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Functional differential equations of a type similar to $f'(x)=2f(2x+1)-2f(2x-1)$ and its application to Poisson's equation

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This report is devoted to the precise formulation on the solution operator in terms of the quarkonial decomposition. We apply the quarkonial decomposition to the delay equation and the Poisson equation.

In this report we intend to apply the special solution of

$$f'(x)=2f(2x+1)-2f(2x-1),\ f\in\mathcal{S},\ f(0)=1$$

to the Poisson equation. This solution will be denoted by ψ and we use the quarkonial decomposition method. Before we go into the detail, we will describe the quarkonial decomposition.

1 Besov and Triebel-Lizorkin spaces

In this section we will make a brief sketch of the Besov and Triebel Lizorkin spaces. First, pick $\psi_0,\psi_1\in\mathcal{S}$ that satisfy

$$\chi_{B(2)}\le\psi_0\le\chi_{B(4)},\ \chi_{B(4)\setminus B(2)}\le\psi_1\le\chi_{B(8)\setminus B(1)}.$$

Set $\phi_j(x):=\phi(2^{-j+1}x)$ for $j\ge 2$. In general given $\phi\in\mathcal{S}$, we set $\phi(D)f=\mathcal{F}^{-1}(\phi\cdot\mathcal{F}f)$. Note that if ϕ is a compactly supported function and $f\in\mathcal{S}'$, then $\phi(D)f$ is a smooth function. Thus the norm of $\phi(D)f$ makes sense. Next, given a sequence of Lebesgue measurable functions $\{f_j\}_{j\in\mathbb{N}_0}$ we define

$$\|f_j\ :\ L_p(l_q)\| = \left(\int_{\mathbb{R}^n}\left(\sum_{j=0}^{\infty}|f_j(x)|^q\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}$$

$$\|f_j\ :\ l_q(L_p)\| = \left(\sum_{j=0}^{\infty}\left(\int_{\mathbb{R}^n}|f_j(x)|^p\,dx\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

With this preparation in mind, we define the norms. For $f\in\mathcal{S}'$ we define

$$\|f\ :\ B^s_{pq}\| = \|2^{js}\phi_j(D)f\ :\ l_q(L_p)\|,\ 0<p\le\infty,\ 0<q\le\infty,\ s\in\mathbb{R}$$

$$\|f\ :\ F^s_{pq}\| = \|2^{js}\phi_j(D)f\ :\ L_p(l_q)\|,\ 0<p<\infty,\ 0<q\le\infty,\ s\in\mathbb{R}.$$

B^s_{pq} and F^s_{pq} are function spaces consisting of $f\in\mathcal{S}'$ such that the norm of f is finite. If we write A^s_{pq}, then we mean that $A^s_{pq}=B^s_{pq}$ or F^s_{pq}. If $A=F$, then we tacitly exclude the case when $p=\infty$.

We list key properties of this norms.

Theorem 1.1. *Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.*

1. *The definition of the function space A^s_{pq} does not depend on the choice of ϕ_0 and ϕ_1.*
2. *$S \subset A^s_{pq} \subset S'$ in the sense of continous embedding.*
3. *If $p, q < \infty$, then S is dense in A^s_{pq}.*
4. *A^s_{pq} is a quasi-Banach space. That is for $f, g \in A^s_{pq}$ and $k \in \mathbb{C}$, we have the following assertions.*
   (a) *$\|f : A^s_{pq}\| \geq 0$ and we have the equality precisely when $f = 0$.*
   (b) *$\|k \cdot f : A^s_{pq}\| = |k| \cdot \|f : A^s_{pq}\|$.*
   (c) *$\|f + g : A^s_{pq}\| \leq c(\|f : A^s_{pq}\| + \|g : A^s_{pq}\|)$.*
   (d) *The Cauchy sequence is convergent in B^s_{pq}.*

   We also have c in c can be taken 1, if $p, q \geq 1$.
5. *We have inclusions in the sense of continous embedding.*

$$L_p = F^0_{p2} \text{ if } 1 < p < \infty$$
$$B^0_{p1} \subset L_p \subset B^0_{p\infty} \text{ if } 1 \leq p < \infty$$
$$B^0_{\infty 1} \subset UC \subset L_\infty \subset B^0_{\infty\infty},$$

   where UC denotes the set of all bounded and uniformly continous function.

The Sobolev type embedding is also known.

Theorem 1.2. *Let $0 < p_1, p_2, q \leq \infty$ and $s_1, s_2 \in \mathbb{R}$. Assume that*

$$s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}, \; s_1 > s_2, \; p_1 < p_2.$$

1. *If in addition $p < \infty$, then $F^{s_1}_{p_1\infty} \subset F^{s_2}_{p_2 q}$.*
2. *$B^{s_1}_{p_1 q} \subset B^{s_2}_{p_2 q}$.*

Next, we recall the lift property.

Theorem 1.3. *Let $\sigma \in \mathbb{R}$ and $m \in \mathbb{N}$. Then*

$$\partial_j : A^s_{pq} \to A^{s-1}_{pq}$$

is a continuous mapping. Furthermore the following mappings are all isomorphisms.

1. *$(1 - \Delta)^\sigma : A^s_{pq} \to A^{s-2\sigma}_{pq}$.*
2. *$(1 + (-\Delta)^m) : A^s_{pq} \to A^{s-2m}_{pq}$*
3. *$(1 + {\partial_1}^{4m} + \ldots + {\partial_n}^{4m}) : A^s_{pq} \to A^{s-4m}_{pq}$*

It is not easy to prove Theorem 1.1, But we present a clue to prove it.

Definition 1.4. Let $A \subset \mathbb{R}^n$ be a bounded set. We define $\mathcal{S}'^A$ to be

$$\mathcal{S}'^A := \{f \in \mathcal{S}' \,:\, \text{supp}\,(\mathcal{F}f) \subset \overline{A}\}.$$

We set $L_p^A := L_p \cap \mathcal{S}'^A$.

Remark 1.5. Let $f \in \mathcal{S}'^A$. Take a compactly supported function ψ that takes 1 on A. Then we have $f = \mathcal{F}^{-1}\mathcal{F}f = \mathcal{F}^{-1}(\psi \cdot \mathcal{F}f) = (2\pi)^{\frac{n}{2}}\mathcal{F}^{-1}\psi * f$, which implies $f \in C^\infty(\mathbb{R}^n)$. In particular it is meaningful to evaluate f at $x \in \mathbb{R}^n$.

In view of Remark 1.5 the statement of the following theorem makes sense. The proof can be found in [1, 7]. However, for convenience for readers we include its proof, hoping that this theorem along with its proof motivates the readers to study this field.

Theorem 1.6. *Let $f \in \mathcal{S}'^{B(1)}$. Then we have*

$$\sup_{z\in\mathbb{R}^n} \frac{|\nabla f(x-y)|}{1+|y|^{\frac{n}{r}}} \le c \sup_{z\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}} \tag{1}$$

$$\sup_{z\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}} \le c M^{(r)}f(x), \tag{2}$$

where c depends on r and n.

Proof of (1). To prove this we take $\psi \in \mathcal{S}$ so that

$$\chi_{B(1)} \le \psi \le \chi_{B(2)}.$$

By the similar reasoning as Remark 1.5 we have $f = (2\pi)^{\frac{n}{2}}\mathcal{F}^{-1}\psi * f$. Write it out in full:

$$f(x) = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \mathcal{F}^{-1}\psi(y) f(x-y)\,dy. \tag{3}$$

To prove $\sup_{z\in\mathbb{R}^n} \frac{|\nabla f(x-y)|}{1+|y|^{\frac{n}{r}}} \le c \sup_{z\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}}$ we may replace ∇ by ∂_j for fixed j. That is, we have only to prove it componentwise. Differentiation of (3) then yields

$$\partial_j f(x) = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} [\partial_j \mathcal{F}^{-1}\psi](y) f(x-y)\,dy.$$

Let us write $\partial_j \mathcal{F}^{-1}\psi = \rho$ for simplicity. By the triangle inequality of integral we obtain

$$\frac{|\partial_j f(x-y)|}{1+|y|^{\frac{n}{r}}} \le (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \frac{|\rho(z) f(x-y-z)|}{1+|y|^{\frac{n}{r}}}\,dz.$$

It is well known that

$$(1+|y+z|^{\frac{n}{r}}) \le c(1+|z|^{\frac{n}{r}})(1+|y|^{\frac{n}{r}}).$$

In fact the proof of this inequality is very simple.* Keeping $\rho \in \mathcal{S}$ in mind, we are led to

$$\begin{aligned} \frac{|\partial_j f(x-y)|}{1+|y|^{\frac{n}{r}}} &\le c \int_{\mathbb{R}^n} \frac{(1+|z|^{\frac{n}{r}})|\rho(z) f(x-y-z)|}{1+|y+z|^{\frac{n}{r}}}\,dz \\ &\le c \int_{\mathbb{R}^n} \frac{\{(1+|z|^{\frac{n}{r}})|\rho(z)|\}\,|f(x-y-z)|}{1+|y+z|^{\frac{n}{r}}}\,dz \le c \sup_{z\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}}. \end{aligned}$$

This is the desired inequality. ∎

* We calculate

$$\left(1+|y+z|^{\frac{n}{r}}\right) \le \left(1+(|y|+|z|)^{\frac{n}{r}}\right) \le 2^{\frac{n}{r}}\left(1+|y|^{\frac{n}{r}}+|z|^{\frac{n}{r}}\right) \le 2^{\frac{n}{r}}\left(1+|z|^{\frac{n}{r}}\right)\left(1+|y|^{\frac{n}{r}}\right).$$

Proof of (2) Reduction step. First, we may assume that $f \in \mathcal{S}'^{B(1-\varepsilon)}$ for some $\varepsilon > 0$ by the dilation argument. Let ψ be a smooth function such that

$$\int \mathcal{F}\psi(\xi)\,d\xi = (2\pi)^{\frac{n}{2}},\ \text{supp}\ (\mathcal{F}\psi) \subset \chi_{B(1)}.$$

Set $g_t(x) := \psi(tx)f(x),\ x \in \mathbb{R}^n,\ 0 < t < \dfrac{\varepsilon}{2}$. Then we have

1. $M^{(r)}g_t(x) \leq M^{(r)}f(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$.
2. $\lim_{t\to+0} g_t(x) = f(x)$ for all $x \in \mathbb{R}^n$
3. $\text{supp}\ (\mathcal{F}g_t) \subset t\cdot \text{supp}\ (\psi) + \text{supp}\ (f) \subset B\left(\frac{\varepsilon}{2}\right) + B(1-\varepsilon) \subset B(1)$.
4. $g_t \in \mathcal{S}$ for each $0 < t < \dfrac{\varepsilon}{2}$.

Thus we may assume $f \in \mathcal{S}$. ∎

Proof of (2). To prove this inequality, we first take $v \in \mathbb{R}^n$ and $0 < r < 1$. The constant r will be fixed sufficiently small.

Let $y_v \in \overline{B}(x,r)$ that attains the minimum of $|f(\cdot)|$ in $\overline{B}(v,r)$. Then by the mean value theorem we have

$$|f(v)| \leq |f(y_v)| + |f(v) - f(y_v)| \leq \inf_{z\in B(v,r)} |f(z)| + r \sup_{w\in B(v,r)} |\nabla f(w)|.$$

By replacing v with $x - y$ we obtain

$$|f(x-y)| \leq \inf_{z\in B(x-y,r)} |f(z)| + r \sup_{w\in B(x-y,r)} |\nabla f(w)|,\ x, y \in \mathbb{R}^n$$

Since $|B(1)| \geq 1$, we obtain

$$\inf_{z\in B(x-y,r)} |f(z)| \leq \left(\int_{B(x-y,1)} |f(z)|^r\,dz\right)^{\frac{1}{r}}. \tag{4}$$

Observe that this is where the integral and hence the maximal operator appears. The inclusion $B(x-y,1) \subset B(x,|y|+1)$ together with (4) gives us

$$|f(x-y)| \leq \left(\int_{B(x,|y|+1)} |f(z)|^r\,dz\right)^{\frac{1}{r}} + r \sup_{w\in B(x-y,r)} |\nabla f(w)|.$$

Taking supremum over $y \in \mathbb{R}^n$ we obtain

$$\sup_{y\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}} \leq \frac{1}{1+|y|^{\frac{n}{r}}} \left(\int_{B(x,|y|+1)} |f(z)|^r\,dz\right)^{\frac{1}{r}} + r \sup_{\substack{y,w\in\mathbb{R}^n \\ |x-y-w|<r}} \frac{|\nabla f(w)|}{1+|y|^{\frac{n}{r}}}.$$

Note that, changing variables $w \mapsto z := x - w$, we obtain

$$\sup_{\substack{y,w\in\mathbb{R}^n \\ |x-y-w|<r}} \frac{|\nabla f(w)|}{1+|y|^{\frac{n}{r}}} = \sup_{\substack{y,z\in\mathbb{R}^n \\ |z-y|<r}} \frac{|\nabla f(x-z)|}{1+|y|^{\frac{n}{r}}}$$

and if $z \in B(y,r)$ with $r \le 1$, we obtain $1+|y|^{\frac{n}{r}} \sim 1+|z|^{\frac{n}{r}}$.** Meanwhile it is easy to see

$$\frac{1}{1+|y|^{\frac{n}{r}}}\left(\int_{B(x,|y|+1)} |f(z)|^r \, dz\right)^{\frac{1}{r}} \le c M^{(r)} f(x).$$

Consequently we obtain

$$\sup_{z\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}} \le c \left(M^{(r)} f(x) + r \sup_{z\in B(y,r)} \frac{|\nabla f(x-z)|}{1+|z|^{\frac{n}{r}}} \right).$$

Since we have shown that

$$\sup_{z\in\mathbb{R}^n} \frac{|\nabla f(x-z)|}{1+|z|^{\frac{n}{r}}} \le c \sup_{z\in\mathbb{R}^n} \frac{|f(x-z)|}{1+|z|^{\frac{n}{r}}},$$

it follows that there exists a constant $c_0 > 0$ such that

$$\sup_{z\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}} \le c M^{(r)} f(x) + c_0 \, r \sup_{z\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}}. \tag{5}$$

If we take $r = \min(1, (2c_0)^{-1})$, we can bring the most right side to the left side. Since $f \in \mathcal{S}$, every term in (5) is finite. Thus we are allowed to subtract $c_0 \, r \sup_{z\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}}$ in (5). Consequently we finally obtain

$$\sup_{z\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{r}}} \le c M^{(r)} f(x).$$

This is the desired result. ∎

To deal with the Poisson equation and the delay equation, it is not suitable to consider global L_p-solution. To deal with the properties of functions we consider the localized function spaces.

Definition 1.7. We define

$$A^s_{pq,loc}(\mathbb{R}^n) := \{f \in D'(\mathbb{R}^n) \,:\, \phi \cdot f \in A^s_{pq}(\mathbb{R}^n)\}.$$

1.1 Quarkonial decomposition

Having set down the elementary properties of the function spaces, we now turn to describe the quarkonial decomposition. For details we refer to [3, 4, 8, 9, 10].

Definition 1.8. $\psi \in \mathcal{S}$ is a function satisfying

$$\sum_{m\in\mathbb{Z}^n} \psi(x-m) \equiv 1$$

for all $x \in \mathbb{R}^n$. Accordingly the number $r > 0$ is fixed so that

$$\text{supp}\,(\psi) \subset \{|x| \le 2^r\}. \tag{6}$$

** See the footnote of the previous page for the similar calculation.

Definition 1.9. Let $0 < p, q \le \infty$.

1. Let $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then we define

$$Q_{\nu m} := \prod_{j=1}^{n} \left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu}\right).$$

2. Let $0 < p \le \infty$, $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then we define the p-normalized indicator $\chi_{\nu m}^{(p)}$ by

$$\chi_{\nu m}^{(p)} := 2^{n\nu/p}\chi_{Q_{\nu m}}.$$

3. Then, given a complex sequence $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbf{N}_0,\ m \in \mathbf{Z}^n}$, we define

$$\begin{aligned} \|\lambda : b_{pq}\| &= \left\| \sum_{m \in \mathbf{Z}} \lambda_{\nu m}\chi_{\nu m}^{(p)} : l_q(L_p) \right\| \\ \|\lambda : f_{pq}\| &= \left\| \sum_{m \in \mathbf{Z}} \lambda_{\nu m}\chi_{\nu m}^{(p)} : L_p(l_q) \right\|, \end{aligned}$$

Now we define the quark.

Definition 1.10. Let $\beta \in \mathrm{N}_0{}^n$, $\nu \in \mathrm{N}_0$, $m \in \mathbb{Z}^n$ and $\rho > r$, where r is a positive number specified in (6).

1. $\psi^\beta(x) := x^\beta \psi(x)$.
2. $(\beta qu)_{\nu m}(x) = 2^{-\nu(s-\frac{n}{p})}\psi^\beta(2^\nu x - m)$.
3. Let the parameters p, q, u satisfy

$$0 < u \le p < \infty,\ 0 < q \le \infty.$$

Given a triply parameterized sequence $\lambda = \{\lambda^\beta\}_{\beta \in \mathrm{N}_0{}^n} = \{\lambda_{\nu m}^\beta\}_{\beta \in \mathrm{N}_0{}^n,\ \nu \in \mathrm{N}_0,\ m \in \mathbf{Z}^n}$, we define

$$\|\lambda\,;\, a_{pq}\|_\rho := \sup_{\beta \in \mathrm{N}_0{}^n} 2^{\rho|\beta|}\|\lambda^\beta : a_{pq}\|.$$

Here we tacitly exclude the case when $p = \infty$ if we consider f_{pq}.

We assume

$$0 < u \le p \le \infty,\ 0 < q \le \infty,\ s > \sigma_p \tag{7}$$

for F-scale and

$$0 < u \le p < \infty,\ 0 < q \le \infty,\ s > \sigma_{pq} \tag{8}$$

for F-scale.

With this preparation in mind, we state the quakonial decomposition.

Theorem 1.11. *Suppose that the parameters p, q, u, s satisfy (7) for B-scale and (8) for F-scale. Let $f \in S'$. Then $f \in A_{pq}^s$ if and only if there exists a triply indexed sequence $\lambda = \{\lambda_{\nu m}^\beta\}_{\beta \in \mathrm{N}_0{}^n,\ \nu \in \mathrm{N}_0,\ m \in \mathbf{Z}^n}$ such that f can be expressed as*

$$f = \sum_{\beta \in \mathrm{N}_0{}^n} \sum_{\nu \in \mathrm{N}_0} \sum_{m \in \mathbf{Z}^n} \lambda_{\nu m}^\beta (\beta qu)_{\nu m}$$

with

$$\|\lambda : a_{pq}\| < \infty. \tag{9}$$

If this is the case, then λ can be taken so that

$$\|\lambda : a_{pq}\| \simeq \|f : A^s_{pq}\|. \tag{10}$$

2 Integral operation in terms of quarks

Here the function space $A^s_{pq}((0,1))$ is given by

$$A^s_{pq}((0,1)) := \{f \in D'((0,1)) : \exists g \in A^s_{pq}(\mathbb{R})\, s.t.\, g|_{(0,1)} = f\},$$

which is quasi-normed by

$$\|f : A^s_{pq}((0,1))\| := \inf_{\substack{g \in A^s_{pq}(\mathbb{R}) \\ g|_{(0,1)} = f}} \|g : A^s_{pq}(\mathbb{R})\|.$$

It is shown in [7] that there is a "canonical" representative. For all $f \in A^s_{pq}((0,1))$, there exists $g \in A^s_{pq}(\mathbb{R})$ such that $g|_{(0,1)} = f$ and $\|g : A^s_{pq}(\mathbb{R})\| \leq C\,\|f : A^s_{pq}((0,1))\|$, which is denoted by $extf$. Thus our problem can be restated as

Problem 2.1. *Solve the following functional-differential equation in* $[1,2]$:

$$f'(x) = f(x-1),\ x \geq 1,\ f(x) = \phi(x),\ x \in [0,1],\ \phi \in A^s_{pq}(\mathbb{R}).$$

Here the parameter satisfies $p, q > 1$ and $s > \frac{1}{p}$.

Since $A^s_{pq}(\mathbb{R})$ is embedded continuously to $C(\mathbb{R})$, our solution operator in Introduction makes sense. We shall calculate

$$g(1) + \int_0^{x-1} g(u)\,du, x \in [1,2]$$

for $g \in A^s_{pq}(\mathbb{R})$.

As is often the case, a parallel argument to F-scale works for B-scale and F-scale is somehow more difficult. Thus, in what follows we let $A^s_{pq} = F^s_{pq}$.

2.1 Solution operator

Since the functions are written as the sum of quarks, we have only to derive a solution formula for each quark. Now that ψ is specified as Y-function, we can obtain a solution formula explicitly.

Definition 2.2. Let $\beta \in \mathbb{N}_0$.

1. We put the moment of β-degree by $c_\beta := \int_{\mathbb{R}} x^\beta \phi(x)\,dx$.

2. We set $\Psi^\beta(x) := \int_{-\infty}^{x} u^\beta \phi(u)\, du - c_\beta \sum_{l=2}^{\infty} (0qu)_{\nu,l}(x)$.

3. We define an auxiliary quark by $(\beta qu)^*_{\nu,m} := 2^{-\nu\left(s-\frac{1}{p}\right)} \Psi^\beta(2^\nu x - m)$.

By support condition we have $\int_{-\infty}^{x} y^\beta \phi(y)\, dy$ is constant, if $x \geq 1$. We also have

$$\sum_{l=2}^{\infty} \phi(x-l) = 1,$$

if $x \geq 2$. As a result it follows that $\Psi^\beta(x)$ has compact support.

As for c_β, we have the following recurrence formula to calculate c_β inductively.

Lemma 2.3. *Let $\beta \in \mathbb{N}_0$. Then we have*

(1) $|c_\beta| \leq \dfrac{2}{\beta+1}$ *and* $c_0 = 1$. $c_\beta = 0$ *if β is odd.*

(2) $c_{2\beta}$ *satisfies the following recurrence formula.*

$$c_{2\beta} = \frac{1}{(2\beta+1)(4^\beta - 1)} \sum_{\gamma=0}^{\beta-1} {}_{2\beta+1}C_{2\gamma} \cdot c_{2\gamma}$$

for $\beta \geq 1$. In particular $c_2 = \dfrac{1}{9}$.

Proof. The fact that $c_0 = 1$ can be proved from the nomalization condition of the equation

$$u'(x) = 2u(2x+1) - 2u(2x-1).$$

It can be also proved that $0 \leq u(x) \leq 1$ and that u is positive and supported in supp $(u) = [-1, 1]$. Because u can be expressed in terms of infinite convolution. For details we refer to [5, 6]. Thus the estimate $|c_\beta| \leq \dfrac{2}{\beta+1}$ is immediate. The the last part of assertion (1) is clear because u is even. Let $\beta \geq 1$ and prove (2). By integration by parts and the functional differential equation we have

$$\begin{aligned}
c_{2\beta} &= \int_{\mathbb{R}} \left(\frac{x^{2\beta+1}}{2\beta+1}\right)' u(x)\, dx \\
&= -\int_{\mathbb{R}} \left(\frac{x^{2\beta+1}}{2\beta+1}\right) (2u(2x+1) - 2u(2x-1))\, dx \\
&= \frac{1}{2\beta+1} \int_{\mathbb{R}} \left(\frac{x+1}{2}\right)^{2\beta+1} u(x)\, dx - \frac{1}{2\beta+1} \int_{\mathbb{R}} \left(\frac{x-1}{2}\right)^{2\beta+1} u(x)\, dx \\
&= \frac{1}{2\beta+1} \sum_{\gamma=0}^{\beta} 2^{-2\beta} {}_{2\beta+1}C_{2\gamma} \cdot c_{2\gamma} = 2^{-2\beta} c_{2\beta} + \frac{2^{-2\beta}}{2\beta+1} \sum_{\gamma=0}^{\beta-1} {}_{2\beta+1}C_{2\gamma} \cdot c_{2\gamma}.
\end{aligned}$$

Equating with respect to $c_{2\beta}$, we have the desired result. ∎

Proposition 2.4. *We have*

$$\int_{-\infty}^{x} (\beta qu)_{\nu,m}(y)\, dy = 2^{-\nu}(\beta qu)^{*}_{\nu,m}(x) + c_\beta 2^{-\nu} \sum_{l=2}^{\infty} (0qu)_{\nu,m+l}(x).$$

Proof. By the change of variable the lemma follows easily. ∎

Although we cannot tell that $(\beta qu)^{*}_{\nu,m}$ is used to decompose the function, we still have a nice convergence.

Proposition 2.5. *Let $s > 0$ and $\rho > 1$. Suppose that $\|\lambda : a_{pq}\|_\rho \leq C$. Then*

$$\sum_{\beta \in \mathbf{N}_0} \sum_{\nu \in \mathbf{N}_0} \sum_{m=0}^{2^\nu} 2^{-\nu} \lambda^{\beta}_{\nu,m} (\beta qu)^{*}_{\nu,m}$$

is convergent in $A^s_{pq}(\mathbb{R})$ and satisfies the norm estimate

$$\left\| \sum_{\nu \in \mathbf{N}_0} \sum_{m=0}^{2^\nu} 2^{-\nu} \lambda^{\beta}_{\nu,m} (\beta qu)^{*}_{\nu,m} \ : \ A^s_{pq}(\mathbb{R}) \right\| \leq C_\varepsilon \, 2^{-(\rho-\varepsilon)\beta} \|\lambda^\beta \ : \ a_{pq}\|_\rho$$

for all $\varepsilon > 0$.

Proof. To prove this assertion we have only to check that $2^{-\nu-\rho\beta}(\beta qu)^{*}_{\nu,m}$ satisfies the requirement of the atom described in [8]. This is easily checked and as a result the desired norm estimate follows. ∎

2.2 Calculation of $(\beta qu)^{*}_{\nu,m}(x)$.

By using the functional-differential equation $\phi'(x) = 2\phi(2x+1) - 2\phi(2x-1)$, we can calculate $\Psi^\beta(x)$ directly.

Lemma 2.6. *Define $I_\beta(\phi)$ inductively by the following formula:*

$$I_0(\phi)(x) = \int_{-\infty}^{x} \phi(u)\, du, \quad I_\beta(\phi)(x) = \int_{-\infty}^{x} I_{\beta-1}(\phi)(u)\, du. \ (\beta = 1, 2, \ldots)$$

Then

$$I_\beta(\phi)(x) = \sum_{j_{\beta+1}=0}^{\infty} \sum_{j_\beta=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} 2^{\frac{\beta(\beta+1)}{2}} \phi\left(\frac{x - 2^{\beta+1} + 1}{2^{\beta+1}} - \sum_{\gamma=1}^{\beta+1} \frac{j_\gamma}{2^{\gamma-1}} \right).$$

In particular we have $I_0(\phi)(x) = \displaystyle\sum_{j=0}^{\infty} \phi\left(\frac{x-1-2j}{2}\right)$.

Proof. By the functional-differential equation and the size of supp (ϕ), we have $\phi^{(\beta-1)}(2x+1) = \displaystyle\sum_{j=0}^{\infty} \frac{1}{2^\beta} \phi^{(\beta)}(x-j)$. Thus it follows that

$$\phi^{(\beta-1)}(x) = \sum_{j=0}^{\infty} \frac{1}{2^\beta} \phi^{(\beta)} \left(\frac{x-1-2j}{2} \right).$$

If we use this formula inductively, we have

$$\begin{aligned}\phi(x) &= \sum_{j_1=0}^{\infty} \frac{1}{2}\phi^{(1)}\left(\frac{x-1-2j_1}{2}\right)\\ &= \sum_{j_2=0}^{\infty}\sum_{j_1=0}^{\infty}\frac{1}{2}\cdot\frac{1}{2^2}\phi^{(2)}\left(\frac{x-1-2j_1-2-4j_2}{4}\right)\\ &= \cdots\\ &= \sum_{j_{\beta+1}=0}^{\infty}\sum_{j_\beta=0}^{\infty}\cdots\sum_{j_1=0}^{\infty}\frac{1}{2^{\frac{(\beta+1)(\beta+2)}{2}}}\phi^{(\beta+1)}\left(\frac{x-2^{\beta+1}+1}{2^{\beta+1}}-\sum_{\gamma=1}^{\beta+1}\frac{j_\gamma}{2^{\beta+1-\gamma}}\right).\end{aligned}$$

Thus integrating $\beta+1$-times over $(-\infty, x)$, we obtain

$$I_\beta(\phi)(x) = \sum_{j_{\beta+1}=0}^{\infty}\sum_{j_\beta=0}^{\infty}\cdots\sum_{j_1=0}^{\infty} 2^{\frac{\beta(\beta+1)}{2}}\phi\left(\frac{x-2^{\beta+1}+1}{2^{\beta+1}}-\sum_{\gamma=1}^{\beta+1}\frac{j_\gamma}{2^{\gamma-1}}\right).$$

The proof is now complete. ∎

Using $I_\beta(\phi)$, we can express Ψ^β as an infinite sum of quarks.

Lemma 2.7. *We have*

$$\Psi^\beta(x) = \left(\sum_{\gamma=0}^{\beta}(-1)^\gamma {}_\beta P_\gamma x^{\beta-\gamma} I_\gamma(\phi)(x)\right) - c_\beta\sum_{l=2}^{\infty}(0qu)_{\nu,l}(x),$$

where ${}_\beta P_\gamma$ denotes the permutation from β to γ.

Proof. Noticing that $\frac{d^{\beta+1}}{dx^{\beta+1}}I_\beta(\phi)(x) = \phi(x)$, we have the desired result by integration by parts. ∎

It is easy to see that the differential of $(\beta qu)_{\nu,m}$ can be written as a finite sum of other quarks. As a conclusion we can say that quarks generated by Y-function are closed under differentiation and integration.

2.3 Convergence of the quarkonial decomposition

Finally we consider the convergence of the constructed solution. We will obtain an explicit formula in terms of quarks. Put a solution operator

$$T : A^s_{pq}((0,1)) \to A^s_{pq}((1,2)),\ f \mapsto f(1) + \int_0^{*-1} f(u)\,du.$$

We shall decompose this operator in terms of quark and decompose T to each β-level. Define

$$t^\beta : a_{pq}((0,1)) \to A^s_{pq}((1,2))$$

by the formula

$$\begin{aligned}\{\lambda_{\nu,m}\}_{\nu\in\mathbf{N}_0,\, m\in\mathbf{Z}} &\mapsto \sum_{\nu\in\mathbf{N}_0}\sum_{m=0}^{2^\nu} 2^{-\nu}\lambda_{\nu,m}(\beta qu)^*_{\nu,m}(x-1)\\ &+ c_\beta \sum_{\nu\in\mathbf{N}_0}\sum_{m=2}^{2^\nu}\left(\sum_{l=0}^{m-2}\lambda_{\nu,l}2^{-\nu}\right)(0qu)_{\nu,m}(x-1).\end{aligned}$$

Here we defined $a_{pq}((0,1))$ as

$$a_{pq}((0,1)) := \{\{\lambda_{\nu,m}\}_{\nu\in\mathbf{N}_0,\, m\in\mathbf{Z}} \in a_{pq}(\mathbb{R}) \,:\, \lambda_{\nu,m}=0, \text{ if } Q_{\nu,m}\cap(0,1)=\emptyset\}.$$

We also define $a_{pq}((1,2))$ similarly.

In view of preceding subsection by our notation the solution of

$$\begin{aligned}f'(x) &= f(x-1)\\ f|_{[0,1]} &= \phi|_{[0,1]} = \left(\sum_{\beta\in\mathbf{N}_0}\sum_{\nu\in\mathbf{N}_0}\sum_{m=0}^{2^\nu}\lambda^\beta_{\nu,m}(\beta qu)_{\nu,m}\right)_{|[0,1]}.\end{aligned}$$

can be described explicitly in $[1,2]$ as

$$\begin{aligned}f(x) &= f(1)+\sum_{\beta\in\mathbf{N}_0}\sum_{\nu\in\mathbf{N}_0}\sum_{m=0}^{2^\nu}2^{-\nu}\lambda^\beta_{\nu,m}(\beta qu)^*_{\nu,m}(x-1)\\ &+ \sum_{\beta\in\mathbf{N}_0}\sum_{\nu\in\mathbf{N}_0}\sum_{m=2}^{2^\nu}c_\beta\left(\sum_{l=0}^{m-2}\lambda^\beta_{\nu,l}2^{-\nu}\right)(0qu)_{\nu,m}(x-1). \qquad (11)\end{aligned}$$

Namely, we can express

$$f|_{[1,2]} = \left(\sum_{\beta\in\mathbf{N}_0}\sum_{\nu\in\mathbf{N}_0}\sum_{m=0}^{2^\nu}\mu^\beta_{\nu,m}(\beta qu)_{\nu,m}\right)_{|[1,2]}.$$

$\{\mu^\beta_{\nu,m}\}$ has apprpriate condition. By Proposition 2.5, the first sum is convergent. In this subsection we mainly consider the convergence of the second sum. Define

$$\tau^\beta_{\nu,m} := c_\beta\left(\sum_{l=0}^{m-2}2^{-\nu}\lambda^\beta_{\nu,l}\right).$$

We intend to show

Theorem 2.8. *There is a constant C independent on β so that*

$$\|\{\tau^\beta_{\nu,m}\}_{\nu,m} \,:\, f_{pq}((0,1))\| \le C\,\|\{\lambda^\beta_{\nu,m}\}_{\nu,m} \,:\, f_{pq}((0,1))\|.$$

Hence the series in (11) *converges in* $F^s_{pq}((1,2))$.

Proof. We set $\rho^\beta_{\nu,m} := 2^{-\nu}\sum_{j=0}^{2^\nu}|\lambda^\beta_{\nu,j}|$ for $m=1,2,\ldots,2^\nu$. By their definitions we have $|\tau^\beta_{\nu,m}| \le \rho^\beta_{\nu,m}$. Thus for fixed β we have

$$\|\{\tau^\beta_{\nu,m}\}_{\nu,m} \,:\, f_{pq}((0,1))\| \le \|\{\rho^\beta_{\nu,m}\}_{\nu,m} \,:\, f_{pq}((0,1))\|.$$

We write $\|\{\rho^{\beta}_{\nu,m}\}_{\nu,m} : f_{pq}((0,1))\|$ out in full:

$$\|\{\rho^{\beta}_{\nu,m}\}_{\nu,m} : f_{pq}((0,1))\| = \left\| \sum_{m=0}^{2^{\nu}} \rho^{\beta}_{\nu,m}\chi^{(p)}_{\nu,m} : L^p(l^q) \right\|.$$

Note that since $\{Q_{\nu,m}\}_{m\in\mathbf{Z}}$ is disjoint hence we have

$$\begin{aligned}\rho^{\beta}_{\nu,m} &\leq 2\int_0^1 \left| \sum_{j=0}^{2^{\nu}} \lambda^{\beta}_{\nu,j}\chi_{Q_{\nu,j}} \right| dx \leq 2\int_0^1 \sum_{j=0}^{2^{\nu}} |\lambda^{\beta}_{\nu,j}|\chi_{Q_{\nu,j}}(x)\,dx \\ &\leq 6\cdot\frac{1}{|[-1,2]|}\int_{-1}^2 \sum_{j=0}^{2^{\nu}} |\lambda^{\beta}_{\nu,j}|\chi_{Q_{\nu,j}}(x)\,dx \leq 6M\left(\sum_{j=0}^{2^{\nu}} \lambda^{\beta}_{\nu,j}\chi_{Q_{\nu,j}}\right)(y)\end{aligned}$$

for all $y \in [0,1]$. Here M is the Hardy-Littlewood maximal operator. Again by noting that $\{Q_{\nu,m}\}_{m\in\mathbf{Z}}$ is disjoint, this estimate can be strengthed to

$$\sum_{m=0}^{2^{\nu}} \rho_{\nu,m}\chi^{(p)}_{\nu,m}(x) \leq M\left(\sum_{m=0}^{2^{\nu}} \lambda^{\beta}_{\nu,m}\chi^{(p)}_{\nu,m}\right)(x).$$

Recall that $p, q > 1$. The Fefferman-Stein vector-valued inequality then yields

$$\begin{aligned}&\|\{\rho^{\beta}_{\nu,m}\}_{\nu,m} : f_{pq}((0,1))\| \\ \leq& \left\| M\left(\sum_{m=0}^{2^{\nu}} \lambda^{\beta}_{\nu,m}\chi^{(p)}_{\nu,m}\right) : L^p(l^q)\right\| \leq C\left\| \left(\sum_{m=0}^{2^{\nu}} \lambda^{\beta}_{\nu,m}\chi^{(p)}_{\nu,m}\right) : L^p(l^q)\right\|.\end{aligned}$$

This is the desired. ∎

As a conclusion we have given an explicit formula. We can write T out in full in terms of quarkonial decomposition, as is announced in Introduction.

$$\begin{aligned}T : \sum_{\beta,\nu,m} \lambda^{\beta}_{\nu,m}(\beta qu)_{\nu,m} \quad\mapsto\quad & \sum_{\beta,\nu,m} \lambda^{\beta}_{\nu,m}(\beta qu)_{\nu,m}(1) \\ &+ \sum_{\nu\in\mathbf{N}_0}\sum_{m=0}^{2^{\nu}} 2^{-\nu}\lambda^{\beta}_{\nu,m}(\beta qu)^{*}_{\nu,m}(x-1) \\ &+ c_{\beta}\sum_{\nu\in\mathbf{N}_0}\sum_{m=2}^{2^{\nu}}\left(\sum_{l=0}^{m-2}\lambda^{\beta}_{\nu,l}2^{-\nu}\right)(0qu)_{\nu,m}(x-1).\end{aligned}$$

From Proposition 2.5 the second term is convergent in $A^s_{pq}(\mathbb{R})$. By using $\Psi^{\beta}_{\nu,m} \in \S$ we set

$$\theta^{\beta}_{\nu,m} = c_{\beta}\left\langle \sum_{\nu\in\mathbf{N}_0}\sum_{m=0}^{2^{\nu}} 2^{-\nu}\lambda^{\beta}_{\nu,m}(\beta qu)^{*}_{\nu,m}(\cdot-1), \Psi^{\beta}_{\nu,m}\right\rangle.$$

Then the first term can expressed as

$$\sum_{\nu\in\mathbf{N}_0}\sum_{m=0}^{2^{\nu}} 2^{-\nu}\lambda^{\beta}_{\nu,m}(\beta qu)^{*}_{\nu,m}(x-1) = \sum_{\beta\in\mathbf{N}_0}\sum_{\nu\in\mathbf{N}_0}\sum_{m\in\mathbf{Z}} \theta^{\beta}_{\nu,m-2^{\nu}}(\beta qu)_{\nu,m}(x).$$

Thus we can have another expression of the solution operator $T : A^s_{p,q}((0,1)) \to A^s_{p,q}((1,2))$

$$T : \sum_{\beta,\nu,m} \lambda^{\beta}_{\nu,m}(\beta qu)_{\nu,m}$$
$$\mapsto \sum_{\beta\in\mathbb{N}_0}\left(c_\beta \sum_{\nu\in\mathbb{N}_0}\sum_{m=2}^{2^\nu}\sum_{l=0}^{m-2}\lambda^{\beta}_{\nu,l}2^{-\nu}(0qu)_{\nu,m+2^\nu}(x) + \sum_{m\in\mathbb{Z}}\theta^{\beta}_{\nu,m+2^\nu}(\beta qu)_{\nu,m}(x)\right).$$

3 1-dimensional Poisson equation

In the same way as above we can construct the solution operator of the 1-dimensional Poisson equation

$$\frac{d^2}{dx^2}f(x) = g(x)$$

and prove the following.

Theorem 3.1. *Suppose that $g \in A^s_{pq}(\mathbb{R})$ is given. Then we can construct the solution operator $g \in A^s_{pq}(\mathbb{R})_{loc} \mapsto S(g) \in A^s_{pq}(\mathbb{R})$ of the Poisson equation:*

$$\frac{d^2}{dx^2}f(x) = g(x), x \in \mathbb{R}.$$

Furthermore S is a continuous operator in $A^s_{pq}(\mathbb{R})_{loc}$.

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