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DYNAMICAL ZETA FUNCTIONS FOR EXPANDING SEMI-FLOWS

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1. Introduction

The aim of this article is to exhibit some results about dynamical zeta functions for suspension semi-flows of the angle-multiplying maps on the circle. The proofs and more detailed argument will be provided in the forthcoming paper with the same title[15].

For an Anosov flow, the dynamical zeta function is the function defined by

$$\zeta(s)=\prod_{\gamma}(1-e^{-s|\gamma|})^{-1} \tag{1}$$

where the product is taken over all prime periodic orbits γ and $|\gamma|$ denotes the prime period of γ. Though the right hand side is well-defined for complex numbers $s\in\mathbb{C}$ with real part larger than the topological entropy h_{top}, it extends meromorphically to much larger region. For instance, D. Fried[7] showed that $\zeta(s)$ extends to a meromorphic function on the complex plane $\mathbb{C}$ provided that the Anosov flow is real-analytic.

In this paper, we are interested in the singularities, *zeros* and *poles*, of such meromorphic extension. It is well known that, $\zeta(s)$ is a holomorphic function without zeros on the region $\Re(s)>h_{top}$ and, if the flow is mixing, h_{top} is the unique pole on the line $\Re(s)=h_{top}$. More recently, D. Dolgopyat showed, under some reasonable conditions, that there exists some $\epsilon>0$ such that $\zeta(s)$ has no pole or zero other than h_{top} on the region $\Re(s)>h_{top}-\epsilon$.

The geodesic flows of closed surfaces with constant negative curvature are types of the Anosov flow. And the famous results[12, 9] of Selberg give much more precise description to the singularities of $\zeta(s)$ for such flows, which motivated the study of dynamical zeta functions: $\zeta(s)$ has only finitely many poles on the region $\Re(s)>h_{top}/2$ and countably many poles on the line $\Re(s)=h_{top}/2$ while no zeros on $\Re(s)\geq 0$. In view of this result, we pose a question how small we may take a real number h so that $\zeta(s)$ has only finitely many poles on the

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region $\Re(s) > h$. In this paper, we study this question in a simplified setting: the suspension semi-flows of angle-multiplying maps on the circle. This class of expanding semi-flows was studied by D. Ruelle[11] as a simplified model of Anosov flows and then by M. Pollicott[10] and the author[14] more recently.

Let $\ell \geq 2$ and $r \geq 3$ be integers. Let $\tau : S^1 \to S^1$ be the angle-multiplying map on $S^1 = \mathbb{R}/\mathbb{Z}$ defined by $\tau(x) = \ell x$. Let $C^r_+(S^1)$ be the set of positive-valued C^rfunctions on S^1. For each $f \in C^r_+(S^1)$, we consider the subset

$$X_f = \{(x,s) \in S^1 \times \mathbb{R} \mid 0 \leq s < f(x)\}$$

of the cylinder $S^1 \times \mathbb{R}$. The suspension semi-flow $\mathbf{T}_f = \{T^t_f : X_f \to X_f\}_{t\geq 0}$ for the ceiling function f is the semi-flow on X_f, in which each point on X_f moves upward with unit speed and, at the instant it reaches to the upper boundary of X_f, it jumps down to the bottom side with the x-coordinate transfered by τ. (Figure 1) The time-t-map $T^t_f : X_f \to X_f$ of the semi-flow $\mathbf{T}_f$ is given as

$$T^t_f(x,s) = \left(\tau^{n(x,s+t;f)}(x), s + t - f^{(n(x,s+t;f))}(x)\right)$$

where $f^{(n)}(x) = \sum_{i=0}^{n-1} f(\tau^i(x))$ for $n \geq 0$ and

$$n(x,t;f) := \max\{n \geq 0 \mid f^{(n)}(x) \leq t\}.$$

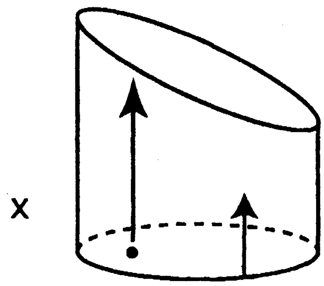

FIGURE 1. The semiflow $\mathbf{T}_f$

Let $m = m_f$ be the normalization of the standard Lebesgue measure on X_f. This is an ergodic invariant measure for $\mathbf{T}_f$. For a point $z = (x,s)$ and $t \geq 0$, we put $E(z,t;f) := \ell^{n(x,s+t;f)}$. Note that

$$(2) \qquad \sum_{w:T^t(w)=z} E(w,t;f)^{-1} = 1 \qquad \text{for any } z \in X_f \text{ and } t \geq 0.$$

For the semi-flow $\mathbf{T}_f$ and a real number α, we define

$$\lambda_{\min}(\mathbf{T}_f, \alpha) := \lim_{t\to\infty} \left(\inf_{z\in X_f} E(z,t;f)^\alpha \right)^{1/t}$$

and

$$\lambda_{\max}(\mathbf{T}_f, \alpha) := \lim_{t\to\infty} \left(\sup_{z\in X_f} E(z,t;f)^\alpha \right)^{1/t} .$$

Then the minimum and maximum expansion rate of the semi-flow $\mathbf{T}_f$ is given as $\lambda_{\min}(\mathbf{T}_f) := \lambda_{\min}(\mathbf{T}_f, 1)$ and $\lambda_{\max}(\mathbf{T}_f) := \lambda_{\max}(\mathbf{T}_f, 1)$ respectively.

Let $\zeta_f(s)$ be the dynamical zeta function of the semi-flow $\mathbf{T}_f$ defined by (1). The main results in the next section will give

Theorem 1.1. *Under a C^r generic condition on the ceiling function $f \in C^r_+(S^1)$, the function $1/\zeta_f(s)$ extends to the region $\Re(s) > \lambda_{\max}(\mathbf{T}_f)/2$ as a holomorphic function and has only finitely many zeros on the region*

(3) $$\Re(s) > (\lambda_{\max}(\mathbf{T}_f)/2) + \epsilon \qquad \textit{for each } \epsilon > 0.$$

Notice that, for the geodesic flows of closed surfaces with constant negative curvature, the topological entropy of the flow coincides with the constant expansion rate in the unstable direction. So, if we translate the theorem above to such cases, the bound (3) would be best-possible. Still, the bound (3) is definitely not optimal in the case where $\log f$ is far from constant: $\lambda_{\max}(\mathbf{T}_f)/2$ may be larger than the topological entropy $h_{top}(\mathbf{T}_f)$, so that the theorem may be vacuous. We would like to end this introduction by posing a question: *Does the theorem above hold true if we substitute $h_{top}(\mathbf{T}_f)/2$ for $\lambda_{\max}(\mathbf{T}_f)/2$?*

2. Results

We consider the semi-flow $\mathbf{T}_f$ for $f \in C^r_+(S^1)$ defined in the previous section.

2.1. Ruelle transfer operators and Selberg functions.

For $\alpha \in \mathbb{R}$, we define the semi-group of Ruelle transfer operators

$$\mathcal{L}^t_{f,\alpha} : L^1(X_f, m_f) \to L^1(X_f, m_f), \quad t \geq 0,$$

by

$$\mathcal{L}^t_{f,\alpha}(u)(z) = \sum_{w\in(T^t_f)^{-1}(z)} E(w,t;f)^{-\alpha} \cdot u(w).$$

The Selberg functions for these semi-groups are defined by

$$d_{f,\alpha}(s) = \exp\left(\sum_{\gamma}\sum_{n=1}^{\infty} \frac{\exp(-sn|\gamma|)\cdot E(\gamma)^{(-\alpha+1)n}}{n\cdot(E(\gamma)^n - 1)}\right)$$

for $s \in \mathbb{C}$ with sufficiently large real part, where the sum $\sum_\gamma$ is taken over all prime periodic orbit γ of the semi-flow $\mathbf{T}_f$, $|\gamma|$ denotes the prime period of γ and $E(\gamma) := E(z, |\gamma|; f)$ for any (or some) $z \in \gamma$. In a heuristical sense, we may regard this function as

$$\exp\left(\mathrm{Tr}\left(\int_{+0}^{\infty} \frac{1}{t} e^{-st} \mathcal{L}_{f,\alpha}^t dt\right)\right). \tag{4}$$

And the dynamical zeta funtion $\zeta_f(s)$ defined in the previous section is given by

$$\zeta_f(s) = d_{f,1}(s)/d_{f,0}(s). \tag{5}$$

So we expect that the zeros (resp. poles) of the meromorphic extension of $\zeta_f(s)$ are related to the eigenvalues of the operators $\mathcal{L}_{f,1}^t$ (resp. $\mathcal{L}_{f,0}^t$).

2.2. **Transversality exponent.** For $t \geq 0$ and $z \in X_f$, we define the differential $(DT_f^t)_z : \mathbb{R}^2 \to \mathbb{R}^2$ of T_f^t at $z \in X_f$ in the usual way if $z \in X_f^\circ \cap (T_f^t)^{-1}(X_f^\circ)$ and, otherwise, as the limit $(DT_f^t)_z = \lim_{\epsilon\to+0}(DT_f^t)_{z+(0,\epsilon)}$. Let $(DT_f^t)_z^{tr}$ be the transpose of $(DT_f^t)_z$. We will consider the cone

$$\mathbf{C}^*(\theta) = \{(x,y) \in \mathbb{R}^2 \mid |x| \leq \theta|y|\} \qquad \text{for } \theta > 0.$$

We henceforth fix a real number $\ell^{-1} < \gamma_0 < 1$ and put

$$\mathbf{C}_f^* := \mathbf{C}^*(\theta_f) = \{(x,y) \in \mathbb{R}^2 \mid |x| \leq \theta_f|y|\}$$

where

$$\theta_f = (1/(\gamma_0\ell - 1)) \cdot \max_{x\in S^1}|f'(x)|.$$

Then the cone $\mathbf{C}_f^*$ is forward-invariant for the differentials of $\mathbf{T}_f$ in the sense that

(6)

$$((DT_f^t)_z^{tr})^{-1}(\mathbf{C}_f^*) \Subset \mathbf{C}(\gamma_0\theta_f) \Subset \mathbf{C}_f^* \quad \text{for } z = (x,s) \in X_f \text{ and } t \geq f(x) - s$$

where $\mathbf{C} \Subset \mathbf{C}'$ implies that $\mathbf{C} \setminus \{0\}$ is contained in the interior of $\mathbf{C}'$.

Suppose that $T^t(\zeta) = T^t(\zeta')$ for some $\zeta, \zeta' \in X_f$ and $t \geq 0$. We will write $\zeta \pitchfork \eta$ if the cones $((DT_f^t)_\zeta^{tr})^{-1}(\mathbf{C}_f^*)$ and $((DT_f^t)_\eta^{tr})^{-1}(\mathbf{C}_f^*)$ intersect only at the origin and write $\zeta \not\pitchfork \eta$ otherwise. (Note that this relation does not depend on the choice of t.) Using this relation, we define two exponents

$$\mathbf{m}_\alpha(f) = \limsup_{t\to\infty} \mathbf{m}_\alpha(t; f)^{1/t} \qquad \text{for } \alpha = 0, 1,$$

where

$$\mathbf{m}_0(t;f) = \max_{z\in X} \max_{\zeta'\in(T_f^t)^{-1}(z)} E(\zeta',t;f)\#\{\zeta \in (T_f^t)^{-1}(z) \mid \zeta \not\pitchfork \zeta'\}$$

and

$$\mathbf{m}_1(t;f) = \max_{z\in X} \max_{\zeta'\in(T_f^t)^{-1}(z)} \sum_{\zeta:\zeta\not\pitchfork\zeta'} \frac{1}{E(\zeta,t;f)}.$$

For the exponent $\mathbf{m}_1(t;f)$, we showed in [14]

Theorem 2.1. *The semi-flow $(\mathbf{T}_f, m_f)$ is mixing if and only if $\mathbf{m}_1(f) < 1$. For any $\rho > 1$, there exists an open and dense subset $\mathcal{R}_1(\rho) \subset C_+^r(S^1)$ such that $\mathbf{m}_1(f) < \rho \cdot \lambda_{\min}^{-1}$ for $f \in \mathcal{R}_1(\rho)$.*

In the same spirit, we can show

Theorem 2.2. *For any $\rho > 0$, there exists an open and dense subset $\mathcal{R}_0(\epsilon) \subset C_+^r(S^1)$ such that $\mathbf{m}_0(f) < \rho \cdot \lambda_{\max}$ for $f \in \mathcal{R}_0(\rho)$.*

Thus we have

Corollary 2.3. $\mathbf{m}_0(f) = \lambda_{\max}$ *and* $\mathbf{m}_1(f) = \lambda_{\min}^{-1}$ *for C^r generic $f \in C_+^r(S^1)$.*

2.3. **Statement of the main results.** Let $C_0^r(X_f)$ be the set of functions on X_f whose pull-back by T_f^t for any $t \geq 0$ is C^r on X_f°. Similarly let $C_1^r(X_f)$ be the set of functions on X_f whose image by $\mathcal{L}_{f,\alpha}^t$ for any $t \geq 0$ is C^r on X_f°. [1)]

Now we state the main results. The first one is about spectral properties of the operators $\mathcal{L}_{f,\alpha}^t$. This is a slight generalization of the main result of the paper[14].

Theorem 2.4. *For $\alpha \in \{0,1\}$, there exists a Hilbert space*

$$C_\alpha^1(X_f) \subset W_\alpha(X_f) \subset L^2(X_f)$$

such that $\mathcal{L}_{f,\alpha}^t$ for sufficiently large $t > 0$ restricts to a bounded operator

$$\mathcal{L}_{f,\alpha}^t : W_\dagger(X_f) \to W_\dagger(X_f)$$

whose essential spectral radius is bounded by $\mathbf{m}_\alpha(f)^{t/2}$. Further, for any $\epsilon > 0$, there exists a decomposition of the Banach space $W_\dagger(X_f) = W_0 \oplus W_1$ into closed subspaces such that

(a) *W_0 and W_1 are invariant with respect to $\mathcal{L}_{f,\alpha}^t$.*
(b) $\dim W_0 < \infty$.

[1)] The functions in $C_0^r(X_f)$ and those in $C_1^r(X_f)$ are C^r on the interior of X_f and satisfies different restrictions on their behavior near the boundary of X_f.

(c) *The restriction* $\mathcal{L}^t_{f,\alpha}|_{W_0}$ *is written as* e^{tA} *for a linear map* $A : W_0 \to W_0$*, whose real parts of eigenvalues are not smaller than* $(1/2)\log \mathbf{m}_\alpha(f) + \epsilon$.

(d) $\|\mathcal{L}^t_{f,\alpha}|_{W_1}\| \le C \cdot e^{\epsilon t} \cdot \mathbf{m}_\alpha(f)^{t/2}$ *for some* $C > 0$.

The second one is about the Selberg function.

Theorem 2.5. *For* $\alpha \in \{0, 1\}$*, the Selberg function* $d_{f,\alpha}(s)$ *extends to the region* $\Re(s) > (1/2)\log \mathbf{m}_\alpha(f)$ *as a holomorphic function. Further, for any* $\epsilon > 0$*, the zeros of the extension in the region* $\Re(s) > (1/2)\log \mathbf{m}_\alpha(f) + \epsilon$ *is in one-to-one correspondence to the eigenvalues of the linear map* $A : W_0 \to W_0$ *in the last theorem, the order of the zero coinciding with the algebraic multiplicity of the eigenvalue.*

Theorem 1.1 in the previous section is an immediate consequence of these theorems, Corollary 2.3 and the relation (5). The proofs of theorem 2.4 and 2.5 will be provided in the paper [15] in preparation.

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