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Vanishing cycles over general bases after P. Deligne, O. Gabber, G. Laumon and F. Orgogozo (*)

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1. Historical sketch

1.1. Almost forty years have elapsed since Milnor introduced, for a germ of analytic map $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, having an isolated singularity at the origin, what is now called the *Milnor ball* B and the *Milnor fibration* $t \mapsto V_t = f^{-1}(t) \cap B$ of f at 0. About at the same time, Grothendieck defined the functors $R\Psi$ and $R\Phi$, globalizing Milnor's constructions and proving in passing his conjecture that the eigenvalues of the monodromy acting on $H^*(V_t, \mathbb{Z})$ are roots of unity. He also constructed and studied analogues in étale cohomology. His theory, written up by Deligne in [SGA 7], has had immense applications.

Before turning to the main topic of this talk, let me briefly recall the definition of the functors $R\Psi$ and $R\Phi$ in the étale context. Let $f : X \to S$ be a morphism of schemes, where S is a henselian trait, with closed point s, generic point η, and generic geometric point $\overline{\eta}$ defining $\overline{s}$ over s. Fix some coefficients ring Λ, which for simplicity we choose to be $\mathbb{Z}/\ell^{\nu}$ (ℓ a prime invertible on S, $\nu \geq 1$) (there are variants with $\mathbb{Z}_{\ell}$, $\mathbb{Q}_{\ell}$, $\overline{\mathbb{Q}_{\ell}}$). For $F \in D^{+}(X_{\eta}, \Lambda)$, the complex of *nearby cycles* $R\Psi F$ (of (F,f)) is a complex on the geometric special fiber $X_{\overline{s}}$ defined by

$$R\Psi F = \overline{i}^{*} R\overline{j}_{*} F, \tag{1.1.1}$$

where $\overline{i} : X_{\overline{s}} \to X_{(\overline{s})}$ and $\overline{j} : X_{\overline{\eta}} \to X_{(\overline{s})}$ are deduced by pull-back from the maps $\overline{i} : \overline{s} \to S_{(\overline{s})}$, $\overline{j} : \overline{\eta} \to S_{(\overline{s})}$, with $S_{(\overline{s})}$ the strict localization of S at $\overline{s}$. If $\overline{x}$ is a geometric point of X over $\overline{s}$, and $X_{(\overline{x})}$ denotes the strict localization of X at $\overline{x}$ (the analogue of the Milnor ball B), the fibre $(X_{(\overline{x})})_{\overline{\eta}}$ of $X_{(\overline{x})} \to S_{(\overline{s})}$ at $\overline{\eta}$ plays the role of a Milnor fibre, and

$$(R\Psi F)_{\overline{x}} = R\Gamma((X_{(\overline{x})})_{\overline{\eta}}, F). \tag{1.1.2}$$

The complex $R\Psi F$ comes equipped with more structure : if G is the Galois group of $\overline{\eta}$ over η, $R\Psi F$ underlies a complex of sheaves of Λ-modules endowed with a continuous action of G compatible with its action on $X_{\overline{s}}$ (an object of $D^{+}(X_s \times_s \eta, \Lambda)$ in Deligne's notations in [SGA 7 XIII]). This action plays the role of the monodromy action of π_1 of the punctured disc on the cohomology of a Milnor fiber V_t.

For $F \in D^{+}(X, \Lambda)$, the adjunction map defines an equivariant triangle

$$F|X_{\overline{s}} \to R\Psi F \to R\Phi F \to, \tag{1.1.3}$$

where $R\Phi F$ is by definition the complex of *vanishing cycles* of (F,f). This complex measures the non local acyclicity of (F,f) : by definition, the stalk of $R\Phi F$ at $\overline{x}$ vanishes

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if and only if (F,f) is locally acyclic at $\overline{x}$; this is the case, for example, if both F and f are smooth at $\overline{x}$.

The functor $R\Psi$ enjoys the following basic properties :

(a) Let $f : X \to S$, $g : Y \to S$, $h : X \to Y$ be S-morphisms, with $gh = f$.

(i) If h is *proper*, then, for $F \in D^+(X_\eta, \Lambda)$,

$$R\Psi(R(h_\eta)_*F) \xrightarrow{\sim} R(h_{\overline{s}})_*R\Psi F. \tag{1.1.4}$$

This is an immediate consequence of the proper base change theorem [SGA 4 XI].

(ii) If h is *locally acyclic* (e. g. smooth), then, for $F \in D^+(Y_\eta, \Lambda)$,

$$h_s^* R\Psi F \xrightarrow{\sim} R\Psi(h_\eta^* F). \tag{1.1.5}$$

(b) Suppose f of finite type. Then $R\Psi$ preserves constructibility, i. e. sends $D_c^+(X_\eta, \Lambda)$ (resp. $D_c^b(X_\eta, \Lambda)$) into $D_c^+(X_{\overline{s}}, \Lambda)$ (resp. $D_c^b(X_{\overline{s}}, \Lambda)$), where D_c^* denotes the full subcategory consisting of complexes with constructible cohomology, and the formation of $R\Psi F$ commutes with any dominant change of traits $S' \to S$ ([SGA 4 1/2, Th. finitude]).

1.2. A natural question arises : how do complexes of nearby cycles vary in families ? The answer is : in general, not so well, as is shown by the following elementary example, discussed by Deligne [D]. Let Y be the affine plane $\mathbb{A}^2_{\mathbb{C}}$, $f : X \to Y$ the blow-up of the origin in Y, $E = f^{-1}(0) = \mathbb{P}^1_{\mathbb{C}}$ the exceptional divisor. Lines in Y passing through 0 are parametrized by E ; for t in E, let D_t be the corresponding line. Fix $t \in E$. Let U be an open neighborhood of $t \in E$ in X, sent by f into some open neighborhood V of the origin in Y. Then $R(f|U)_*\mathbb{Z}|V = (f|U)_*\mathbb{Z}|V$ is $\mathbb{Z}$ on some sector around $D_t \cap V$ and zero elsewhere. These sectors shrink as U, V do. Therefore the cohomologies of the (generalized) Milnor fibers around the origin in Y do not form a nice family : the inductive system $R(f|U)_*\mathbb{Z}|V$ around 0 in Y, is not essentially constant (and, in addition, each member is, in general, not analytically constructible, because of the shape of these sectors).

The morphism f above is not flat, but other examples with f flat and with the same bad properties were given by Deligne [D] and Lê [Lê]. Lê (*loc. cit*) showed that morphisms "without blow-ups", i. e. admitting suitable Thom-Whitney stratifications, had a good theory of "punctual" nearby cycles, i. e. the inductive systems considered above were constructible and essentially constant. Deligne [D] asked whether such good properties could be obtained after a suitable modification of the base (for instance, in the example discussed above, the modification by f itself works). This was proven by Sabbah [S], at least for f proper.

It was not at all clear how to proceed in the étale set-up. While trying to prove the product formula for the constants of the functional equations of L functions for function fields, in the tamely ramified case, Deligne conceived a theory of nearby and vanishing cycles valid over general bases. A short summary, without proofs, was written by Laumon [L1]. This topic long remained untouched, because Laumon's proof of the product formula using the ℓ-adic Fourier transform [L2] rendered Deligne's approach useless. But it has been recently revisited by Orgogozo [O], who (with the help of Gabber for certain points) proved an analogue of Sabbah's theorem in the étale context. Some applications have already been obtained (more, I think, are looming in the background). This is what I am going to report on.

I wish to thank O. Gabber and F. Orgogozo for very helpful comments and discussions in the preparation of these notes.

2. Oriented products and vanishing toposes

2.1. Let $f : X \to S$ be a morphism of schemes, $\Lambda = \mathbb{Z}/\ell^\nu$ ($\nu \geq 1$), with ℓ a prime invertible on S. Let $F \in D^+(X, \Lambda)$. Let x be a geometric point of X, with image s in S, and let $t \to s$ be a map of geometric points in S, i. e. a geometric point $t \to S_{(s)}$ of the strict localization of S at s. As in 1.1, the strict localization $X_{(x)}$ of X at x plays the role of a Milnor ball and $(X_{(x)})_t$ the role of a Milnor fiber. One would like to construct a complex of sheaves of "nearby cycles" (resp. "vanishing cycles") (on a suitable space) whose stalk at each (x, t) would be $R\Gamma((X_{(x)})_t, F)$ (resp. a cone of $F_x \to R\Gamma((X_{(x)})_t, F)$). In order to do that, Deligne slightly changes the viewpoint. Instead of working with Milnor fibers, he prefers to work with "Milnor tubes" $X_{(x,t)}$. Here $X_{(x,t)}$ is the inverse image in the Milnor ball $X_{(x)}$ of the strict localization $S_{(t)}$ of S (or $S_{(s)}$) at t, i. e.

$$X_{(x,t)} = X_{(x)} \times_{S_{(s)}} S_{(t)}.$$

The "center" $X_{(x)} \times_{S_{(s)}} t$ of the Milnor tube is the Milnor fiber $(X_{(x)})_t$. It coincides with the tube when t is localized at a maximal point of $S_{(s)}$. It turns out that pairs $(x, t \to s = f(x))$ are points of a certain topos $X \overleftarrow{\times}_S S$ *under* X, called the *vanishing topos* of f, Minor tubes $X_{(x,t)}$ the localizations at them, and the complexes $R\Gamma(X_{(x,t)}, F)$ the stalks at them of a certain complex $R\Psi(F)$ on $X \overleftarrow{\times}_S S$.

2.2. The construction of $X \overleftarrow{\times}_S S$ is a particular case of a general construction of "oriented" fiber products of toposes, which we will, for simplicity, explain only in the case of étale toposes of schemes, see [I2] for the general case. Let $f : X \to S$, $g : Y \to S$ be morphism of schemes. The *oriented fiber product* of the étale toposes of X and Y over S is a topos

(2.2.1) $$X \overleftarrow{\times}_S Y,$$

equipped with morphisms $p : X \overleftarrow{\times}_S Y \to X$, $q : X \overleftarrow{\times}_S Y \to Y$, sometimes denoted p_1, p_2 (or p_X, p_Y), and a morphism $\tau : gq \to fp$, and which is universal for these data, i. e. having the following property : for any topos T equipped with a triple $(a : T \to X, b : T \to Y, t : gb \to fa)$, there exists a unique morphism $h : T \to X \overleftarrow{\times}_S Y$ together with isomorphisms $ph \xrightarrow{\sim} a$, $qh \xrightarrow{\sim} b$ making the composition $\tau * h$ equal to t. We shall sometimes omit the mention of these isomorphisms.

A defining site $(X \overleftarrow{\times}_S Y)_{\text{ét}}$ for $X \overleftarrow{\times}_S Y$ (cf. [L1, 3.1.3]) is the category of pairs of morphisms $U \to V \leftarrow W$ étale above $X \to S \leftarrow Y$, with the topology generated by covering families $(U_i \to V_i \leftarrow W_i) \to (U \to V \leftarrow W)$ ($i \in I$) of the following types :

(a) $V_i = V$, $W_i = W$ for all i and $(U_i \to U)$ is an étale covering,
(b) $U_i = U$, $V_i = V$ for all i and $(W_i \to W)$ is an étale covering,
(c) $(U' \to V' \leftarrow W') \to (U \to V \leftarrow W)$, where $U' = U$ and $W' \to W$ is obtained by base change from a map $V' \to V$ of the étale site of S (not necessarily covering).

By the universal property, the projections $pr_1 : X \times_S Y \to X$, $pr_2 : X \times_S Y \to Y$ define a morphism

$$\Psi : X \times_S Y \to X \overleftarrow{\times}_S Y, \tag{2.2.2}$$

such that $\Psi^{-1}(U \to V \leftarrow W) = U \times_V W$. By the universal property again, a pair of geometric points $(x \to X, y \to Y)$ and a map $c : g(y) \to f(x)$ define a point (x, y, c) of $X \overleftarrow{\times}_S Y$. There are enough such points. For a sheaf F on $X \times_S Y$, the stalk of $\Psi_* F$ at (x, y, c) is $\Gamma(X_{(x)} \times_{S_{(f(x))}} Y_{(y)}, F)$, where $Y_{(y)} \to S_{(f(x))}$ is the composition $Y_{(y)} \to S_{(g(y))} \to S_{(f(x))}$, the second map being given by c. This identification is derived into an isomorphism

$$R\Psi_* F_{(x,y,c)} \simeq R\Gamma(X_{(x)} \times_{S_{(f(x))}} Y_{(y)}, F) \tag{2.2.3}$$

for $F \in D^+(X \times_S Y, \Lambda)$. We will abbreviate (x, y, c) to (x, y) when no confusion can arise. For brevity, we will usually write $R\Psi$ instead of $R\Psi_*$.

2.3. For $Y = S$ and $g = Id_S$, the oriented product $X \overleftarrow{\times}_S S$ (2.2.1) is called the *vanishing topos* of f. In order to avoid confusion, the morphism Ψ (2.2.2) will be sometimes denoted Ψ_f (or Ψ_X), and the projection p denoted p_X. Here we have $X \times_S S = X$ and (2.2.3) is the formula announced at the end of 2.1, namely, for a point (x, t) of $X \overleftarrow{\times}_S S$,

$$R\Psi F_{(x,t)} = R\Gamma(X_{(x,t)}, F). \tag{2.3.1}$$

One can also think of $R\Psi F$ as putting together all the $i^* Rj_* F$ for $i : X_s \to X \times_S S_{(s)}$ and $j : X \times_S S_{(t)} \to X \times_S S_{(s)}$, as at a point x of X_s we have

$$(i^* Rj_* F)_x = R\Psi F_{(x,t)}. \tag{2.3.2}$$

Note again that the stalks of $R\Psi F$ are cohomologies of Milnor *tubes*, not Milnor fibers, and that the restriction to the cohomology of the corresponding Milnor fiber is not in general an isomorphism. This question is addressed in §3.

By construction one has $p\Psi = Id$, and one shows that the map

$$p_* \to \Psi^* \tag{2.3.3}$$

obtained by applying p_* to the adjunction map $Id \to \Psi_* \Psi^*$ is an isomorphism. The identity of F, for $F \in D^+(X, \Lambda)$, gives a canonical map

$$p^* F \to R\Psi F,$$

and (with the usual caution) a canonical triangle

$$p^*F \to R\Psi F \to R\Phi F \to . \tag{2.3.4}$$

The complex $R\Psi F$ (resp. $R\Phi F$) is called the complex of *nearby cycles* (resp. *vanishing cycles*) of f.

When S is the spectrum of a field, the morphism Ψ is an equivalence. In particular, $R\Phi F = 0$ for all $F \in D^+(X, \Lambda)$.

When S is a trait, as in 1.1, the topos denoted $X_s \times_s S$ (resp. $X_s \times_s \eta$) by Deligne in [SGA 7 XIII] can be identified with the sub-topos $X_s \overleftarrow{\times}_S S$ (resp. $X_s \overleftarrow{\times}_S \eta$) of $X \overleftarrow{\times}_S S$, and (2.3.4) induces the usual triangle relating (classical) nearby and vanishing cycles.

2.4. Oriented products and vanishing toposes satisfy various formal properties with respect to composition and base change, which we will not describe in detail. Let us just note the following.

(a) Let $f : X \to S$ and $g : Y \to S$ be morphisms and let $h : X \to Y$ be an S-morphism. Then h defines a morphism

$$\overleftarrow{h} : X \overleftarrow{\times}_S S \to Y \overleftarrow{\times}_S S,$$

compatible with the projections to X and Y and with the morphisms Ψ_f and Ψ_g, i. e. giving rise to 2-commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\Psi_X"] \arrow[d, "h"] & X \overleftarrow{\times}_S S \arrow[r, "p_X"] \arrow[d, "\overleftarrow{h}"'] & X \arrow[d, "h"] \\ Y \arrow[r, "\Psi_Y"] & Y \overleftarrow{\times}_S S \arrow[r, "p_Y"] & Y. \end{tikzcd} \tag{2.4.1}$$

(b) For $f : X \to S$ and $f' : X' \to S'$ deduced from f by a base change $g : S' \to S$, there is a natural morphism $\overleftarrow{g} : X' \overleftarrow{\times}_{S'} S' \to X \overleftarrow{\times}_S S$ inserting itself in a 2-commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "\Psi_{X'}"] & X \arrow[d, "\Psi_X"'] \\ X' \overleftarrow{\times}_{S'} S' \arrow[r, "\overleftarrow{g}"] \arrow[d, "p_{X'}"] & X \overleftarrow{\times}_S S \arrow[d, "p_X"] \\ X' \arrow[r, "g"] & X. \end{tikzcd} \tag{2.4.2}$$

For $F \in D^+(X, \Lambda)$ we say that *the formation of $R\Psi_f F$ commutes with the base change g* if the base change morphism

$$\overleftarrow{g}^* R\Psi_X F \to R\Psi_{X'} g^* F$$

given by the upper square of (2.4.2) is an isomorphism.

2.5. On toposes of the form (2.2.1) there are good notions of finiteness and constructibility ([O, 7]). Assume, for simplicity, that S is noetherian and X and Y are of finite type over S. A sheaf F of Λ-modules on $X \overleftarrow{\times}_S Y$ is called *constructible* if there exist finite partitions of X and Y into locally closed subsets : $X = \cup X_i$, $Y = \cup Y_j$, such that, for all (i, j), the restriction of F to the sub-topos $X_i \overleftarrow{\times}_S Y_j$ is locally constant of finite type. The constructible sheaves of Λ-modules form a thick subcategory of the category of all sheaves of Λ-modules, so that the full subcategory $D^b_c(X \overleftarrow{\times}_S Y, \Lambda)$ consisting of complexes with bounded, constructible cohomology sheaves is a triangulated subcategory. Constructible sheaves are the noetherian objects of the category of Λ-modules and any sheaf of Λ-modules is a filtering inductive limit of constructible sheaves.

If $S' \to S$ is proper and surjective, a sheaf of Λ-modules on $X \overleftarrow{\times}_S S$ is constructible if and only if its inverse image on $X' \overleftarrow{\times}_{S'} S'$ is, where $X' = X \times_S S'$.

3. Main results ([O])

3.1. *First basic properties.*

There are (easy) generalizations of properties (a) (i) and (ii) of 1.1, already mentioned in [L1, 3.2]. Let S be a scheme and Λ be as in 2.1.

Diagram (2.4.1) yields an isomorphism

$$R\Psi_Y Rh_* F \xrightarrow{\sim} R\overleftarrow{h}_* R\Psi_X F \tag{3.1.1}$$

for *any* $F \in D^+(X, \Lambda)$. When h is *proper*, the formation of $R\overleftarrow{h}_*$ commutes with base changes $Y' \to Y$, $S' \to S$ ([O, 8.1.1]). In particular, in the situation of 1.1, base changing by $Y_s \to Y$, $\eta \to S$ one recovers (1.1.4). Moreover, when h is proper, the second square of (2.4.1) gives an isomorphism

$$p_Y^* Rh_* F \xrightarrow{\sim} R\overleftarrow{h}_* p_X^* F \tag{3.1.2}$$

for any $F \in D^+(X, \Lambda)$, which implies an isomorphism analogous to (3.1.1) for $R\Phi$.

When h is locally acyclic, the base change map given by the first square of (2.4.1)

$$\overleftarrow{h}^* R\Psi_Y F \to R\Psi_X (h^* F) \tag{3.1.3}$$

is an isomorphism. This generalizes (1.1.5).

Properties (b) of 1.1 turn out to be false in general. As shown by Orgogozo [O, 9], in the case of the blow-up discussed in 1.2, $R\Psi_f \Lambda$ has not constructible cohomology, and its formation does not commute with the base change by f itself. Additional assumptions are necessary.

Recall that a morphism $g : S' \to S$ is called a *modification* (resp. an *alteration*) if g is proper, surjective and induces an isomorphism (resp. a finite morphism) over an everywhere dense open subscheme, with the property that each maximal point of S' is sent to a maximal point of S. Orgogozo's main result is the following theorem, conjectured by Deligne :

Theorem 3.2 [O, 1.1, 5.1, 6.1]. *Let S be a noetherian scheme and $f : X \to S$ be a morphism of finite type. Let $F \in D^b_c(X, \Lambda)$. Then there exists a modification $g : S' \to S$ such that if f' (resp. F') is deduced from f (resp. F) by base change by g, $R\Psi_{f'}F'$ belongs to $D^b_c(X' \overleftarrow{\times}_{S'} S', \Lambda)$ and the formation of $R\Psi_{f'}F'$ commutes with any base change $S" \to S'$ (cf. (2.4.2)).*

In particular, after base change by g, the cohomology of the Milnor tube restricts isomorphically to that of the Milnor fiber : for any point (x, y) of $X' \overleftarrow{\times}_{S'} S'$, the restriction map

$$R\Psi_{f'}F_{(x,y)} \to R\Gamma(X'_{(x)} \times_{S'_{(f(x))}} y, F).$$

is an isomorphism.

Remarks 3.3. (a) When S is the spectrum of a field, 3.2 says that, after any base change $S' \to S$, $R\Phi_{X'}F' = 0$: in view of 3.2.1, this is Deligne's universal local acyclicity theorem [SGA 4 1/2 Th. finitude, 2.13].

(b) When S is a trait, as in 1.1, one can take g to be the identity, and one recovers Deligne's results 1.1 (b).

(c) A slightly more general result is stated in (*loc. cit.*) : the conclusion is valid if S is quasi-compact and quasi-separated and has a finite number of irreducible components, and f is of finite presentation. In particular, if S is the spectrum of a valuative ring, as any modification of a valuative scheme admits a section, one recovers a result of Huber [H, 4.2.4].

(d) Theorem 3.2 can be viewed as an analogue (and strengthening), in the étale context, of Sabbah's theorem [S, th.1, th. 2].

(e) One has unfortunately no control on the modification g. For example, if f is smooth over some open subset V of S and F has locally constant cohomology sheaves on $f^{-1}(V)$, one doesn't know if one can impose to g to be an isomorphism over V.

(f) One can ask : when is (f, F) already good for $R\Psi$, i. e. $R\Psi F$ belongs to D^b_c and is base change compatible ? Here is an important example, a particular case of which is stated in [L1, 3.2.5] :

Proposition 3.4 ([O, 5.1]). *Let S be a noetherian scheme and $f : X \to S$ be a separated and of finite type morphism. Let $F \in D^b_c(X, \Lambda)$. Let Σ be the complement in X of the largest open subset U such that $F|U$ is universally locally acyclic over S [SGA 4 1/2 Th. finitude, 2.12]. Assume that Σ is quasi-finite over S. Then $R\Psi F$ (resp. $R\Phi F$) belongs to $D^b_c(X \overleftarrow{\times}_S S, \Lambda)$ (2.5) and its formation commutes with any base change $S' \to S$ (cf. (2.4.2)) (in particular, for any point (x, y) of $X \overleftarrow{\times}_S S$, the cohomology of the Milnor tube at (x, y) restricts isomorphically to the cohomology of the Milnor fiber). Moreover, $R\Phi F$ is concentrated on $\Sigma \overleftarrow{\times}_S S$.*

The proof of the commutation with base change is a standard local to global argument (cf. Deligne's theorem [L0, 4.1.2]). A similar argument shows the constructibility when Σ is finite over S. The general case follows from 3.2.

For $\Sigma = \emptyset$, we get :

Corollary 3.5. *Let $f : X \to S$ be separated and of finite type, with S noetherian, and let $F \in D_c^b(X, \Lambda)$. Then (f, F) is universally locally acyclic if and only if, after any base change $g : S' \to S$, $R\Phi_{X'}(F') = 0$, where X' is deduced from X by base change by g and $F' = g^*F$.*

4. Outline of proofs

The idea is to reduce 3.2 to the particular case of 3.4 where f is a proper semistable curve and $F = \Lambda_X$, using cohomological descent and de Jong's alterations. There are several steps.

In what follows, we will sometimes say "constructible" for "belongs to D_c^b".

4.1. *Preliminary reductions.*

(a) If there exists a finite surjective morphism $g : S' \to S$ such that (with the notations of 3.2), $R\Psi_{f'}F'$ is constructible and of formation compatible with base change, then $R\Psi_f F$ enjoys the same properties.

This relies on a simple cohomological descent argument for g.

(b) By standard reductions we may assume S affine, integral and of finite type over $\mathbb{Z}$ (in particular, excellent and finite dimensional).

On the other hand, as the problem is local upstairs, we may assume X affine. Embedding X into a projective space over S and extending F by zero, we may assume f *proper*.

(c) It is enough to show that there exists an *alteration* $g : S' \to S$ (cf. 3.1 for the definition) such that $R\Psi_{f'}F'$ satisfies the properties of the conclusion of 3.2.

This is, thanks to (a), a consequence of Gruson-Raynaud's flattening theorem [GR, I 5.2.2].

(d) If N is an integer bounding the dimension of the fibers of f, then $R\Psi_f$ is of cohomological dimension $\leq 2N$ on $D^+(X, \Lambda)$.

By an argument of Gabber, this follows from a result of Artin on the join of henselian rings [A, 3.4].

4.2. *Key lemmas.*

Recall that a morphism h is called *plurinodal* [dJ, 5.8] if h is a finite composition of proper semistable curves, i. e. proper and flat morphisms whose geometric fibers have at most ordinary quadratic singularities (this definition is slightly less restrictive than that in (*loc. cit.*) as we don't require the curves to be quasi-split nor have sections).

Recall, on the other hand, that a *proper hypercovering* of a scheme S is a simplicial scheme $S.$ over S such that for any $n \in \mathbb{N}$, the natural map $S_{n+1} \to \operatorname{cosk}_n^S(S.)_{n+1}$ is proper and surjective. If $\varepsilon : S. \to S$ is the augmentation, then, for any $F \in D^+(S, \Lambda)$, the adjunction map

$$F \to R\varepsilon_*\varepsilon^*F$$

is an isomorphism (Deligne's cohomological descent theorem).

The reduction to the case where f is a proper semistable curve and $F = \Lambda_X$ relies on the following three lemmas.

Lemma 4.2.1 ([O, 3.3.1]). *Let $f : X \to S$ be a proper morphism between noetherian, integral, excellent schemes. Then there exists an alteration $g : S' \to S$, with S' integral, and a proper hypercovering $X'_\bullet \to X' = X_{S'}$ such that each connected component of the S'-scheme X'_0 is either integral and plurinodal over S', or of image a strict closed subset of S', and that, furthermore, the generic relative dimension of X'_0 over S' is at most that of X over S.*

This is an easy consequence of one of de Jong's main theorems [dJ, 5.10], namely that under the assumptions of 4.2.1, there exists a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S, \end{tikzcd}$$

where g and h are alterations, with Z integral, and f' is plurinodal.

Lemma 4.2.2 (Gabber). *Let $g : Y \to S$ be a proper morphism between reduced noetherian schemes, and let $T = g(Y)$. Then there exists a commutative diagram*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ T_{S'} \arrow[r] \arrow[d] & T \arrow[d] \\ S' \arrow[r, "p"] & S, \end{tikzcd}$$

where the right vertical composition is g, the bottom square is cartesian, p is a blow-up whose center is nowhere dense in S, and the morphism $W \to T_{S'}$ is finite and surjective.

See [O, 3.2.1] for the proof. We will apply 4.2.2 to the case where $Y \to T$ is a modification : the meaning of 4.2.2 then is that such a modification is - up to a finite surjective morphism - dominated by a modification of T induced from one of S ("strict" domination is in general impossible, as an example of Kollár shows [O, 3.2.3]).

Lemma 4.2.3. *Let S be a noetherian scheme and let $f : X \to Y$ be a* proper *morphism between S-schemes of finite type. Then $R\overleftarrow{f}_*$ sends $D^b_c(X \overleftarrow{\times}_S S, \Lambda)$ to $D^b_c(Y \overleftarrow{\times}_S S, \Lambda)$.*

This is an easy consequence of the classical finiteness theorem and of (3.1.2). In view of Gabber's recent results [Ga], one may expect that if S is excellent the properness assumption on f is superfluous.

4.3. *The triple inductions.*

It turns out to be more convenient to treat constructibility and commutation with base change separately.

4.3.1. *Commutation with base change.*

One proves by induction on the triples of integers $t = (\delta, r, d)$, lexicographically ordered, with $\delta \geq 0,\ r \geq -2,\ d \geq 0$, the following assertion :

$A(t)$: *for every morphism* $f : X \to S$, *with* S *excellent,* $\dim S \leq \delta$, f *of finite type and relative dimension* $\dim(f) \leq d$ *and every* $F \in D^b_c(X, \Lambda)$ *such that* $H^i F = 0$ *for* $i < 0$, *there exists an alteration* $g : S' \to S$ *such that for any* S'*-morphism* $T' \to T$, *if we denote by* $K_{TT'} R\Psi_f F$ *the cone of the base change map*

$$R\Psi_{f'_T}(F_T)|T' \to R\Psi_{f'_{T'}}(F_{T'}),$$

where $f' : X' \to S'$ *is deduced from* f *by base change by* g, *one has*

$$\tau_{\leq r} K_{TT'} R\Psi_f F = 0.$$

In view of 4.1 (b), (c) and (d), the existence of a modification g in 3.2 such that the formation of $R\Psi_{f'} F'$ commutes with any base change follows from $A(t)$ for all t.

The assertion $A(\delta, -2, d)$ holds trivially for any δ and d, in particular $A(0, -2, 0)$ holds, which starts the induction. Let $t = (\delta, r, d) > (0, -2, 0)$, assume that $A(t')$ holds for any $t' < t$ and let us prove that $A(t)$ holds. We first treat a particular case :

(a) *Composition with a semistable curve.*

Suppose that f can be factored as $f = ba$, where $b : Y \to S$ is proper, of relative dimension $\leq d - 1$, and $a : X \to Y$ is a proper semistable curve. Let us show that after a suitable alteration $g : S' \to S$, we have $\tau_{\leq r} K_{TT'} R\Psi_f \Lambda = 0$ for every S'-morphism $T' \to T$.

Let $U \subset X$ be the open subset of smoothness of a. The complement $\Sigma = X - U$ is finite over Y. As b is of relative dimension $\leq d - 1$, by the induction assumption we may assume, up to base changing by an alteration $S' \to S$, that $\tau_{\leq r} K_{TT'} R\Psi_b \Lambda = 0$. By the basic property (3.1.3), as $a_U = a|U$ is smooth, we then have

$$0 = \overleftarrow{a_U}^{*} \tau_{\leq r} K_{TT'} R\Psi_b \Lambda = \tau_{\leq r} K_{TT'} R\Psi_f \Lambda | U.$$

Hence $\tau_{\leq r} K_{TT'} R\Psi_f \Lambda$ is concentrated on Σ, and it suffices to show that

$$\overleftarrow{a_\Sigma}_{*} \tau_{\leq r} K_{TT'} R\Psi_f \Lambda = 0.$$

Now

$$\overleftarrow{a_\Sigma}_{*} \tau_{\leq r} K_{TT'} R\Psi_f \Lambda = \tau_{\leq r} R \overleftarrow{a}_{*} K_{TT'} R\Psi_f \Lambda,$$

and by the basic property (3.1.1) we have

$$R \overleftarrow{a}_{*} K_{TT'} R\Psi_f \Lambda = K_{TT'} R\Psi_b (Ra_* \Lambda).$$

As $Ra_*\Lambda$ is in $D^b_c(Y, \Lambda)$ and cohomologically concentrated in nonnegative degrees, by the induction assumption again, we may assume, up to base changing S by an alteration that $\tau_{\leq r} K_{TT'} R\Psi_b (Ra_* \Lambda) = 0$, hence $\tau_{\leq r} K_{TT'} R\Psi_f \Lambda = 0$ as required.

(b) *The general case.*

We may assume F concentrated in degree zero. By 4.1 (b) we may assume furthermore that S is integral, affine, of finite type over $\mathbb{Z}$ and f is proper. Embedding F into a finite sum of sheaves of the form p_*C for $p : Z \to X$ finite and C constant and using $A(\delta, r-1, d)$ and (3.1.1) we reduce to the case F is constant, and finally to $F = \Lambda_X$. Using cohomological descent by a proper hypercovering of X whose components are disjoint sums of integral schemes finite over X, we may assume X integral. Now apply the key lemma 4.2.1 to f. Up to changing notations we may assume that $S = S'$, $X = X_{S'}$. By cohomological descent we have $\Lambda_X = R\varepsilon_{.*}\Lambda_{X_.}$, hence (by (3.1.1))

$$K_{TT'}R\Psi_X\Lambda = R\overleftarrow{\varepsilon}_{.*}K_{TT'}R\Psi_{X'_.}\Lambda.$$

It suffices to show that, for $n \geq 0$,

$$(*) \qquad \tau_{\leq r-n}K_{TT'}R\Psi_{X'_n}\Lambda = 0.$$

Recall that X'_0 is a disjoint sum of schemes $(X'_0)_i$ such that $(X'_0)_i \to S$ is either plurinodal or of image a proper closed subset of S. The first case is disposed of by (a). The second one follows from the induction assumption ($\dim f'((X'_0)_i) < \delta$, thanks to 4.1 (a) and key lemma 4.2.2. This proves (*) for $n = 0$. The case $n > 0$ follows from the induction assumption $(r - n < r)$.

4.3.2. *Constructibility.*

In order to prove that there exists a modification $g : S' \to S$ such that, with the notations of 3.2, $R\Psi_{f'}F'$ belongs to $D^b_c(X' \overleftarrow{\times}_{S'} S', \Lambda)$, it is again enough to show, by induction on the triples of integers $t = (\delta, r, d)$, lexicographically ordered, with $\delta \geq 0$, $r \geq -2$, $d \geq 0$, the following assertion :

$B(t)$: *for every* proper *morphism* $f : X \to S$, *with* S *excellent,* $\dim S \leq \delta$*,* f *of finite type and relative dimension* $\dim(f) \leq d$ *and every* $F \in D^b_c(X, \Lambda)$ *such that* $H^iF = 0$ *for* $i < 0$*, there exists an alteration* $g : S' \to S$ *such that* $\tau_{\leq r}R\Psi_{f'}F' \in D^b_c(X' \overleftarrow{\times}_{S'} S', \Lambda)$ *(where* $f' : X' \to S'$ *is deduced from* f *by base change by* g*).*

Again $B(\delta, -2, d)$ is trivially true, which starts the induction. Let $t = (\delta, r, d) > (0, -2, 0)$, assume that $B(t')$ holds for any $t' < t$ and let us prove that $B(t)$ holds. The strategy is the same as for $A(t)$.

(a) *Composition with a semistable curve.*

Suppose that f can be factored as $f = ba$, where $b : Y \to S$ is proper, of relative dimension $\leq d - 1$, and $a : X \to Y$ is a proper semistable curve. Let us show that after a suitable alteration $g : S' \to S$, $\tau_{\leq r}R\Psi f'\Lambda$ is in D^b_c. By the induction assumption, we may assume, up to base changing by a suitable alteration, that $\tau_{\leq r}R\Psi_b\Lambda$ is constructible. Therefore it suffices to show that, after base changing by a suitable alteration, a cone $K(a)$ of the canonical map

$$\overleftarrow{a}^*\tau_{\leq r}R\Psi_b\Lambda \to \tau_{\leq r}R\Psi_a\Lambda$$

is constructible. By the basic property (3.1.3) $K(a)$ is concentrated on $\Sigma \overleftarrow{\times}_S S$, where Σ is the non smoothness locus of a, which is finite over Y. It follows that it suffices to check that $\tau_{\leq r} R\overleftarrow{a}_* K(a)$ is constructible. By 4.2.3 we know that $R\overleftarrow{a}_* \overleftarrow{a}^* \tau_{\leq r} R\Psi_b \Lambda$ is constructible, so we are reduced to showing that $\tau_{\leq r} R\overleftarrow{a}_* \tau_{\leq r} R\Psi_f \Lambda$ is in D_c^b. But

$$\tau_{\leq r} R\overleftarrow{a}_* \tau_{\leq r} R\Psi_f \Lambda = \tau_{\leq r} R\overleftarrow{a}_* R\Psi_f \Lambda,$$

and by the basic property (3.1.1),

$$R\overleftarrow{a}_* R\Psi_f \Lambda = R\Psi_b(Ra_*\Lambda).$$

As $Ra_*\Lambda$ is in $D_c^b(Y, \Lambda)$, by the induction assumption, up to base changing by a suitable alteration, $\tau_{\leq r} R\Psi_b(Ra_*\Lambda)$ is constructible and we are done.

(b) *The general case.*

The proof follows the lines of 4.3.1 (b).

5. Lefschetz pencils

This section is an application of 3.4, in a situation where the complex of vanishing cycles can be somehow explicitly calculated.

5.1. Let k be an algebraically closed field of characteristic p, P a projective space of dimension ≥ 1 over k, $X \subset P$ a smooth, connected, closed subscheme of P, of dimension $n+1$. Let $\check{P}$ be the dual projective space, parametrizing the hyperplanes in P. Let D be a pencil of hyperplanes, i. e. a line in $\check{P}$. For $t \in D$ we denote by H_t the corresponding hyperplane, defining a hyperplane section $X_t = X \cap H_t$ of X, and by $\check{D}$ the *axis* of D, a codimension 2 linear subvariety of P through which all the H_t pass. Recall ([SGA 7 XVII], [D1 5.6], [D2 4.2]) that one says that $(X_t)_{t \in D}$ (or D) is a *Lefschetz pencil* if the following conditions are satisfied :

(i) the axis $\check{D}$ is *transverse* to X ;

(ii) there exists a finite closed subset S of D such that X_t is smooth for $t \notin S$, and for $s \in S$, X_s is smooth outside a single closed point x_s, which is an ordinary quadratic singularity of X_s.

If D is a Lefschetz pencil, the sections X_t are the fibers of the projection $\tilde{X} \to D$, where $\tilde{X} \to D$ is the *incidence scheme*, consisting of pairs of points $(x, t) \in X \times D$ such that $x \in X_t$; $\tilde{X}$ is smooth over k, and coincides with the blow-up of $\check{D} \cap X$ in X.

Let D be a Lefschetz pencil. Put $n = 2n'$ (resp. $2n'+1$) is n is even (resp. odd). Let ℓ be a prime $\neq p$ and Λ as in 2.1. For each $s \in S$, let $\overline{\eta}_s$ be a geometric generic point of $D_{(s)}$. Choose also a generic geometric point u of $D - S$ and morphisms $c : u \to D_{(s)}$. Recall (*loc. cit.*) that, for each s, there is an element

(5.1.1) $$\delta_s \in H^n(X_{\overline{\eta}_s}, \Lambda)(n'),$$

well defined up to sign, called the *vanishing cycle at s*. Thanks to c, δ_s defines an element

(5.1.2) $$\delta_c \in H^n(X_u, \Lambda)(n'),$$

also called "vanishing cycle". Thanks to the Picard-Lefschetz formula, these vanishing cycles (together with some local characters of order 2 when n is even and $p = 2$) determine the local and global monodromy of the family $H^n(X_t, \Lambda)_{t\in D}$. In [SGA 7 XVIII 6.6] it is proven that if n is odd or $p \neq 2$, the vanishing cycles $\pm\delta_c$ are conjugate under the action of $\pi_1(D - S, u)$ on $H^n(X_u, \Lambda)(n')$, i. e. under the action of the *monodromy group*, image of $\pi_1(D-S,u)$ in $\mathrm{GL}(H^n(X_u,\Lambda)(n'))$. In [D2, 4.2.8], Deligne showed that if n is even and $p = 2$, the same is true provided that $\Lambda = \mathbb{Q}_\ell$ and D is sufficiently general (i. e. belongs to a suitable open subset of the grassmannian of lines in $\check{P}$). He also added :

"Il devrait résulter d'une théorie - non écrite - des cycles évanescents pour une base de dimension > 1 que, pour un pinceau de Lefschetz transverse, les cycles évanescents pris au signe près $\pm\delta$ sont tous conjugués sous le groupe de monodromie et que la géométrie de la situation (en particulier, le groupe de monodromie) ne dépend pas du pinceau de Lefschetz transverse choisi."

Recall that a Lefschetz pencil D is said to be *transverse* if it is transverse to the smooth locus of the *dual variety* $\check{X}$, an integral closed subscheme of $\check{P}$ of dimension $\leq \dim P - 1$, which is the set of points $t \in \check{P}$ such that H_t is not transverse to X. In other words, $\check{X}$ is the image of the projection to $\check{P}$ of the *nonsmoothness locus* over $\check{P}$ of the *incidence scheme* $Z \subset X \times \check{P}$ (Z consists of pairs of points (x,t) such that $x \in X_t$). If $p = 2$ and n is even, a Lefschetz pencil is not necessarily transverse.

Deligne's expectation proved to be true. Orgogozo indeed shows :

Theorem 5.2 ([O, 11.2]. *Let $X \subset P$ and $\Lambda = \mathbb{Z}/\ell^\nu$ be as in 5.1, and let $D \in \check{P}$ be a Lefschetz pencil. Then, with the above notations :*

(i) The vanishing cycles $\pm\delta_c \in H^n(X_u,\Lambda)(n')$ are conjugate under the monodromy group $G = \operatorname{Im} \pi_1(D - S, u) \to \mathrm{GL}(H^n(X_u, \Lambda)(n'))$;

(ii) G is the image of the homomorphism $h : \pi_1(\check{P} - \check{X}, u) \to \mathrm{GL}(H^n(X_u,\Lambda)(n'))$, in particular, is independent of D.

Note that here D is not assumed to be transverse, and one doesn't need to take $\Lambda = \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$. The proof of (ii) is due to Gabber (letter to Orgogozo, 3/14/2005).

5.3. *Sketch of proof.*

Let G_0 be the image of h. It is equivalent to show (i) and (ii), or (ii) (i. e. $G = G_0$) and

(i') *The vanishing cycles $\pm\delta_c \in H^n(X_u, \Lambda)(n')$ are conjugate under G_0.*

(a) We will first sketch the proof of (i'). Let $Z \subset X \times \check{P}$ be the incidence scheme considered above, and

$$f : Z \to \check{P}$$

be the projection, so that, for $t \in \check{P}$, $f^{-1}(t) = X \cap H_t = X_t$. Using vanishing cycles for f, the proof closely follows Lefschetz's proof in the transcendental case, explained in [D1, 5.4]. Let Σ be the nonsmoothness locus of f. This is a smooth closed subscheme of Z, consisting of points (x,t) such that H_t is tangent to X at x, in other words, $\Sigma = \mathbf{P}(N)$ where N is the normal bundle of X in P. By definition, $\check{X} = f(\Sigma)$. Let $\check{X}_0$ be the open subset of $\check{X}$ whose closed points s are such that X_s has a unique singular point x_s, which is ordinary quadratic, and Σ_0 the inverse image of $\check{X}_0$ in Σ. By definition $\check{X}_0$ contains

$D \cap \check{X}$, and we may assume that $D \cap \check{X}$ is nonempty. Set $U := \check{P} - \check{X}$ and let $f_U : Z_U \to U$ be the restriction of f. As f_U is proper and smooth, the sheaf $R^n f_{U*}\Lambda$ is lisse. Let

$$p_2 : \check{X}_0 \overleftarrow{\times}_{\check{P}} U \to U$$

be the second projection. Applying 3.2 to the restriction of f over $\check{P} - (\check{X} - \check{X}_0)$, whose nonsmoothness locus Σ_0 is radicial and surjective over $\check{X}_0$, and using the classical local Lefschetz theory [SGA 7 XV §§2, 3], one defines a set-theoretic local system of order 1 or 2

$$\pm\delta \subset p_2^* R^n f_{U*}\Lambda(n'),$$

which plays the role of a *universal local vanishing cycle*. In particular, its fiber at each point $c : u \to s$ of $\check{X}_0 \overleftarrow{\times}_{\check{P}} U$ chosen above is the vanishing cycle $\pm\delta_c \in H^n(X_u, \Lambda(n')) = (R^n f_{U*}\Lambda(n'))_u = (p_2^* R^n f_{U*}\Lambda(n'))_c$ (5.1.2). As $\check{X}_0$ is nonempty, hence irreducible, $\check{X}_0 \overleftarrow{\times}_{\check{P}} U$ is connected. If $c : u \to s$, $c' : u \to s'$ are two points, one can choose a path $d : c \to c'$. As $\pm\delta$ is a local system, d sends $\pm\delta_c$ to $\pm\delta_{c'}$. The image of d by p_2 is a path from u to u in U, i. e. an element g of $\pi_1(U, u)$. The isomorphism from $(p_2^* R^n f_{U*}\Lambda(n'))_c$ to $(p_2^* R^n f_{U*}\Lambda(n'))_{c'}$ defined by d coincides with the automorphism of $H^n(X_u, \Lambda(n'))$ defined by g. In other words, there is an element $g \in \pi_1(U, u)$ such that $g\delta_c = \pm\delta_{c'}$.

(b) Let us now sketch the proof of (ii). We may assume that $\check{X}$ is a hypersurface (otherwise $G = G_0 = \{1\}$). When $p \neq 2$ or n is odd, $\pi_1(D - S, u)$ acts on $H^n(X_u, \Lambda)$ through its tame quotient $\pi_1^t(D - S, u)$, and the corresponding homomorphism on the tame quotients $\pi_1^t(D - S, u) \to \pi_1^t(\check{P} - \check{X}, u)$ is surjective [SGA 7 XVIII 6.1], so the conclusion is immediate in this case. When $p = 2$ and n is even, the actions on $H^n(X_u, \Lambda)$ of the inertia groups of D around the points of S are non necessarily tame : they are given by characters of order two associated to quadratic extensions which can be wildly ramified. Moreover, the homomorphism $\pi_1(D - S, u) \to \pi_1(\check{P} - \check{X}, u)$ is not surjective. Thus, in this case, a new argument is needed. Here is the idea. Let Y be the (connected) Galois étale cover of U given by G_0. One has to show that the pull-back $Y_D := Y \times_{\check{P}} (D - S)$ of Y to $D - S$ is still connected. Let $\overline{Y}$ be the normalization of $\check{P}$ in Y, and let $W = \check{P} - (\check{X} - \check{X}_0)$. A local study of Y around points of $\check{X}_0$ shows that the restriction of $\overline{Y}$ to W is *smooth* over k, as well as $\overline{Y}_D := \overline{Y} \times_{\check{P}} D$. These schemes $\overline{Y}_D$ are the fibers of a proper and smooth morphism $g : A \to B$ where B is the open subset of the Grasmannian $\mathrm{Gr}(1, \check{P})$ consisting of Lefschetz pencils. By a theorem of Bertini, the generic fiber of g is geometrically connected, hence each $\overline{Y}_D$ is connected by Zariski's theorem, and in particular Y_D is connected. The main point in the local study is a refinement of [SGA 7 XV 1.3.2], which says that étale locally on W, f defines a flat double cover of W, étale outside $\check{X}$, which is unique up to isomorphism, and smooth over k.

6. Open problems

6.0. *Variation.*

Let $f : X \to S$ be separated and of finite type, with S noetherian, and Λ be as in 2.1.

If S is a henselian trait, for $F \in D_c^b(X, \Lambda)$ and $\sigma \in G$ (with the notations of 1.1), there is defined a morphism, called the *variation*,

$$\mathrm{Var}(\sigma) : R\Phi F \to R\Psi F,$$

deduced from the action of $\sigma - 1$ on $R\Psi F$ [SGA 7 XIII]. This morphism is a finer invariant than the action of σ and its determination in the case of ordinary quadratic singularities (Picard-Lefschetz formula) is the key to the study of the monodromy of Lefschetz pencils. What could replace this morphism in the general situation of 2.1 ?

6.1. *Duality, perversity.*

Let $f : X \to S$ and Λ be as in 6.0.

If S is a henselian trait, according to a theorem of Gabber, the functor $R\Psi$: $D_c^b(X_\eta, \Lambda) \to D_c^b(X_s \times_s \eta, \Lambda)$ commutes with the dualizing functors D_η and D_s on X_η and $X_s \times_s \eta$ respectively, defined by $RHom(-, a^!\Lambda)$, where a is the projection to the base, i. e. there is a natural isomorphism [I1, 4.2] :

$$R\Psi D_\eta F \xrightarrow{\sim} D_s R\Psi F \tag{6.1.1}$$

for $F \in D_c^b(X_\eta, \Lambda)$ (the proof given in (*loc. cit.*) for F in D_{ctf}^b works also for F in D_c^b). This fact, combined with the right t-exactness of $R\Psi$ [BBD, 4.2] (a consequence of the affine Lefschetz theorem [SGA 4 XIV]), implies (*loc. cit.*) that $R\Psi$ is t-exact, and in particular transforms perverse sheaves into perverse sheaves. Moreover, by another theorem of Gabber (*loc. cit*, 4.6), if F is a perverse Λ-sheaf on X (for the standard t-structure), $R\Phi F[-1]$ is perverse.

In general, can one expect similar statements after a suitable modification of S ? Even the formulation of these requires some preliminary work on the definition of "operations" in the derived category involving oriented products, which haven't been considered yet.

6.2. *Comparison with the complex analytic case.*

For a complex analytic space $\mathcal{X}$ over a disc D of center 0, there is defined a functor $R\Psi$ similar to (1.1.1) [SGA 7 XIV 1.3], going from $D^*(\mathcal{X}^*, \mathbb{Z})$ to $D^+(\mathcal{X}_0, \mathbb{Z}[\pi])$, where $\mathcal{X}_0$ is the central fiber, $\mathcal{X}^*$ the restriction of $\mathcal{X}$ over the punctured disc $D^* = D - \{0\}$ and π ($\simeq \mathbb{Z}$) the fundamental group of D^* ;

$$R\Psi F = i^* R\overline{j}_*(F|\overline{\mathcal{X}}^*), \tag{6.2.1}$$

where $i : \mathcal{X}_0 \to \mathcal{X}$ is the inclusion and $\overline{j} : \overline{\mathcal{X}}^* \to \mathcal{X}$ the natural projection, $\overline{\mathcal{X}}^*$ being the pull-back of $\mathcal{X}^*$ to a universal cover of D^*.

Moreover, for a scheme X separated and of finite type over a smooth curve Y over $\mathbb{C}$ and s a closed point of Y, there is a comparison theorem between the functor $R\Psi$ associated to the restriction of X to the strict localization S of Y at s and the functor $R\Psi$ associated to the complex analytic space $\mathcal{X}$ over a disc D centered at s defined by X/S [SGA 7 XIV 2.8].

For a general morphism of complex analytic spaces there is as yet no analogue of the vanishing topos and of the functor $R\Psi$ considered in 2.3, and *a fortiori* no comparison theorem. It is even perhaps too optimistic to hope for such a general theory. The only positive indications toward the existence of "some" theory seem to be the following :

(a) In [S] Sabbah showed that if $f : \mathcal{X} \to S$ is a proper morphism between complex analytic spaces, then, after base change by a suitable modification, f has, at each point x of $\mathcal{X}$, a "theory of vanishing cycles" (for the constant sheaf $\mathbb{Z}$), in the sense explained in 1.2.

(b) In complex analytic *logarithmic* geometry, a variant $R\Psi^{\log}$ of the functor $R\Psi$ (6.2.1) is constructed. This functor $R\Psi^{\log}$ is defined for *any* morphism $f : X \to Y$ of fs log (= fine and saturated logarithmic) analytic spaces. It goes from $D^+(X^{log}, \mathbb{Z})$ to $D^+((X')^{\log}, \mathbb{Z})$, where X' is the log space having the log structure inverse image by f from that of Y and the superscript "log" means the associated Kato-Nakayama space (some kind of a "real blow-up" of the underlying analytic space). By definition ([IKN, 8.1]),

$$R\Psi^{\log} = R\nu_*^{\log}, \tag{6.2.2}$$

where $\nu : X \to X'$ is the natural projection. In the case where Y is a disc, with the standard log structure given by the origin, X is log smooth over $\mathbb{C}$ (i.e. of "toroidal type") and f "vertical", which means that the log structure of X is given by the special fiber (and trivial outside), then one can recover $R\Psi F$ from $R\Psi^{\log} F^{\log}$ for *locally constant* abelian sheaves F on $X - X_0$ ($F^{\log}$ denoting the natural extension of F on $X^{\log}$) ([IKN, 8.3]). It doesn't seem, however, that, in the general case, this functor $R\Psi^{\log}$ should yield a suitable analogue of the functor $R\Psi$ of §2.

6.3. *Non-abelian variants.*

So far there doesn't seem to exist any theory of nearby cycles in the non-abelian setting : sheaves of sets, of groups, or - which encompasses both - stacks in groupoids, even in the classical case, over a trait, as in 1.1. It is not unreasonable to expect that such a theory - in the "prime to p" case - could shed light on specialization theorems for the (prime to p quotients) of the fundamental groups, and their possible jumps between generic and special fibers. In the situation of 2.3, for a morphism $f : X \to S$ and a stack in groupoids F on X, one can consider the stack in groupoids $\Psi_* F$ on $X \overleftarrow{\times}_S S$, the *stack of nearby cycles* of F. It is not clear, however, how to define a *stack of vanishing cycles* $R\Phi F$, as already in the commutative case, the cone of a morphism of Picard stacks is rather a 2-stack. One can of course ask for analogues of 3.2 and 3.4, when F is constructible and its local automorphism sheaves are of order invertible on S.

6.4. *Tubular neighborhoods and rigid geometry.*

Let S be a scheme. The topos

$$\overleftarrow{S} := S \overleftarrow{\times}_S S, \tag{6.4.1}$$

which might be called the *vanishing topos* of S, is of special interest. Its points are maps $y \to x$ of geometric points of S. The defining site considered in 2.2 can be replaced by the following simpler one, namely the category of morphisms $U \leftarrow V$, with U and V étale over S, with the topology generated by the covering families of the following types : (i) $(U \leftarrow V_i) \to (U \leftarrow V)$, where $(V_i \to V)$ is an étale covering ; (ii) $(U_i \leftarrow V_i) \to (U \leftarrow V)$, where $(U_i \to U)$ in an étale covering and $V_i = U_i \times_U V$. A sheaf $F : (U \leftarrow V) \mapsto F(U \leftarrow V)$ on $\overleftarrow{S}$ can be viewed as a (nonnecessarily cartesian) section $U \mapsto F(U, -)$ of the stack of sheaves on S satisfying a certain descent condition. If p_1, p_2 are the projections from $\overleftarrow{S}$ to S, in addition to $p_{1*} = \Psi^*$ (2.3.3), we have here $p_2^* = \Psi_*$. The oriented fiber product considered in (2.2.1) can be recovered from $\overleftarrow{S}$ by "usual pull-backs", i.e. fiber products of toposes (cf. [L1, 3.1.3]) :

$$X \overleftarrow{\times}_S Y = X \times_{\overleftarrow{S}} Y. \tag{6.4.2}$$

The existence of fiber products of toposes is proven in [Gi]. Gabber recently observed that a more direct construction can be given in terms of oriented products, see [I2].

Let now X be a noetherian scheme, Y a closed subscheme, and $U = X - Y$. The topos

$$Y \overleftarrow{\times}_X U, \tag{6.4.3}$$

plays the role of a *punctured tubular neighborhood* of Y in X. Its points are maps $x \to y$ of geometric points of X, with y localized in Y and x in U. Such a topos appeared in the proof of 5.2. If Y consists of a single closed point y, with $k(y)$ separably closed, then $Y \overleftarrow{\times}_X U$ is the punctured Milnor ball $X_{(y)} - \{y\}$.

Let $p_1 : Y \overleftarrow{\times}_X U \to Y$ and $p_2 : Y \overleftarrow{\times}_X U \to U$ be the projections. Let $\Lambda = \mathbb{Z}/\ell^\nu$ $(\nu \geq 1)$, with ℓ a prime invertible on X. For $F \in D_c^b(U, \Lambda)$ there is a natural base change map

$$i^* Rj_* F \to Rp_{1*} p_2^* F, \tag{6.4.4}$$

where $i : Y \to X$ and $j : U \to X$ are the inclusions. It is easy to check that this is an isomorphism. This is reminiscent of the *(deleted) tubular neighborhood* $T_{X/Y}$ of X along Y defined by Fujiwara [F, 6], a topos equipped with projections q_1 and q_2 to Y and U respectively, and a map $jq_2 \to iq_1$ giving rise to an isomorphism similar to (6.4.4). By the universal property of the topos $Y \overleftarrow{\times}_X U$, there is a morphisms

$$\varepsilon_X : T_{X/Y} \to Y \overleftarrow{\times}_X U.$$

More generally, if $X' \to X$ is a modification inducing an isomorphism over U and $Y' = Y \times_X X'$, there is a morphism

$$\varepsilon_{X'} : T_{X/Y} \to Y' \overleftarrow{\times}_{X'} U,$$

and these morphisms form a compatible system in a natural way. We thus obtain a (nonnecessarily coherent) morphism

$$\varepsilon : T_{X/Y} \to \tilde{Y} \overleftarrow{\times}_{\tilde{X}} U, \tag{6.4.5}$$

where $\tilde{Y} \overleftarrow{\times}_{\tilde{X}} U$ is the 2-limit of the toposes $Y' \overleftarrow{\times}_{X'} U$. What can be said of ε ? For example, is it true that $R\varepsilon_* \Lambda = \Lambda$, assuming X excellent ? Recently, the tubular neighborhood $T_{X/Y}$ has been used (indirectly) by Abbes and T. Saito [AS] in their theory of microlocalization. Could $\tilde{Y} \overleftarrow{\times}_{\tilde{X}} U$ be a substitute for $T_{X/Y}$?

6.5. *Families of pencils.*

Let S be a noetherian scheme, $f : X \to Y$ an S-morphism between schemes separated and of finite type over S. Assume that Y is smooth over S with geometrically connected fibers of relative dimension 1. One can view f as a family of (generalized) *pencils* parametrized by S. Let Z be a closed subscheme of Y, proper over S. Finally let $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$, with ℓ invertible on S, and let $F \in D^b_c(X, \Lambda)$. It follows from Orgogozo's theorem 3.2 that there exists a partition of S into locally closed subschemes S_i ($i \in I$), and, for each $i \in I$, an open neighborhood U_i of Z_i in Y_i (denoting by the subscript i the restriction to S_i), such that, for $f_i : f^{-1}(U_i) \to U_i$ induced by f and $F_i = F|f^{-1}(U_i)$, $R\Psi_{f_i}(F_i)$ is in $D^b_c(f^{-1}(U_i) \overleftarrow{\times}_{U_i} U_i, \Lambda)$ and commutes with base changes $T \to U_i$. The proof is easy, by noetherian induction, observing that for S irreducible, a modification of Y is an isomorphism on the generic fiber.

This result, suitably reinforced to take into account the jumps from one stratum to another, might have some interest in ramification theory, in view of Deligne's theorem [L0].

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