数理解析研究所講究録
1521 巻 2006 年 21-26

Galois covers of degree p and semi-stable reduction of curves in mixed characteristics

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The following is a survey on the main results that I discussed in my talk in the annual number theory meeting at RIMS, Kyoto (2005). The proof of these results will be published elsewhere ([9]).

We study the semi-stable reduction of Galois covers of degree p above curves over a complete discrete valuation ring of mixed characteristics $(0,p)$.

Let $p>0$ be a prime integer. Let R be a complete discrete valuation ring, with fraction field K of characteristic 0, and residue field k of characteristic p, which we assume to be algebraically closed. Let $\mathcal{X}$ be a proper and smooth R-curve, with generic fibre $\mathcal{X}_K := \mathcal{X} \times_R K$, and special fibre $\mathcal{X}_k := \mathcal{X} \times_R k$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a finite Galois cover with Galois group G, and with $\mathcal{Y}$ normal. Let $\mathcal{Y}_K := \mathcal{Y} \times_R K$ be the generic fibre of $\mathcal{Y}$, and let $\mathcal{Y}_k := \mathcal{Y} \times_R k$ be its special fibre, which we assume to be reduced (this condition is always satisfied after a finite extension of R). If the cardinality of G is prime to p, and if the cover $f_K : \mathcal{Y}_K \to \mathcal{X}_K$ between generic fibres is étale, then it follows from the purity theorem that $\mathcal{Y}$ is smooth (cf. [10]). If the cardinality of G is divisible by p then $\mathcal{Y}$ is not smooth in general (even if the cover f_K between generic fibres is étale). However, it follows from the theorem of semi-stable reduction of curves (cf. [2]) that $\mathcal{Y}$ admits potentially semi-stable reduction, i.e. there exists (possibly after extending R) a proper and birational morphism $\tilde{\mathcal{Y}} \to \mathcal{Y}$, where $\tilde{\mathcal{Y}}$ is a semi-stable R-curve. Moreover, there exists such a semi-stable model $\tilde{\mathcal{Y}}$ which is minimal. We are interested in the study of the geometry (of the special fibre) of a minimal semi-stable model $\tilde{\mathcal{Y}}$, under the assumption that p divides the cardinality of G. The first result in this direction is the following, which is due to Raynaud (cf. [6]):

Theorem: (Raynaud) *Assume that G is a p-group, and that the cover f is étale above the generic fibre $\mathcal{X}_K$ of $\mathcal{X}$. Then the configuration of the special fibre $\tilde{\mathcal{Y}}_k := \tilde{\mathcal{Y}} \times_R k$, of a minimal semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$, is tree-like.*

Though this result is important, it is still rather "qualitative" and doesn't provide much information, say on the type of the "new components" that appear in $\tilde{\mathcal{Y}}_k$. Also the assumption that the cover f is étale above the generic fibre $\mathcal{X}_K$ of $\mathcal{X}$ plays a crucial role in the proof. In fact the above result is not true if this condition is not satisfied. Of course one expects the geometry (of the special fibre) of a minimal semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ to depend on the structure of the group G. We are interested in the case where $G \simeq \mathbb{Z}/p\mathbb{Z}$, and with no restriction on the ramification in the morphism f.

Our approach to study this case is based on (known) results on the degeneration of μ_p-torsors from 0 to positive characteristic (cf. e.g. [7]) (resp. the computation of vanishing cycles in a Galois cover of degree p between formal germs of R-curves, which was established by the author in [8]). As a consequence of these results we can determine the singular points of $\mathcal{Y}_k$, and we can compute the arithmetic genus of these singularities. More precisely, suppose that some branched points in the morphism $f_K : \mathcal{Y}_K \to \mathcal{X}_K$ specialize in the set $B_k \subset \mathcal{X}_k$, and let $U'_k := \mathcal{X}_k - B_k$. Then f induces (by restriction to U'_k) a finite cover $f'_k : V'_k \to U'_k$, which has the structure of a torsor under a finite and flat k-group scheme of rank p. Suppose for example that this torsor is radicial (this is the most difficult case to treat), and let ω be the associated differential form (cf. [7], 1). Let Z_k be the set of zeros of ω, and let $\mathrm{Crit}(f) := Z_k \cup B_k$. If y is a singular point of $\mathcal{Y}_k$, then $f(y) \in \mathrm{Crit}(f)$. Further, let $m_y := \mathrm{ord}_{f(y)}(\omega)$. Then the arithmetic genus of y (cf. [18], 3.1) equals $(r_y + m_y)(p-1)/2$, where r_y is the number of branched points of f in the generic fibre $\mathcal{X}_K$ which specialize in $f(y)$ ($r_y = 0$, if $f(y) \in \mathrm{Crit}(f) - B_k$).

In order to understand the geometry of $\tilde{\mathcal{Y}}$ one needs to understand the fibre of a singular point y of $\mathcal{Y}_k$ in the minimal semi-stable model $\tilde{\mathcal{Y}}$. This indeed is a local problem. We consider a finite Galois p-cover $f_x : \mathcal{Y}_y \to \mathcal{X}_x$ between formal germs of R-curves at a closed point y (resp. x), where x is a smooth point (i.e. $\mathcal{X}_x \simeq \mathrm{Spf}\, R[[T]]$) and we study the geometry of a minimal semi-stable model $\tilde{\mathcal{Y}}_y$ of $\mathcal{Y}_y$. We exhibit what we call "simple degeneration data of rank p", comprising a tree Γ of k-projective lines which is endowed with some data of geometric and combinatorial nature, and which completely describe the geometry of $\tilde{\mathcal{Y}}_y$. These degeneration data are defined as follows:

Definition 1. **K'-simple degeneration data** Deg(x) of type $(r,(n,m))$, and rank p, where K' is a finite extension of K, consist of the following:

Deg.1. $r \geq 0$ is an integer, m is an integer prime to p such that $r - m - 1 \geq 0$, and $0 \leq n \leq v_{K'}(\lambda)$ is an integer. Further, G_k is a commutative finite and flat k-group scheme of rank p which is either étale if $n = v_K(\lambda)$, radicial of type α_p if $0 < n < v_K(\lambda)$, or radicial of type μ_p if $n = 0$.

Deg.2. $\Gamma := X_k$ is an oriented tree of k-projective lines, with set of vertices $\mathrm{Vert}(\Gamma) := \{X_i\}_{i \in I}$, which is endowed with an origin vertex X_{i_0}, and a marked point $x := x_{i_0,j_0}$ on X_{i_0}. We denote by $\{z_{i,j}\}_{j \in D_i}$ the set of double points, or (non oriented) edges of Γ, which are supported by X_i. Further, we assume that the orientation of Γ is in the direction going from X_{i_0} towards its ends.

Deg.3. For each vertex X_i of Γ there is a set (which may be empty) of smooth marked points $\{x_{i,j}\}_{j\in S_i}$.

Deg.4. For each $i \in I$, there is a torsor $f_i : V_i \to U_i := X_i - \{\{x_{i,j}\}_{j\in S_i} \cup \{z_{i,j}\}_{j\in D_i}\}$ under a finite commutative and flat k-group scheme $G_{i,k}$ of rank p, which is either étale or radicial of type α_p or μ_p, with V_i *smooth.* Moreover, for each $i \in I$ there is an integer $0 \le n_i \le v_{K'}(\lambda)$ which equals $v_{K'}(\lambda)$ if and only if f_i is étale, and equals 0 if and only if $G_{i,k} \simeq \mu_p$. If S_i is non empty we assume that $G_{i,k} \simeq \mu_p$.

Deg.5. For each $i \in I$, and $j \in S_i$, there is a pair of integers $(m_{i,j}, h_{i,j})$, where $m_{i,j}$ (resp. $h_{i,j}$) is the conductor (resp. the residue) of the torsor f_i at the point $x_{i,j}$ (cf. [7] I). Further, we assume that $m_{i_0,j_0} = -m$, and $m_{i,j} = 0$ otherwise, and $\sum_{j\in S_i} h_{i,j} = 0$.

Deg.6. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$, there is an integer $m_{i,j}$ (resp. $m_{i',j'}$) prime to p, where $m_{i,j}$ (resp. $m_{i',j'}$) is the conductor of the torsor f_i (resp. $f_{i'}$) at the point $z_{i,j}$ (resp. $z_{i',j'}$) (cf. [9] 1.3 and 1.5). These data must satisfy $m_{i,j} + m_{i',j'} = 0$.

Deg.7. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of Γ, with origin vertex X_i, there is an integer $e_{i,j} = pt_{i,j}$ divisible by p such that, with the same notation as above, we have $n_i - n_{i'} = m_{i,j} t_{i,j}$. Moreover, associated with x is an integer $e = pt$ such that $n - n_{i_0} = mt$.

Deg.8. Let I_{et} be the subset of I consisting of those i for which $G_{i,k}$ is étale. Then the following equality should hold: $(r - m - 1)(p-1)/2 = \sum_{i\in I_{\text{et}}}(-2 + \sum_{j\in S_i}(m_{i,j}+1) + \sum_{j\in D_i}(m_{i,j}+1))(p-1)/2$. The integer $g := (r + m - 1)(p-1)/2$ is called the *genus* of the degeneration data Deg(x).

Note that if K'' is a finite extension of K', then K'-simple degeneration data $\mathrm{Deg}(x)$ can be naturally considered as K''-degeneration data, by multiplying all integers n, n_i, and $e_{i,j}$, by the ramification index of K'' over K'.

Let $\mathbf{Deg_p}$ be the set of "isomorphism classes" of such data. Then we construct a canonical specialization map $\mathrm{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}/p\mathbb{Z}) \to \mathbf{Deg_p}$, where L is the function field of the geometric fibre $\bar{\mathcal{X}}_x := \mathcal{X}_x \times_R \bar{R}$ of $\mathcal{X}_x$, and $\bar{R}$ is the integral closure of R in an algebraic closure of K. This map is constructed in such a way that given a Galois cover of degree p $f_x : \mathcal{Y}_y \to \mathcal{X}_x$ between formal germs of R-curves, as above, the image of (the isomorphism class of) this cover via Sp describes completely the geometry of semi-stable reduction of $\mathcal{Y}_y$. Our first main result is the following realization result for simple degeneration data.

Theorem 1. *The specialization map* $\mathrm{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}/p\mathbb{Z}) \to \mathbf{Deg_p}$ *is surjective.*

In other words we are able to reconstruct Galois covers of degree p above open p-adic discs, starting from (the) degeneration data which describe the semi-stable reduction of such a cover. The proof of this result relies on the technique of formal patching initiated by Harbater and Raynaud (cf. [8], 1). The above theorem was proved in [3] under the assumption that $\mathcal{Y}_y$ is *smooth*, and where $\tilde{\mathcal{Y}}_y$ is the minimal semi-stable model in which the ramified points (on the generic fibre) specialize in smooth distinct points.

Let's return to the above global situation of a Galois p-cover $f : \mathcal{Y} \to \mathcal{X}$. The above local results allow us to associate with each critical point $x_i = f(y_i) \in \text{Crit}(f)$, simple degeneration data $\text{Deg}(x_i)$ of rank p which describe the preimage of the singular point y_i in $\tilde{\mathcal{Y}}_k$. These simple degeneration data, plus the data given by the torsor $f'_k : V'_k \to U'_k$, lead to the definition of "smooth degeneration data" $\text{Deg}(\mathcal{X}_k)$ of rank p, which are associated with the special fibre $\mathcal{X}_k$ of $\mathcal{X}$, and which describe the geometry of the semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$. These are defined as follows:

Definition 2. Smooth K'**-degeneration data** Deg(X), of rank p, consist of the following data:

Deg.1. K' is a finite extension of K. X is a proper and smooth k-curve, endowed with a finite set B_k of closed (mutually distinct) marked points. Let $U := X - B_k$.

Deg.2. $\bar{f} : V \to U$ is a torsor under a finite and flat k-group scheme G_k of rank p, and $0 \leq n \leq v_{K'}(\lambda)$ is an integer which equals 0 (resp. equals $v_{K'}(\lambda)$) if and only if G_k is of multiplicative type (resp. if and only if G_k is étale).

Deg.3. Let $\text{Crit}(\bar{f}) = \{x_i\}_{i \in I}$ be the set B_k if $\bar{f}$ is étale (resp. the set $\text{Crit}(\bar{f}) = B_k \cup Z_k$ if $\bar{f}$ is radicial, where Z_k is the set of zeros of the corresponding differential form). For each $i \in I$, let m_i be the conductor of the above torsor $\bar{f}$ at the point x_i (cf. [7], I). We assume that we are given K'-simple degeneration data $\text{Deg}(x_i)$ of type $(r_i, (n, m_i))$.

Let $\mathbf{DEG_p}(\mathcal{X}_\mathbf{k})$ be the set of isomorphism classes of smooth degeneration data of rank p associated with $\mathcal{X}_k$. We construct a canonical "specialization" map $\text{Sp} : H^1_{\text{et}}(\text{Spec}\, L, \mathbb{Z}/p\mathbb{Z}) \to \mathbf{DEG_p}(\mathcal{X}_\mathbf{k})$, where L is the function field of the geometric fibre $\bar{\mathcal{X}} := \mathcal{X} \times_R \bar{R}$ of $\mathcal{X}$, and $\bar{R}$ is the integral closure of R in an algebraic closure of K. This map is constructed in such a way that given a Galois cover of degree p $f : \mathcal{Y} \to \mathcal{X}$ between proper R-curves, as above, the image of (the class of) this cover via Sp describes completely the geometry of semi-stable reduction of $\mathcal{Y}_y$.

Our second main result is the realization of smooth degeneration data associated with $\mathcal{X}_k$, if necessary after modifying the R-curve $\mathcal{X}$ into another R-curve $\mathcal{X}'$ with special fibre $\mathcal{X}'_k$ isomorphic to $\mathcal{X}_k$. More precisely, we have the following.

Theorem 2. *Let* $\text{Deg}(\mathcal{X}_k) \in \mathbf{DEG_p}(\mathcal{X}_\mathbf{k})$ *be smooth degeneration data of rank p, associated with* $\mathcal{X}_k$. *Then there exists a smooth and proper R-curve* $\mathcal{X}'$, *with special fibre isomorphic to* $\mathcal{X}_k$, *such that* $\text{Deg}(\mathcal{X}_k)$ *is in the image of the specialization map* Sp :

$H^1_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}/p\mathbb{Z}) \to \mathbf{DEG}_{\mathrm{p}}(\mathcal{X}_{\mathrm{k}})$, *where* L *is the function field of the geometric fibre* $\mathcal{X}' \times_R \bar{R}$ *of* $\mathcal{X}'$, *and* $\bar{R}$ *is the integral closure of* R *in an algebraic closure of* K.

As another application of our techniques, we prove the following result of lifting of torsors under finite and flat group schemes of rank p (this result is also proved in [1] using different methods).

Theorem 3. *Let* X *be a smooth and proper* k*-curve, and let* $f : Y \to X$ *be a torsor under a finite and flat* k*-group scheme* G_k *of rank* p*. Then there exists a smooth and proper* R*-curve* $\mathcal{X}$*, with special fibre isomorphic to* X*, and a torsor* $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ *under an* R*-group scheme* G_R*, which is commutative finite and flat of rank* p*, such that the torsor induced on the level of special fibres* $\tilde{f}_k : \mathcal{Y}_k \to \mathcal{X}_k$ *is isomorphic to the torsor* f*. In other words the torsor* $\tilde{f}$ *lifts* f.

In our study we do not adress questions of "effectiveness". Namely is it possible for a given Galois p-cover $f : \mathcal{Y} \to \mathcal{X}$ as above (say given by explicit equations), to determine explicitly the smooth degeneration data which describes the geometry of a semi-stable model of $\mathcal{Y}$? This question is studied in [4] and [5], for the case where $\mathcal{X}$ is the R-projective line, and under some (restrictive) conditions on the branch locus. However, it is not clear whether the methods used in [4] and [5] can be used to treat this question in general.

It is important to be able to extend the results of this paper to the more general case where the Galois group $G \simeq \mathbb{Z}/p^n\mathbb{Z}$ is cyclic of order p^n. However, what is really missing is to describe the way μ_{p^n}-torsors degenerate from characteristic 0 to characteristic p. Examples in the case $n = 2$ already illustrate the complexity of the situation, by comparison with the case $n = 1$

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