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ON BOUNDED ANALYTIC FUNCTIONS ON TWO-SHEETED COVERING SURFACES

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In this note, we pose some problems which is related to the algebras of bounded analytic functions on two-sheeted covering surfaces $(\widetilde{R}, R, \pi)$, where the base domain R is a *Zalcman domain* (or an L-domain in the terminology of [5]). In [5], L. Zalcman showed some theorems related to the algebra $H^\infty(R)$ of bounded analytic functions on a domain R of infinite connectivity. Especially, the *distinguished homomorphism* is of our interest. We summarize Zalcman's results in §1.

For the covering surface $(\widetilde{R}, R, \pi)$, the *point separation problem* was studied in [2] and [3]. We review this problem in §2.

1 Zalcman's results

Let Δ be the open unit disc and $\Delta_0 = \{0 < |z| < 1\}$ the punctured unit disc. Let $\{c_n\}$ and $\{r_n\}$ be sequences satisfying:

$$\begin{cases} 1 > c_1 > c_2 > \cdots > 0 \ , & \lim_{n\to\infty} c_n = 0 \ , \\ 1 > r_1 > r_2 > \cdots > 0 \ , & \lim_{n\to\infty} r_n = 0 \ , \\ c_{n+1} + r_{n+1} < c_n - r_n \ , & c_1 + r_1 < 1 \ . \end{cases} \tag{1}$$

These coditions simply say that closed discs $\{\bar{\Delta}_n\}$ are contained in Δ_0, are mutually disjoint, and accumulate only at the origin. Consider the domain

$$R = \Delta_0 \setminus \bigcup_{n=1}^{\infty} \bar{\Delta}(c_n, r_n) \ , \tag{2}$$

which is a simplest example of bounded infinitely connected domains in the complex plane $\mathbb{C}$. We call a domain R of the form (2) a Zalcman domain.

Each $f \in H^\infty(R)$ has nontangential boundary values at almost every point of $\Gamma = \partial R$. And the Cauchy integral formula holds;

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta \ , \quad z \in R \ .$$

Let $\mathcal{M} = \mathcal{M}(R)$ be the maximal ideal space of $H^\infty(R)$, the set of all non-zero multiplicative linear functionals on $H^\infty(R)$. The topology of $\mathcal{M}$ is the weak-* topology which it inherits from $H^\infty(R)^*$. With this topology, we can regard the functions in $H^\infty(R)$ as continuous functions on $\mathcal{M}$ by setting $f(\varphi) = \varphi(f)$ ($\varphi \in \mathcal{M}, f \in H^\infty(R)$). In particular, the coordinate function z can be regarded as a continuous function on $\mathcal{M}$. And we have $z(\mathcal{M}) = \bar{R}$. The set $\mathcal{M}_\zeta = z^{-1}(\{\zeta\})$ is called the fiber over ζ ($\zeta \in \bar{R}$).

For $\zeta \in R$, $\mathcal{M}_\zeta = \{\varphi_\zeta\}$, where φ_ζ is the point evaluation homomorphism ($\varphi_\zeta(f) = f(\zeta)$). And, for $\zeta \in \Gamma \setminus \{0\}$, $\mathcal{M}_\zeta$ is homeomorphic to $\mathcal{M}_1(\Delta)$. So, we are interested in the fiber $\mathcal{M}_0$.

Suppose that the sequeces $\{c_n\}$ and $\{r_n\}$ satisfy the condition

$$\sum_{n=1}^{\infty} \frac{r_n}{c_n} < \infty \tag{3}$$

in addition to (1). Then $d\zeta/\zeta$ is a finite measure on Γ. By Lebesgue's theorem, we have that $\lim_{x \nearrow 0} f(x)$ exists for all $f \in H^\infty(R)$. Set $\varphi_0(f) = \lim_{x \nearrow 0} f(x)$. Then we have

(i) $\varphi_0 \in \mathcal{M}_0$,

(ii) φ_0 does not lie in the Shilov boundary of $H^\infty(R)$,

(iii) φ_0 lies in the same Gleason part as R.

The homomorphism φ_0 is called the *distinguished homomorphism.*

2 Covering surfaces

Let $(\widetilde{\Delta}_0, \Delta_0, \pi)$ be the unlimited two-sheeted covering surface whose branch points are those over $\{c_n\}$ (Fig. 1). In 1949, Myrberg pointed out that $H^\infty(\widetilde{\Delta}_0) = H^\infty(\Delta_0) \circ \pi$. This means that for any point $z \in \Delta_0 \setminus \{c_n\}$, the points of the fiber $\pi^{-1}(z) = \{z_+, z_-\}$ can not be separated by $H^\infty(\widetilde{\Delta}_0)$.

Myrberg's proof goes as follows. Let $F \in H^\infty(\widetilde{\Delta}_0)$, and consider the function f on Δ_0 defined by $f(z) = (F(z_+) - F(z_-))^2$. Then $f \in H^\infty(\Delta_0)$ and, by Riemann's theorem, $f \in H^\infty(\Delta)$. Since $f(c_n) = 0$ and $c_n \to 0$, we have $f \equiv 0$.

Restricting the base domain Δ_0 of the covering surface to R, and setting $\widetilde{R} = \pi^{-1}(R)$, we obtain the two-sheeted smooth covering surface $(\widetilde{R}, R, \pi)$ (Fig. 2). In spite of complete lack of branch points, it is shown in [2] and [3] that non-separating phenomenon may occur for $(\widetilde{R}, R, \pi)$ depending on $\{c_n\}$ and $\{r_n\}$. Roughly speaking,

(i) if $r_n \to 0$ "rapidly", then $H^\infty(\widetilde{R}) = H^\infty(R) \circ \pi$,

(ii) if $r_n \to 0$ "slowly", then $H^\infty(\widetilde{R}) \supsetneqq H^\infty(R) \circ \pi$.

(Unfortunately, the necessary and sufficient condition for $H^\infty(\widetilde{R}) = H^\infty(R) \circ \pi$ is not known.)

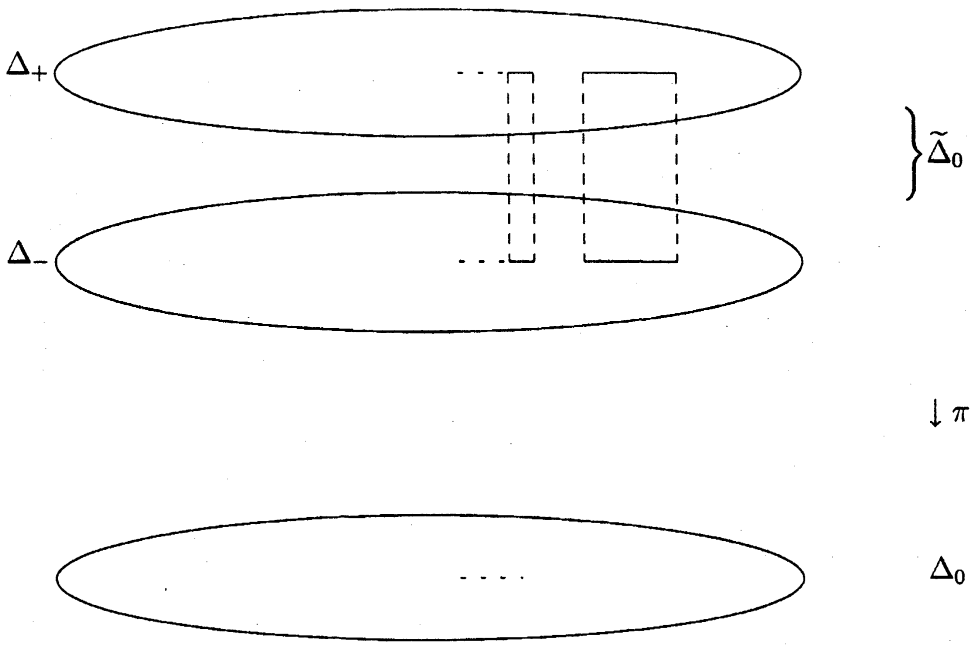

Figure 1: $(\widetilde{\Delta}_0, \Delta_0, \pi)$

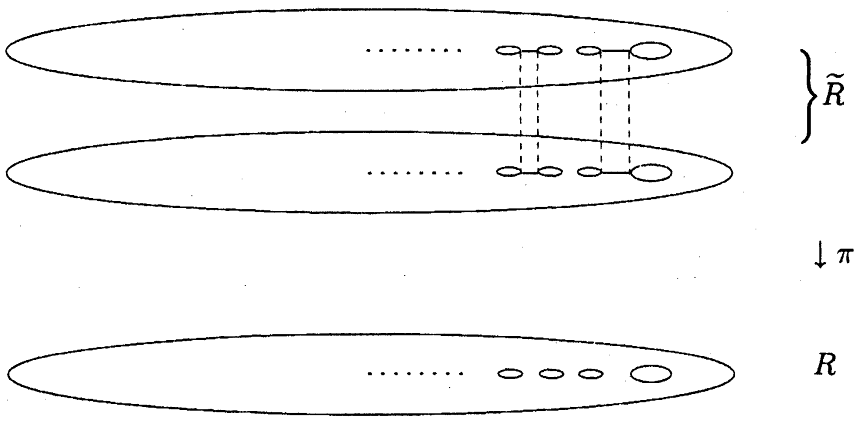

Figure 2: $(\widetilde{R}, R, \pi)$

3 Problems

The covering surface $(\widetilde{R}, R, \pi)$ induces the covering space $(\widetilde{\mathcal{M}}, \mathcal{M}, \tau)$, where $\widetilde{\mathcal{M}}$ is the maximal ideal space of $H^\infty(\widetilde{R})$ and the map τ is defined by

$$\tau(\tilde{\varphi})(f) = \tilde{\varphi}(f \circ \pi) \ , \quad \tilde{\varphi} \in \widetilde{\mathcal{M}}, \ f \in H^\infty \ .$$

Let $\iota : R \to \mathcal{M}$ and $\tilde{\iota} : \widetilde{R} \to \widetilde{\mathcal{M}}$ be natural maps. Then we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{R} & \xrightarrow{\tilde{\iota}} & \widetilde{\mathcal{M}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tau} \\ R & \xrightarrow[\iota]{} & \mathcal{M} \end{array}$$

By Nakai's theorem ([4]), we see that the map τ is surjective and the fiber $\tau^{-1}(\varphi)$ over any point $\varphi \in \mathcal{M}$ consists of at most two points, i.e., the number $\#(\tau^{-1}(\varphi))$ of points of the fiber $\tau^{-1}(\varphi)$ is 1 or 2 for all $\varphi \in \mathcal{M}$.

Consider the problem to determine $\#(\tau^{-1}(\varphi))$. The following partial answer is trivial.

Proposition. (i) *If* $H^\infty(\widetilde{R}) = H^\infty(R) \circ \pi$, *then* $\#(\tau^{-1}(\varphi)) = 1$ *for all* $\varphi \in \mathcal{M}$.
(ii) *If* $H^\infty(\widetilde{R}) \supsetneqq H^\infty(R) \circ \pi$, *then* $\#(\tau^{-1}(\varphi_z)) = 2$ *for all* $z \in R$

Now we pose some problems related to the fiber over the distinguished homomorphism.

3.1. Suppose that $H^\infty(\widetilde{R}) \supsetneqq H^\infty(R) \circ \pi$. Determine $\#(\tau^{-1}(\varphi_0))$

The distinguished homomorphism was defined by $\varphi_0(f) = \lim_{x \nearrow} f(x)$ for $f \in H^\infty(R)$. In view of this, the following problem is posed.

3.2. Does $\lim_{x \nearrow 0} F(x_+)$ (or $\lim_{x \nearrow 0} F(x_-)$) exist for all $F \in H^\infty(\widetilde{R})$?

Note that $\lim_{x \nearrow 0}(F(x_+) + F(x_-))$ exists for all $F \in H^\infty(\widetilde{R})$ because $F(z_+) + F(z_-) \in H^\infty(R)$. Therefore, the existence of one of the limits in the above problem implies the existence of the other.

Set $J = [-1/2, 0)$. Then Zalcman's result can be restated as $\bar{J} = J \cup \{\varphi_0\}$ in $\mathcal{M}$. Related to this statement, the following problem is posed.

3.3 Let $\pi^{-1}(J) = J^+ \cup J^-$. ($J^+ = \pi^{-1}(J) \cap \Delta_+$, $J^- = \pi^{-1}(J) \cap \Delta_-$.) Determine the closures $\bar{J}^+, \bar{J}^-$ and $\overline{J^+ \cup J^-}$ in $\widetilde{\mathcal{M}}$.

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