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A certain Galois action on modular forms with respect to any unitary group and the arithmeticity of Petersson inner products

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0 Introduction

Let us consider holomorphic modular forms for any symplectic group $\mathrm{Sp}(l, F)$, where F is a totally real algebraic number field of finite degree. In this case, a holomorphic modular form f on $\mathfrak{H}_l^{\mathbf{a}}$ (Hilbert-Siegel domain) has a Fourier expansion of the following form:

$$f((z_v)_{v\in\mathbf{a}}) = \sum_h c_h \exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}} \mathrm{tr}(h_v z_v)\right), \qquad (0.1)$$

where $\mathbf{a}$ denotes the set of all archimedean primes of F, and h runs over the points in a certain lattice in symmetric matrices of degree l with coefficients in F. Shimura showed that, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, there exists a holomorphic modular form f^σ whose Fourier expansion is given by

$$f^\sigma((z_v)_{v\in\mathbf{a}}) = \sum_h c_h^\sigma \exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}} \mathrm{tr}(h_v z_v)\right). \qquad (0.2)$$

It is also proved that this Galois action is compatible with Hecke operators.

In this lecture we will construct such a Galois action on holomorphic modular forms for an arbitrary unitary group over any CM-field K, which is the result of [12] and a natural generalization of [11]. This is essentially the same as the conjugate of automorphic vector bundles on Shimura varieties, which was researched in [4] or [1]. But the action was not *explicitly* written in those papers. In this lecture, the Galois action will be given explicitly. Moreover, we can obtain the relation between the Galois action and Petersson inner products, which is stated in [13].

1 Modular forms for an arbitrary unitary group

In this lecture, we treat scalar-valued holomorphic modular forms on hermitian unitary groups for any CM-fields.

Let F be a totally real algebraic number field of finite degree and K be its CM-extension (namely, a totally imaginary quadratic extension of F). Such a field K is called a CM-field. As is well known, the non-trivial element of $\mathrm{Gal}(K/F)$ is the complex conjugate for any embedding of K into $\mathbb{C}$. We denote this by ρ. Let $\mathbf{a}$ be the set of all archimedean primes of F, which can be identified with those of K. For each $v \in \mathbf{a}$, there are two embeddings of K into $\mathbb{C}$ which lie above v. By a CM-type of K, we mean a set $\Psi = (\Psi_v)_{v\in\mathbf{a}}$ where each Ψ_v is an embedding of K into $\mathbb{C}$ which lies above v. We can view a CM-type Ψ as an embedding of K into $\mathbb{C}^{\mathbf{a}}$ such that $b^{\Psi} = (b^{\Psi_v})_{v\in\mathbf{a}}$ for any $b \in K$. Via Ψ, we can view K as a dense subset of $\mathbb{C}^{\mathbf{a}}$. When $b \in F$, we drop the symbol Ψ (since b^{Ψ} does not depend on Ψ) and regard b as the element $(b_v)_{v\in\mathbf{a}}$ in $\mathbb{R}^{\mathbf{a}}$. We identify $\mathbb{Z}^{\mathbf{a}}$ with the free module $\sum_{v\in\mathbf{a}}\mathbb{Z}\cdot v$ by putting $(k_v)_{v\in\mathbf{a}} = \sum_{v\in\mathbf{a}} k_v v$. Also put $\mathbf{1} = (1)_{v\in\mathbf{a}} = \sum_{v\in\mathbf{a}} v$. We can define the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on $\mathbb{Z}^{\mathbf{a}}$ by $\left(\sum_{v\in\mathbf{a}} k_v v\right)^{\sigma} = \sum_{v\in\mathbf{a}} k_v(v\sigma)$.

For a positive integer m, take a non-degenerate skew-hermitian matrix T of dimension m with coefficients in K, *i.e.* $\det(T) \neq 0$ and ${}^tT^{\rho} = -T$. We view T as a skew-hermitian form on K^m by $(x_1, x_2) \to {}^tx_1Tx_2^{\rho}$ and denote by q the dimension of maximal isotropic subspace of K^m with respect to T. Take a CM-type $\Psi = (\Psi_v)_{v\in\mathbf{a}}$ of K so that each hermitian matrix $-\sqrt{-1}T^{\Psi_v}$ has signature (r_v, s_v) $(r_v + s_v = m)$ with $r_v \geq s_v$. The choice of Ψ is unique if and only if $r_v \neq s_v$ for each $v \in \mathbf{a}$. Choosing a suitable basis of K^m, we can express T as

$$T = \begin{pmatrix} \tau 1_q & & & & & \\ & t_1 & & & & \\ & & t_2 & & & \\ & & & \ddots & & \\ & & & & t_{m-2q} & \\ & & & & & \tau^{\rho}1_q \end{pmatrix}, \tag{1.1}$$

where $\tau, t_j \in K^{\times}$ so that $\tau^{\rho} = -\tau$, $t_j^{\rho} = -t_j$ $(1 \leq j \leq m-2q)$ and $\mathrm{Im}(\tau^{\Psi_v}) > 0$. Here we take t_j $(1 \leq j \leq m-2q)$ so that $\mathrm{Im}(t_j^{\Psi_v}) > 0$ if $1 \leq j \leq r_v - q$ and $\mathrm{Im}(t_j^{\Psi_v}) < 0$ if $r_v - q + 1 \leq j \leq m - 2q$ for each $v \in \mathbf{a}$. We call such T a "normal" skew-hermitian matrix with respect to Ψ. For T

as in (1.1) and $1 \le j \le m-2q$, we denote by $\Psi(T,j) = (\Psi(T,j)_v)_{v\in\mathbf{a}}$, the CM-type of K such that $\mathrm{Im}(t_j^{\Psi(T,j)_v}) > 0$ for each $v \in \mathbf{a}$. Clearly, we have $\Psi(T,j) = \Psi$ if $j \le \frac{m}{2} - q$.

Note that, for each $v \in \mathbf{a}$, a "normal" skew-hermitian matrix T with respect to Ψ can be written as

$$T = \begin{pmatrix} T_{1,v} & \\ & T_{2,v} \end{pmatrix} \tag{1.2}$$

with diagonal matrices $T_{1,v}$ and $T_{2,v}$ of degree r_v and s_v which satisfy $-\sqrt{-1}T_{1,v}^{\Psi_v} > 0$ and $-\sqrt{-1}T_{2,v}^{\Psi_v} < 0$. (The symbol > 0 means positive definite.) In case $r_v = s_v = \frac{m}{2}$ for any $v \in \mathbf{a}$, we have $q = \frac{m}{2}$ if $\det(T) \in N_{K/F}(K^\times)$ and $q = \frac{m}{2} - 1$ if $\det(T) \notin N_{K/F}(K^\times)$. In case $r_v > s_v$ for some $v \in \mathbf{a}$, the minimum of $\{s_v\}_{v\in\mathbf{a}}$ is equal to q.

Let $T \in K_m^m$ be a "normal" skew-hermitian matrix with respect to a CM-type $\Psi = (\Psi_v)_{v\in\mathbf{a}}$. Then we can define the algebraic groups corresponding to T and Ψ as follows.

$$\begin{aligned} \mathrm{U}(T,\Psi) &= \{\alpha \in \mathrm{GL}(m,K)\,|\alpha T {}^t\alpha^\rho = T\}, \\ \mathrm{U}_1(T,\Psi) &= \{\alpha \in \mathrm{GL}(m,K)\,|\alpha T {}^t\alpha^\rho = T,\ \det(\alpha) = 1\}. \end{aligned}$$

As is well known, the algebraic group $\mathrm{U}_1(T,\Psi)$ has the strong approximation property.

For each $v \in \mathbf{a}$, we can define the v-components of these algebraic groups as follows.

$$\begin{aligned} \mathrm{U}(T,\Psi)_v &= \left\{\alpha \in \mathrm{GL}(m,\mathbb{C})\,\middle|\alpha T^{\Psi_v} {}^t\overline{\alpha} = T^{\Psi_v}\right\}, \\ \mathrm{U}_1(T,\Psi)_v &= \left\{\alpha \in \mathrm{GL}(m,\mathbb{C})\,\middle|\alpha T^{\Psi_v} {}^t\overline{\alpha} = T^{\Psi_v},\ \det(\alpha) = 1\right\}. \end{aligned}$$

Now we can define the corresponding symmetric domain $\mathfrak{D}_v = \mathfrak{D}(T,\Psi)_v$ as

$$\mathfrak{D}(T,\Psi)_v = \{\mathfrak{z}_v \in \mathbb{C}_{s_v}^{r_v}\,|-\sqrt{-1}\left((T_{2,v}^{\Psi_v})^{-1} + {}^t\overline{\mathfrak{z}_v}(T_{1,v}^{\Psi_v})^{-1}\mathfrak{z}_v\right) > 0\},$$

where $T_{1,v}$, $T_{2,v}$ are as in (1.2) and > 0 means positive definite. For any $\mathfrak{z}_v \in \mathfrak{D}(T,\Psi)_v$ and any $\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{pmatrix} \in \mathrm{U}(T,\Psi)_v$ (where $A_\alpha \in \mathbb{C}_{r_v}^{r_v}$, $B_\alpha \in \mathbb{C}_{s_v}^{r_v}$, $C_\alpha \in \mathbb{C}_{r_v}^{s_v}$, $D_\alpha \in \mathbb{C}_{s_v}^{s_v}$), put

$$\alpha(\mathfrak{z}_v) = (A_\alpha \mathfrak{z}_v + B_\alpha)(C_\alpha \mathfrak{z}_v + D_\alpha)^{-1}.$$

Then the group $\mathrm{U}(T,\Psi)_v$ acts on $\mathfrak{D}(T,\Psi)_v$ as a group of holomorphic automorphism by $\mathfrak{z}_v \to \alpha(\mathfrak{z}_v)$. The automorphic factors are

$$\begin{aligned} \mu_v(\alpha,\mathfrak{z}_v) &= C_\alpha \mathfrak{z}_v + D_\alpha, \\ \lambda_v(\alpha,\mathfrak{z}_v) &= \overline{A_\alpha} - \overline{B_\alpha} T_{2,v}^{\Psi_v}{}^t\mathfrak{z}_v (T_{1,v}^{\Psi_v})^{-1}. \end{aligned}$$

We have

$$\begin{aligned} \mu_v(\beta\alpha,\mathfrak{z}_v) &= \mu_v(\beta,\alpha(\mathfrak{z}_v))\mu_v(\alpha,\mathfrak{z}_v), \\ \lambda_v(\beta\alpha,\mathfrak{z}_v) &= \lambda_v(\beta,\alpha(\mathfrak{z}_v))\lambda_v(\alpha,\mathfrak{z}_v), \\ \det(\alpha)\det(\lambda_v(\alpha,\mathfrak{z}_v)) &= \det(\mu_v(\alpha,\mathfrak{z}_v)), \end{aligned}$$

for any $\alpha,\beta \in \mathrm{U}(T,\Psi)_v$ and any $\mathfrak{z}_v \in \mathfrak{D}(T,\Psi)_v$. Clearly, $\det(\mu_v(\alpha,\mathfrak{z}_v)) \neq 0$ for any $\alpha \in \mathrm{U}(T,\Psi)_v$ and $\mathfrak{z}_v \in \mathfrak{D}(T,\Psi)_v$.

Set

$$\begin{aligned} \mathrm{U}(T,\Psi)_{\mathbf{a}} &= \prod_{v\in\mathbf{a}} \mathrm{U}(T,\Psi)_v, \\ \mathfrak{D}(T,\Psi) &= \prod_{v\in\mathbf{a}} \mathfrak{D}(T,\Psi)_v, \end{aligned}$$

and define the action of $\mathrm{U}(T,\Psi)_{\mathbf{a}}$ on $\mathfrak{D}(T,\Psi)$ componentwise.

We define an embedding of $\mathrm{U}(T,\Psi)$ into $\mathrm{U}(T,\Psi)_{\mathbf{a}}$ by $\alpha \to (\alpha^{\Psi_v})_{v\in\mathbf{a}}$ and also define an action of $\mathrm{U}(T,\Psi)$ on $\mathfrak{D}(T,\Psi)$ by

$$\alpha((\mathfrak{z}_v)_{v\in\mathbf{a}}) = \left(\alpha^{\Psi_v}(\mathfrak{z}_v)\right)_{v\in\mathbf{a}},$$

for $\alpha \in \mathrm{U}(T,\Psi)$ and $\mathfrak{z} = (\mathfrak{z}_v)_{v\in\mathbf{a}} \in \mathfrak{D}(T,\Psi)$. We write

$$\begin{aligned} \mu_v(\alpha,\mathfrak{z}) &= \mu_v(\alpha^{\Psi_v},\mathfrak{z}_v), \\ \lambda_v(\alpha,\mathfrak{z}) &= \lambda_v(\alpha^{\Psi_v},\mathfrak{z}_v), \end{aligned}$$

for $\alpha \in \mathrm{U}(T,\Psi)$, $\mathfrak{z} = (\mathfrak{z}_v)_{v\in\mathbf{a}} \in \mathfrak{D}(T,\Psi)$ and $v \in \mathbf{a}$. We denote by $\mathbf{0}$ the point $(0_{s_v}^{r_v})_{v\in\mathbf{a}} \in \mathfrak{D}(T,\Psi)$.

Set $k = (k_v)_{v\in\mathbf{a}} \in \mathbb{Z}^{\mathbf{a}}$. For $\alpha \in \mathrm{U}(T,\Psi)$ and a $\mathbb{C}$-valued function f on $\mathfrak{D}(T,\Psi)$, We define a $\mathbb{C}$-valued function $f|_k\alpha$ on $\mathfrak{D}(T,\Psi)$ by

$$(f|_k\alpha)(\mathfrak{z}) = f(\alpha(\mathfrak{z})) \prod_{v\in\mathbf{a}} \det(\mu_v(\alpha,\mathfrak{z}))^{-k_v}.$$

For any congruence subgroup Γ of $\mathrm{U}(T,\Psi)$, we denote by $\mathcal{M}_k(T,\Psi)(\Gamma)$, the set of all holomorphic functions on $\mathfrak{D}(T,\Psi)$ such that $f|_k\gamma = f$ for any $\gamma \in \Gamma$. An element of $\mathcal{M}_k(T,\Psi)(\Gamma)$ is called a holomorphic modular form of weight k with respect to Γ. We denote by $\mathcal{M}_k(T,\Psi)$ the union of $\mathcal{M}_k(T,\Psi)(\Gamma)$ for all congruence subgroups Γ of $\mathrm{U}(T,\Psi)$.

We need to consider adelizations of algebraic groups. Put

$$\mathrm{U}(T,\Psi)_A = \left\{x \in \mathrm{GL}(m, K_A) \,\middle|\, xT{}^t x^\rho = T\right\}.$$

Note that $x_{\mathfrak{p}}$, the $\mathfrak{p}$-component of x, belongs to $\mathrm{GL}(m,\mathcal{O}_{\mathfrak{p}})$ for almost all non-archimedean primes $\mathfrak{p}$ of K. We also put

$$\mathrm{U}_1(T,\Psi)_A = \{x \in \mathrm{U}(T,\Psi)_A \,|\det(x) = 1\}.$$

We denote by $\mathrm{U}(T,\Psi)_{\mathbf{h}}$ and $\mathrm{U}_1(T,\Psi)_{\mathbf{h}}$, the non-archimedean components of $\mathrm{U}(T,\Psi)_A$ and $\mathrm{U}_1(T,\Psi)_A$, respectively, and view $\mathrm{U}(T,\Psi)_{\mathbf{a}}$ and $\mathrm{U}_1(T,\Psi)_{\mathbf{a}}$, as the archimedean components of $\mathrm{U}(T,\Psi)_A$ and $\mathrm{U}_1(T,\Psi)_A$, respectively. We regard $\mathrm{U}(T,\Psi)$ and $\mathrm{U}_1(T,\Psi)$, as subgroups of $\mathrm{U}(T,\Psi)_A$ and $\mathrm{U}_1(T,\Psi)_A$, by diagonal embeddings. As is well known, the algebraic group $\mathrm{U}_1(T,\Psi)$ has the strong approximation property.

For symplectic group $\mathrm{Sp}(q,F)$, take the corresponding symmetric domain $\mathfrak{H}_q^{\mathbf{a}} = \left\{z = (z_v)_{v\in\mathbf{a}} \in (\mathbb{C}_q^q)^{\mathbf{a}} \,|\, {}^t z_v = z_v,\ \mathrm{Im}(z_v) > 0 \text{ for each } v \in \mathbf{a}\right\}$. For $z = (z_v)_{v\in\mathbf{a}} \in \mathfrak{H}_q^{\mathbf{a}}$, put

$$\varepsilon_0(T,\Psi)(z) = \begin{pmatrix} 0_{s_v-q}^{q} & (z_v - \frac{\tau^{\Psi_v}}{2}\cdot 1_q)\cdot(z_v + \frac{\tau^{\Psi_v}}{2}\cdot 1_q)^{-1} \\ 0_{s_v-q}^{r_v-q} & 0_q^{r_v-q} \end{pmatrix}_{v\in\mathbf{a}},$$

where r_v, s_v are as above. Then $\varepsilon_0(T,\Psi)$ gives a holomorphic embedding of $\mathfrak{H}_q^{\mathbf{a}}$ into $\mathfrak{D}(T,\Psi)$. This is compatible with the injection $I_0(T,\Psi)$ of $\mathrm{Sp}(q,F)$ into $\mathrm{U}_1(T,\Psi)$ defined by

$$\begin{aligned} &I_0(T,\Psi)\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \\ &= \begin{pmatrix} 1_q & 0 & -\frac{\tau}{2}\cdot 1_q \\ 0 & 1_{m-2q} & 0 \\ 1_q & 0 & \frac{\tau}{2}\cdot 1_q \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 & \alpha_2 \\ 0 & 1_{m-2q} & 0 \\ \alpha_3 & 0 & \alpha_4 \end{pmatrix} \begin{pmatrix} 1_q & 0 & -\frac{\tau}{2}\cdot 1_q \\ 0 & 1_{m-2q} & 0 \\ 1_q & 0 & \frac{\tau}{2}\cdot 1_q \end{pmatrix}^{-1}, \end{aligned}$$

where $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in \mathrm{Sp}(q,F)$ with $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in F_q^q$. We have

$$I_0(T,\Psi)(\alpha)\left(\varepsilon_0(T,\Psi)(z)\right) = \varepsilon_0(T,\Psi)\left(\alpha(z)\right)$$

for any $\alpha \in \mathrm{Sp}(q, F)$ and $z \in \mathfrak{H}_q^{\mathbf{a}}$. We can define pull-back of modular forms by $\varepsilon_0(T, \Psi)$. For $k = (k_v)_{v\in\mathbf{a}} \in \mathbb{Z}^{\mathbf{a}}$ and $f \in \mathcal{M}_k(T, \Psi)$, define a function $f|\varepsilon_0(T, \Psi)$ on $\mathfrak{H}_q^{\mathbf{a}}$ as

$$(f|\varepsilon_0(T,\Psi))(z) = f(\varepsilon_0(T,\Psi)(z)) \prod_{v\in\mathbf{a}} \det\left((\tau^{\Psi_v})^{-1} z_v + \frac{1}{2}\cdot 1_q\right)^{-k_v},$$

where $z = (z_v)_{v\in\mathbf{a}} \in \mathfrak{H}_q^{\mathbf{a}}$. Then $f \in \mathcal{M}_k(T, \Psi)$ is a holomorphic modular form on $\mathfrak{H}_q^{\mathbf{a}}$ with respect to some congruence subgroup of $\mathrm{Sp}(q, F)$.

2 Galois action

Though modular forms (in this lecture) have no Fourier expansions, we can give a Galois action on them concretely, using the pull-back by $\varepsilon_0(T, \Psi)$.

For a CM-field K, its CM-type Ψ, and any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we can define another CM-type $\Psi\sigma = \{\psi\sigma \,|\, \psi \in \Psi\}$ of K. We denote by K_Ψ^* (or simply K^* if there is no fear of confusion), the corresponding algebraic number field to $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \,|\, \Psi\sigma = \Psi\}$ which is a finite index subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. As is well known, K_Ψ^* is a CM-field contained in the Galois closure of K. Viewing Ψ as a union of $[F : \mathbb{Q}]$ different right $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-cosets in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we define a CM-type Ψ^* of K_Ψ^* as follows

$$\mathrm{Gal}(\overline{\mathbb{Q}}/K_\Psi^*)\Psi^* = \left(\mathrm{Gal}(\overline{\mathbb{Q}}/K)\Psi\right)^{-1}.$$

We call Ψ^* by "the reflex of Ψ" and the couple (K_Ψ^*, Ψ^*) by "the reflex of (K, Ψ)". From the definition, we have $(K_\Psi^*)^\sigma = K_{\Psi\sigma}^*$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (or $\in \mathrm{Aut}(\mathbb{C})$). By N'_Ψ, we denote the group homomorphism $x \to \prod_{\psi^*\in\Psi^*} x^{\psi^*}$ from $K_\Psi^{*\times}$ to $K^\times$. It is a morphism of algebraic groups if we view $K_\Psi^{*\times}$ and $K^\times$ as algebraic groups defined over $\mathbb{Q}$, and so it can naturally be extended to the homomorphism of $(K_\Psi^*)_A^\times$ to $K_A^\times$.

For a CM-type Ψ and any $\sigma \in \mathrm{Aut}(\mathbb{C})$, a certain idele class $g_\Psi(\sigma) \in K_A^\times / K^\times K_\infty^\times$ is defined in [3] (or essentially in [2]). Take an abelian variety A of type (K, Ψ) with a $\mathcal{O}_K$-lattice L in K and a complex analytic isomorphism Θ of $\mathbb{C}^{\mathbf{a}}/L^\Psi$ onto A. (See, [9].) We denote by A_{tor} the subgroup of all torsion points of A, which coincides with the image of K/L by $\Theta \circ \Psi$. Next take A^σ. Then it is an abelian variety of type $(K, \Psi\sigma)$ and we have the following commutative diagram

$$\begin{array}{ccc} K/L & \xrightarrow{\Theta\circ\Psi} & A_{\mathrm{tor}} \\ {\scriptstyle\times a}\downarrow & & \downarrow{\scriptstyle\sigma} \\ K/aL & \xrightarrow{\Theta_a\circ(\Psi\sigma)} & A^{\sigma}_{\mathrm{tor}} \end{array}$$

with some $a \in K_A^\times$ and complex analytic isomorphism Θ_a of $\mathbf{C}^{\mathbf{a}}/(aL)^{\Psi\sigma}$ onto A^σ. The coset $aK^\times K_\infty^\times$ is uniquely determined only by (K, Ψ) and σ (not depending on A or L). We denote this coset by $g_\Psi(\sigma)$. For $a \in g_\Psi(\sigma)$, we have $aa^\rho \in \chi(\sigma)F^\times F_\infty^\times$, where $\chi(\sigma) \in \prod_p \mathbf{Z}_p^\times \subset \mathbf{Q}_A^\times$ which satisfies $[\chi(\sigma)^{-1}, \mathbf{Q}] = \sigma|_{\mathbf{Q}_{ab}}$. We define $\iota(\sigma, a) \in F^\times$ by $\frac{\chi(\sigma)}{aa^\rho} \in \iota(\sigma, a)F_\infty^\times$. If σ is trivial on K_Ψ^*, we have $g_\Psi(\sigma) = N'_\Psi(b)K^\times K_\infty^\times$ with $b \in (K_\Psi^*)_A^\times$ such that $[b^{-1}, K_\Psi^*] = \sigma|_{K^*_{\Psi ab}}$; this fact is a main theorem of complex multiplication theory of [9]. Note that $g_\Psi(\sigma_1)g_{\Psi\sigma_1}(\sigma_2) = g_\Psi(\sigma_1\sigma_2)$.

Take CM-types $\Psi(T, j)$ $(1 \le j \le m-2q)$ as in section 1, and set

$$C_{(T,\Psi)}(\mathbf{C}) = \left\{ (\sigma; T, \Psi; \underline{a}) \;\middle|\; \begin{array}{l} \sigma \in \mathrm{Aut}(\mathbf{C}), \\ \underline{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-2q} \end{pmatrix} \in (K_{\mathbf{h}}^\times)^{m-2q+1}, \\ \text{where } a_0 \in g_\Psi(\sigma), \\ \text{and } a_j \in g_{\Psi(T,j)}(\sigma) \text{ for } 1 \le j \le m-2q, \end{array} \right\},$$

where $K_{\mathbf{h}}^\times$ denotes the non-archimedean component of the idele group $K_A^\times$. Note that, for any $\sigma \in \mathrm{Aut}(\mathbf{C})$, there exists some $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbf{C})$.

For any $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbf{C})$, take $B(\sigma; T, \Psi; \underline{a}) \in \mathrm{GL}(m, K_{\mathbf{h}})$ as

$$B(\sigma; T, \Psi; \underline{a}) = \begin{pmatrix} (\frac{1}{2} + \frac{a_0a_0^\rho}{2})1_q & & & & (\frac{1}{2} - \frac{a_0a_0^\rho}{2})1_q \\ & a_1^\rho & & & \\ & & \ddots & & \\ & & & a_{m-2q}^\rho & \\ (\frac{1}{2} - \frac{a_0a_0^\rho}{2})1_q & & & & (\frac{1}{2} + \frac{a_0a_0^\rho}{2})1_q \end{pmatrix}.$$

The following theorem is the main theorem of [12].

Theorem *Let T be a "normal" skew-hermitian matrix with respect to a CM-type Ψ, which is expressed as in (1.1). For any $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbf{C})$,* *take* $\tilde{T} \in K_m^m$ *as*

$$\tilde{T} = \begin{pmatrix} \iota(\sigma, a_0)\tau \cdot 1_q & & & & \\ & \iota(\sigma, a_1)t_1 & & & \\ & & \ddots & & \\ & & & \iota(\sigma, a_{m-2q})t_{m-2q} & \\ & & & & \iota(\sigma, a_0)\tau^\rho \cdot 1_q \end{pmatrix}.$$

Then $\tilde{T}$ *is a "normal" skew-hermitian matrix with respect to the CM-type* $\Psi\sigma$. *Given any* $f \in \mathcal{M}_k(T, \Psi)$, *take an open compact subgroup* $C_{\mathbf{h}}$ *of* $\mathrm{U}(T, \Psi)_{\mathbf{h}}$ *so that* $f \in \mathcal{M}_k(T, \Psi)\left((\mathrm{U}(T, \Psi)_{\mathbf{a}} \times C_{\mathbf{h}}) \cap \mathrm{U}(T, \Psi)\right)$. *Then there exists* $f^{(\sigma;T,\Psi;\underline{a})} \in \mathcal{M}_{k^\sigma}(\tilde{T}, \Psi\sigma)$ *which satisfies the following property.*
(i) *In case* $q > 0$, *we have*

$$(f^{(\sigma;T,\Psi;\underline{a})}|_{k^\sigma}\tilde{\alpha})|\varepsilon_0(\tilde{T}, \Psi\sigma) = \{(f|_k\alpha)|\varepsilon_0(T, \Psi)\}^\sigma. \tag{2.1}$$

for any $\alpha \in \mathrm{U}(T, \Psi)$ *and* $\tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma)$ *such that*

$$\alpha_{\mathbf{h}} \in C_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a})\tilde{\alpha}_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a})^{-1} \tag{2.2}$$

where $\alpha_{\mathbf{h}}$ *and* $\tilde{\alpha}_{\mathbf{h}}$ *mean the non-archimedean parts of* α *and* $\tilde{\alpha}$. *The action of* σ *in the right hand side of (2.1) is as defined in (0.2).*
(ii) *In case* $q = 0$, *we have*

$$(f^{(\sigma;T,\Psi;\underline{a})}|_{k^\sigma}\tilde{\alpha})(\mathbf{0}) = \{(f|_k\alpha)(\mathbf{0})\}^\sigma,$$

for any α *and* $\tilde{\alpha}$ *as in (2.2).*

Remark1 We can easily prove that $\tilde{T}$ is "normal" with respect to $\Psi\sigma$. Moreover, the dimension of the maximal isotropic subspace with respect to $\tilde{T}$ is also q, the signature of $-\sqrt{-1} \cdot \tilde{T}^{\Psi_v\sigma}$ is (r_v, s_v) for each $v \in \mathbf{a}$, and we obtain $\Psi(\tilde{T}, j) = \Psi(T, j)\sigma$ for $1 \leq j \leq m - 2q$.

Remark2 For any $\tilde{x}_{\mathbf{h}} \in \mathrm{U}(\tilde{T}, \Psi\sigma)_{\mathbf{h}}$, we can easily verify that

$$B(\sigma; T, \Psi; \underline{a})\tilde{x}_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a})^{-1} \in \mathrm{U}(T, \Psi)_{\mathbf{h}}.$$

It is because we have

$$B(\sigma; T, \Psi; \underline{a})\tilde{T}_{\mathbf{h}}{}^t B(\sigma; T, \Psi; \underline{a})^\rho = \chi(\sigma) T_{\mathbf{h}},$$

where $\tilde{T}_{\mathbf{h}}$ and $T_{\mathbf{h}}$ denote the non-archimedean components of $\tilde{T}$ and T, respectively.

Remark3 For any $\tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma)$, there exists $\alpha \in \mathrm{U}(T, \Psi)$ which satisfies (2.2). Because we have $\begin{pmatrix} \det(\tilde{\alpha}) & \\ & 1_{m-1} \end{pmatrix} \in \mathrm{U}(T, \Psi)$ and

$$\begin{pmatrix} \det(\tilde{\alpha}) & \\ & 1_{m-1} \end{pmatrix}^{-1} B(\sigma; T, \Psi; \underline{a})\tilde{\alpha}_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a})^{-1} \in \mathrm{U}_1(T, \Psi)_A,$$

the strong approximation property of $\mathrm{U}_1(T, \Psi)$ shows that.

Remark4 Clearly the modular form $f^{(\sigma;T,\Psi;\underline{a})}$ is uniquely determined, since the set $\bigcup_{\tilde{\alpha}\in \mathrm{U}(\tilde{T},\Psi\sigma)} \tilde{\alpha} \circ \varepsilon_0(\tilde{T}, \Psi\sigma)(\mathfrak{H}_q^{\mathbf{a}})$ (or $\left\{\tilde{\alpha}(\mathbf{0}) \middle| \tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma)\right\}$ if $q = 0$) is dense in $\mathfrak{D}(\tilde{T}, \Psi\sigma)$.

Remark5 Let $\mathcal{S}_k(T, \Psi)$ be the space of cusp forms contained in $\mathcal{M}_k(T, \Psi)$. Then we have $\mathcal{S}_k(T, \Psi)^{(\sigma;T,\Psi;\underline{a})} = \mathcal{S}_{k^\sigma}(\tilde{T}, \Psi\sigma)$.

3 Relation with Hecke operators

We can easily prove that our Galois action is compatible with Hecke operators.

To define Hecke operators, we have to consider adelized modular forms. Let D be a subgroup of $\mathrm{U}(T, \Psi)_A$ which is written as $D = \mathrm{U}(T, \Psi)_{\mathbf{a}} \times D_{\mathbf{h}}$ with some open compact subgroup $D_{\mathbf{h}}$ of $\mathrm{U}(T, \Psi)_{\mathbf{h}}$. For any $k \in \mathbb{Z}^{\mathbf{a}}$, we denote by $\mathcal{M}_k(T, \Psi)(D)$, the set of all functions $\mathbf{f} : \mathrm{U}(T, \Psi)_A \to \mathbb{C}$ satisfying the following conditions (1)–(3).

(1) $\mathbf{f}(xd_{\mathbf{h}}) = \mathbf{f}(x)$ for any $d_{\mathbf{h}} \in D_{\mathbf{h}}$.

(2) $\mathbf{f}(\beta x) = \mathbf{f}(x)$ for any $\beta \in \mathrm{U}(T, \Psi)$.

(3) For each $p \in \mathrm{U}(T, \Psi)_{\mathbf{h}}$, there exists an element $f_p \in \mathcal{M}_k(T, \Psi)$ such that $\mathbf{f}(py) = (f_p|_k y)(\mathbf{0})$ for any $y \in \mathrm{U}(T, \Psi)_{\mathbf{a}}$.

Then we easily have $f_p \in \mathcal{M}_k(T, \Psi)(pDp^{-1} \cap \mathrm{U}(T, \Psi))$. Using the strong approximation property of $\mathrm{U}_1(T, \Psi)$, we can take a finite subset $\mathcal{B}$ of $\mathrm{U}(T, \Psi)_{\mathbf{h}}$ so that

$$\mathrm{U}(T, \Psi)_A = \bigsqcup_{b\in\mathcal{B}} \mathrm{U}(T, \Psi)bD \qquad \text{(disjoint union).} \tag{3.1}$$

Then the map $\mathbf{f} \to (f_b)_{b\in\mathcal{B}}$ gives a bijection of $\mathcal{M}_k(T, \Psi)(D)$ onto $\prod_{b\in\mathcal{B}} \mathcal{M}_k(T, \Psi)(bDb^{-1} \cap \mathrm{U}(T, \Psi))$.

We write simply $\mathbf{f} \leftrightarrow (f_p)_p$ or $\mathbf{f} \leftrightarrow (f_b)_{b\in\mathcal{B}}$ to indicate that f_p (resp. f_b) is determined by $\mathbf{f}$ for each $p \in \mathrm{U}(T, \Psi)_{\mathbf{h}}$ (resp. $b \in \mathcal{B}$) as in (3) above.

We denote by $\mathcal{S}_k(T,\Psi)(D)$ the set of all $\mathbf{f} \leftrightarrow (f_p)_p \in \mathcal{M}_k(T,\Psi)(D)$ so that $f_p \in \mathcal{S}_k(T,\Psi)$ for each $p \in \mathrm{U}(T,\Psi)_{\mathbf{h}}$.

Let $\mathbf{f} \leftrightarrow (f_b)_{b\in\mathcal{B}}, \mathbf{g} \leftrightarrow (g_b)_{b\in\mathcal{B}} \in \mathcal{M}_k(T,\Psi)(D)$ and assume that either $\mathbf{f}$ or $\mathbf{g}$ belongs to $\mathcal{S}_k(T,\Psi)(D)$. Then we can define the inner product $<,>$ of $\mathbf{f}$ and $\mathbf{g}$ by

$$<\mathbf{f},\mathbf{g}>=|\mathcal{B}|^{-1}\sum_{b\in\mathcal{B}}<f_b,g_b>,$$

where $|\mathcal{B}|$ denotes the number of elements in $\mathcal{B}$. We can easily verify that $<\mathbf{f},\mathbf{g}>$ is independent of the choice of $\mathcal{B}$.

The Galois action can also be constructed on the space of adelized modular forms.

Theorem. *For any* $\mathbf{f} \leftrightarrow (f_p)_p \in \mathcal{M}_k(T,\Psi)(D)$ *and any* $(\sigma;T,\Psi;\underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, *there exists* $f^{(\sigma;T,\Psi;\underline{a})} \leftrightarrow (\tilde{f}_{\tilde{p}})_{\tilde{p}} \in \mathcal{M}_{k^\sigma}(\tilde{T},\Psi\sigma)(\tilde{D})$ *such that* $\tilde{f}_{\tilde{p}} = f_p^{(\sigma;T,\Psi;\underline{a})}$ *if* $\tilde{p} = B(\sigma;T,\Psi;\underline{a})^{-1}pB(\sigma;T,\Psi;\underline{a})$. *Here* $\tilde{D}$ *is a subgroup of* $\mathrm{U}(\tilde{T},\Psi\sigma)_A$ *defined by* $\tilde{D} = \mathrm{U}(\tilde{T},\Psi\sigma)_{\mathbf{a}} \times \tilde{D}_{\mathbf{h}}$, *where*

$$\tilde{D}_{\mathbf{h}} = B(\sigma;T,\Psi;\underline{a})^{-1}D_{\mathbf{h}}B(\sigma;T,\Psi;\underline{a}) \qquad (\subset \mathrm{U}(\tilde{T},\Psi\sigma)_{\mathbf{h}}).$$

We can prove that this action of $(\sigma;T,\Psi;\underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$ is compatible with the action of the Hecke ring, that is,

$$(\mathbf{f}|DxD)^{(\sigma;T,\Psi;\underline{a})} = \mathbf{f}^{(\sigma;T,\Psi;\underline{a})}|\tilde{D}\tilde{x}\tilde{D},$$

where $\mathbf{f} \in \mathcal{M}_k(T,\Psi)(D)$ and DxD, $\tilde{D}\tilde{x}\tilde{D}$ are elements of the both Hecke rings (corresponding to D and $\tilde{D}$), so that

$$\tilde{x} = B(\sigma;T,\Psi;\underline{a})^{-1}xB(\sigma;T,\Psi;\underline{a}).$$

(For details about the Hecke rings, see [8] or [13].)

Let $\mathbf{f} \in \mathcal{S}_k(T,\Psi)(D)$ be a Hecke eigen cusp form corresponding to D, with eigenvalues

$$\mathbf{f}|DxD = \lambda(x,\mathbf{f})\cdot\mathbf{f}.$$

Then $\mathbf{f}^{(\sigma;T,\Psi;\underline{a})} \in \mathcal{S}_{k^\sigma}(\tilde{T},\Psi\sigma)(\tilde{D})$ is a Hecke eigen cusp form (corresponding to $\tilde{D}$) with eigenvalues $\lambda(\tilde{x},\mathbf{f}^{(\sigma;T,\Psi;\underline{a})}) = \lambda(x,\mathbf{f})^\sigma$. Since $\lambda(x^{-1},\mathbf{f}) = \overline{\lambda(x,\mathbf{f})}$ holds, we have $(\overline{\lambda(x,\mathbf{f})})^\sigma = \overline{\lambda(x,\mathbf{f})^\sigma}$ for any $\sigma \in \mathrm{Aut}(\mathbb{C})$. This implies that $\lambda(x,\mathbf{f})$ is contained in a CM-field.

We can easily obtain that $C(\sigma;T,\Psi;\underline{a}) = B(\rho\sigma\rho;T,\Psi;\underline{a}^\rho)^{-1}B(\sigma;T,\Psi;\underline{a}) \in \mathrm{U}(\tilde{T},\Psi\sigma)_{\mathbf{h}}$. Then we have the following conjecture.

Conjecture. *Let* $\mathbf{f}, \mathbf{g}_1, \mathbf{g}_2 \in \mathcal{S}_k(T, \Psi)(D)$ *be Hecke eigen cusp forms having the same eigenvalues. Assume that* $\mathbf{f} \neq 0$. *For any* $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, *we have* $< \mathbf{f}^{(\rho\sigma\rho; T, \Psi; \underline{a}^\rho)} | C(\sigma; T, \Psi; \underline{a}), \mathbf{f}^{(\sigma; T, \Psi; \underline{a})} > \neq 0$ *and*

$$\frac{< \mathbf{g}_1^{(\rho\sigma\rho; T, \Psi; \underline{a}^\rho)} | C(\sigma; T, \Psi; \underline{a}), \mathbf{g}_2^{(\sigma; T, \Psi; \underline{a})} >}{< \mathbf{f}^{(\rho\sigma\rho; T, \Psi; \underline{a}^\rho)} | C(\sigma; T, \Psi; \underline{a}), \mathbf{f}^{(\sigma; T, \Psi; \underline{a})} >} = \left\{ \frac{< \mathbf{g}_1, \mathbf{g}_2 >}{< \mathbf{f}, \mathbf{f} >} \right\}^\sigma,$$

where $|C(\sigma; T, \Psi; \underline{a})$ *denotes the translation by* $|C(\sigma; T, \Psi; \underline{a})$.

The latest result (the main theorem of [13]) is as follows.

Theorem. *The previous conjecture is true if the weight* $k = \kappa\mathbf{1}$ *with even integer* κ *such that* $\kappa > 2m$.

This is proved by so-called "doubling method" introduced in [8].

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