-differential operators and Ψ^*-quantization,* in: Operator Theory: Adv. and Appl. 57, 71-98. Birkhäuser Basel,(1992).

[13] B. GRAMSCH , K.G. KALB, *Pseudo-locality and hypoellipticity in operator algebras,* Semesterbericht Funkt. Anal., Universität Tübingen, 51-61, Sommersemester (1985).

[14] V. GUILLEMIN, *Toeplitz operators in n-dimensions,* Integral equations Operator Theory 7, pp. 145-205, (1984).

[15] R. LAUTER, *Pseudodifferential analysis on conformally compact spaces*, Memoirs of the AMS 163 No. 777, (2003).

[16] R. LAUTER, *Holomorphic functional calculus in several variables and Ψ^*-algebras of totally characteristic operators on manifolds with boundary,* Dissertation, Shaker-Verlag (1996).

[17] R. LAUTER, R. MONTHUBERT, V. NISTOR, *Spectral invariance for certain algebras of pseudodifferential operators*, Jour. of the Inst. of Math. Jussieu 4, No. 3, 405-442, (2005).

[18] E. SCHROHE, *Fréchet algebra techniques for boundary value problems: Fredholm criteria and functional calculus via spectral invariance,* Math. Nachr. 199, 145-185, (1999).

[19] E. SCHROHE, *Boundedness and spectral invariance for standard pseudodifferential operators on anisotropically weighted L^p-Sobolev spaces.* Int. Equations and Operator Theory 13, 271-284, (1990).

[20] J. UEBERBERG, *Zur Spektralinvarianz von Algebren von Pseudodifferentialoperatoren in der L^p-Theorie.* Manuscripta Math. 61 No. 4, 459-475, (1988).

[21] L. WAELBROECK, *Le calcul symbolique dans les algèbres commutatives,* J. Math. Pures Appl., 33 No. 9, 147-186, (1954).