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SOME ASPECTS OF EXTENSIONALITY OF ARROWS IN CARTESIAN CLOSED CATEGORY

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ABSTRACT. We show a kind of separability under the theory $\beta\eta_{\rightarrow}$ of simply typed λ-calculus, which gives a sufficient condition for the theory of the model induced by a cartesian closed category to be identical with $\beta\eta_{\rightarrow}$. Subject to this sufficient condition, we obtain an extensionality of λ-definable arrows of a cartesian closed category.

1. SIMPLY TYPED λ-CALCULUS

We focus our attention to two versions of simply typed λ-calculus. One would be the most basic system which has been studied widely so far in a number of literatures and the other is an extended variant which is naively obtained by adding the device of finite product to the basic system from logical and categorical viewpoints. Below we briefly review the definition of these systems.

In the basic system, we have only one type constant o, from which the set of simple types is generated by the following grammar:

$$\sigma ::= o \mid \sigma \rightarrow \sigma.$$

We use letters $\sigma, \tau, \upsilon, \ldots$ as meta-variables to designate simple types. Based on these types we put a restriction on the formation of terms, which are inductively generated by the following rules:

$$x^{\sigma} : \sigma \qquad \frac{M : \sigma \rightarrow \tau \quad N : \sigma}{MN : \tau} \qquad \frac{M : \tau}{\lambda x^{\sigma}.M : \sigma \rightarrow \tau}$$

where x^{σ} ranges over the set of term-variables of type σ. We use letters $M, N, \ldots$ as meta-variables to designate these typed terms, and specify the unique type σ of a λ-term M by the expression $M : \sigma$. We define $\Lambda_{\rightarrow}$ to be the set of terms of this system. We write $=_{\beta}$ for the smallest congruence relation on $\Lambda_{\rightarrow}$ satisfying

(1) $$\lambda x^{\sigma}.M = \lambda y^{\sigma}.M[x^{\sigma} := y^{\sigma}] \qquad \text{provided } y^{\sigma} \notin \mathrm{FV}(M),$$

(2) $$(\lambda x^{\sigma}.M)N = M[x := N],$$

where $\mathrm{FV}(M)$ stands for the set of free-variables appearing in M, and analogously $=_{\beta\eta}$ for the smallest congruence relation satisfying (1), (2) and

(3) $$\lambda x^{\sigma}.Mx^{\sigma} = M \qquad \text{provided } x^{\sigma} \notin \mathrm{FV}(M).$$

We then denote the formal theories yielding the equalities $=_\beta$ and $=_{\beta\eta}$ by $\beta_\rightarrow$ and $\beta\eta_\rightarrow$ respectively. For detailed explanation on the syntactical properties of the systems $\beta_\rightarrow$ and $\beta\eta_\rightarrow$, see [5, 6] for example.

We next mention the syntax of the extended version of simply typed λ-calculus, which is obtained by incorporating the syntax of finite products into the above-mentioned basic system. So the set of types are defined by the following grammar:

$$\sigma ::= o \mid 1 \mid \sigma \times \sigma \mid \sigma \rightarrow \sigma$$

where 1 is a type constant to formalise a 0-ary product, namely, a certain singleton set. As for terms, besides the rules of the basic system we further adopt the following to represent finite products:

$$\frac{M : \sigma \times \tau}{\mathrm{Fst}\ M : \sigma} \qquad \frac{M : \sigma \times \tau}{\mathrm{Snd}\ M : \tau} \qquad \frac{M : \sigma \quad N : \tau}{\langle M, N \rangle : \sigma \times \tau}$$

Since there is no possibility of confusion, to denote types and terms we use the same meta-variables as those for the basic system. We write $\Lambda_{\times,\rightarrow}$ for the set of extended terms so defined. Between the terms in $\Lambda_{\times,\rightarrow}$, we define a relation $=_{\beta\eta\pi}$ by the smallest congruence relation satisfying the axiom schemes

$$\begin{aligned}
&\mathrm{Fst}\ \langle M, N \rangle = M, \\
&\mathrm{Snd}\ \langle M, N \rangle = N, \\
&\langle \mathrm{Fst}\ M, \mathrm{Snd}\ M \rangle = M,
\end{aligned}$$

as well as (1), (2) and (3). We denote the formal theory yielding the equalities $=_{\beta\eta\pi}$ by $\beta\eta\pi_\rightarrow$. This extension seems to be reasonable both from the correspondence with category theory and from the aspect of Curry-Howard isomorphism, as is mentioned in [1, 7, 9].

2. Models and cartesian closed categories

For all the systems mentioned in the preceding section, mathematical frameworks to serve their denotational semantics have been also well investigated so far. Here, we first review a usual presentation of models of these systems according to [4, 8, 9, 10] and then refer to an alternative description of models based on category theory as is explained in [1, 7].

A typed applicative structure which underlies the semantics of $\beta_\rightarrow$ and $\beta\eta_\rightarrow$ is given by a 2-tuple

$$\langle [\![\,]\!]^{\mathrm{type}}, \mathrm{App} \rangle$$

of mappings such that the first, called type-interpretation, assigns a non-empty set $[\![\sigma]\!]^{\mathrm{type}}$ to each type σ of the basic system, and the second assigns a function

$$\mathrm{App}^{\sigma,\tau} : [\![\sigma \rightarrow \tau]\!]^{\mathrm{type}} \times [\![\sigma]\!]^{\mathrm{type}} \longrightarrow [\![\tau]\!]^{\mathrm{type}}$$

to each pair of simple types σ and τ. In such a typed applicative structure, an interpretation of free-variables of a term is given by a mapping ξ, called an environment, which assigns an element of $[\![\sigma]\!]^{\mathrm{type}}$ to each term-variable. Then we say that a typed applicative structure is a model, or typed λ-algebra, of the system $\beta_\rightarrow$ if we are able to present a mapping $[\![\,]\!]^{\mathrm{term}}$, called term-interpretation, which assigns a member $[\![M]\!]^{\text{term}}_{\xi}$ of $[\![\sigma]\!]^{\text{type}}$ to each pair of an environment ξ and a term $M \in \Lambda_{\rightarrow}$ of type σ, which satisfies

$$\begin{aligned}
&\forall x^{\sigma} \in \mathrm{FV}(M) \quad \xi(x^{\sigma}) = \rho(x^{\sigma}) \;\Rightarrow\; [\![M]\!]^{\text{term}}_{\xi} = [\![M]\!]^{\text{term}}_{\rho},\\
&[\![x^{\sigma}]\!]^{\text{term}}_{\xi} = \xi(x^{\sigma}),\\
&[\![MN]\!]^{\text{term}}_{\xi} = \mathrm{App}^{\sigma,\tau}([\![M]\!]^{\text{term}}_{\xi}, [\![N]\!]^{\text{term}}_{\xi}),\\
&M =_{\beta} N \;\Rightarrow\; \forall\xi \quad [\![M]\!]^{\text{term}}_{\xi} = [\![N]\!]^{\text{term}}_{\xi}.
\end{aligned}$$

In case where the term-interpretation further satisfies

$$M =_{\beta\eta} N \Rightarrow \forall\xi \quad [\![M]\!]^{\text{term}}_{\xi} = [\![N]\!]^{\text{term}}_{\xi}$$

we say that the typed applicative structure is a model of $\beta\eta_{\rightarrow}$.

In order to model the system $\beta\eta\pi_{\rightarrow}$, we need to extend the conditions listed above. So we say that a model of $\beta\eta_{\rightarrow}$ is even a model of $\beta\eta\pi_{\rightarrow}$ if we have not only the mapping App but also the three mappings Fst, Snd and Pair which satisfies

$$\begin{aligned}
\mathrm{Fst}^{\sigma,\tau} &: [\![\sigma\times\tau]\!]^{\text{type}} \rightarrow [\![\sigma]\!]^{\text{type}},\\
\mathrm{Snd}^{\sigma,\tau} &: [\![\sigma\times\tau]\!]^{\text{type}} \rightarrow [\![\tau]\!]^{\text{type}},\\
\mathrm{Pair}^{\sigma,\tau} &: [\![\sigma]\!]^{\text{type}} \times [\![\tau]\!]^{\text{type}} \rightarrow [\![\sigma\times\tau]\!]^{\text{type}},
\end{aligned}$$

for every type σ and τ, by which we can introduce a term-interpretation satisfying

$$\begin{aligned}
&[\![\mathrm{Fst}\ M]\!]^{\text{term}}_{\xi} = \mathrm{Fst}^{\sigma,\tau}([\![M]\!]^{\text{term}}_{\xi}),\\
&[\![\mathrm{Snd}\ M]\!]^{\text{term}}_{\xi} = \mathrm{Snd}^{\sigma,\tau}([\![M]\!]^{\text{term}}_{\xi}),\\
&[\![\langle M, N\rangle]\!]^{\text{term}}_{\xi} = \mathrm{Pair}^{\sigma,\tau}([\![M]\!]^{\text{term}}_{\xi}, [\![N]\!]^{\text{term}}_{\xi}),\\
&M =_{\beta\eta\pi} N \Rightarrow \forall\xi \quad [\![M]\!]^{\text{term}}_{\xi} = [\![N]\!]^{\text{term}}_{\xi}.
\end{aligned}$$

for every $M, N \in \Lambda_{\times,\rightarrow}$.

In general, following the ordinary notation in logic, we write

$$\mathscr{A} \models M = N$$

for a model $\mathscr{A}$ and terms M and N of a system of simply typed λ-calculus if $[\![M]\!]^{\text{term}}_{\xi} = [\![N]\!]^{\text{term}}_{\xi}$ holds for every environment ξ. For the sake of simplicity, we omit the superscripts to distinguish type-interpretation and term-interpretation, denoting both of them simply by $[\![\,]\!]$ in the rest of this paper.

In the discussions on semantics of simply typed λ-calculus, we often force the following condition on the definition of models, called the weak-extensionality,

$$\begin{aligned}
\forall d \in [\![\sigma]\!] \quad \mathrm{App}^{\sigma,\tau}([\![\lambda x^{\sigma}.M]\!]_{\xi}, d) &= \mathrm{App}^{\sigma,\tau}([\![\lambda x^{\sigma}.N]\!]_{\xi}, d)\\
&\Rightarrow\; [\![\lambda x^{\sigma}.M]\!]_{\xi} = [\![\lambda x^{\sigma}.N]\!]_{\xi}
\end{aligned}$$

in which the expression $\xi(x^{\sigma}\colon d)$ designates the environment such that the value of $\xi(x^{\sigma}\colon d)(y^{\tau})$ is defined by d if $y^{\tau} \equiv x^{\sigma}$, and by $\xi(y^{\tau})$ otherwise. Associating this strong property with models enables us to determine the denotation of a λ-abstraction $\lambda x^\sigma.M$ uniquely based on its extensional behaviour and to make the presentation of term-interpretation considerably simpler. This might be a main reason why the semantical discussion in standard literatures, such as [4, 8], deeply depends on the models endowed with the property of weak-extensionality, which we refer to under the name of type-frame or Henkin-model.

In the field of category theory, we can find a neat notion comparable with the models mentioned above. A category $\mathbf{C}$ is said to be cartesian closed if it permit a construction of finite products and exponentials, that is, we have operations

$$
\begin{aligned}
&1 \in \mathrm{Ob}(\mathbf{C}),\\
&A, B \in \mathrm{Ob}(\mathbf{C}) \ \Rightarrow\ A \times B, A^B \in \mathrm{Ob}(\mathbf{C}),
\end{aligned}
$$

for objects and operations

$$
\begin{aligned}
&\mathrm{id}_A \in \mathbf{C}(A, A),\\
&\bigcirc_A \in \mathbf{C}(A, 1),\\
&\mathrm{p}_{A,B} \in \mathbf{C}(A \times B, A), \quad \mathrm{q}_{A,B} \in \mathbf{C}(A \times B, B),\\
&\mathrm{ev}_{A,B} \in \mathbf{C}(B^A \times A, B),\\
&f \in \mathbf{C}(A, B) \ \&\ g \in \mathbf{C}(B, C) \ \Rightarrow\ g \circ f \in \mathbf{C}(A, C),\\
&f \in \mathbf{C}(A, B) \ \&\ g \in \mathbf{C}(A, C) \ \Rightarrow\ \langle f, g\rangle \in \mathbf{C}(A, B \times C),\\
&f \in \mathbf{C}(A \times B, C) \ \Rightarrow\ \mathrm{Cur}(f) \in \mathbf{C}(A, C^B),
\end{aligned}
$$

for arrows which satisfies

$$
\begin{aligned}
&f \circ \mathrm{id}_A = f, \quad \mathrm{id}_B \circ f = f, \quad (f \circ g) \circ h = f \circ (g \circ h),\\
&\bigcirc_B \circ f = \bigcirc_A,\\
&\bigcirc_1 = \mathrm{id}_1,\\
&\mathrm{p}_{A,B} \circ \langle f, g\rangle = f, \quad \mathrm{q}_{A,B} \circ \langle f, g\rangle = g, \quad \langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle,\\
&\langle \mathrm{p}_{A,B}, \mathrm{q}_{A,B}\rangle = \mathrm{id}_{A\times B},\\
&\mathrm{ev}_{B,C} \circ \langle \mathrm{Cur}(f) \circ \mathrm{p}_{A,B}, \mathrm{q}_{A,B}\rangle = f,\\
&\mathrm{Cur}(f) \circ g = \mathrm{Cur}(f \circ \langle g \circ \mathrm{p}_{A,B}, q_{A,B}\rangle),\\
&\mathrm{Cur}(\mathrm{ev}_{A,B}) = \mathrm{id}_{B^A},
\end{aligned}
$$

for every arrows f, g, h. A cartesian closed category $\mathbf{C}$ is said to be well-pointed if it is possible to discriminate arrows by composition with global elements of $\mathbf{C}$, that is,

$$\forall a \in \mathbf{C}(1, A) \quad f \circ a = g \circ a \ \Rightarrow\ f = g$$

holds for every $A, B \in \mathrm{Ob}(\mathbf{C})$ and $f, g \in \mathbf{C}(A, B)$.

By virtue of this algebraic notion, we actually provide an internal semantics of simply typed λ-calculus. To see it, let us suppose $\mathbf{C}$ is a cartesian closed category and $\mathcal{K}$ is an interpretation of the type constant o, that is,

$\mathcal{K}(o)$ specifies an object of $\mathbf{C}$. Then, for each type σ we inductively assign an object $[\sigma] \in \mathrm{Ob}(\mathbf{C})$ by

$$
\begin{aligned}
[o] &= \mathcal{K}(o),\\
[1] &= 1,\\
[\sigma \times \tau] &= [\sigma] \times [\tau],\\
[\sigma \to \tau] &= [\tau]^{[\sigma]}.
\end{aligned}
$$

Denotation of a term $M : \sigma$ is given by an arrow in $\mathbf{C}$ relative to a finite sequence $\Delta = x_1^{\sigma_1}, \ldots, x_m^{\sigma_m}$ of variables which contains all free-variables in M and the components of which are distinct each other. More precisely, we define an object $\times(\Delta)$ by

$$\times(\Delta) = (\cdots(1 \times [\sigma_1]) \times [\sigma_2]) \cdots) \times [\sigma_m]$$

and associate an arrow $[M]_\Delta$ from $\times(\Delta)$ to $[\sigma]$ by induction on the structure of M, as follows:

$$
\begin{aligned}
[x^\sigma]_\Delta &= \mathrm{q} \circ \overbrace{\mathrm{p} \circ \cdots \circ \mathrm{p}}^{m-i \text{ times}} && \text{if } x^\sigma \text{ is the } i \text{ element of } \Delta,\\
[\mathrm{Fst}\, M]_\Delta &= \mathrm{p} \circ [M]_\Delta,\\
[\mathrm{Snd}\, M]_\Delta &= \mathrm{q} \circ [M]_\Delta,\\
[\langle M, N\rangle]_\Delta &= \langle [M]_\Delta, [N]_\Delta \rangle,\\
[MN]_\Delta &= \mathrm{ev} \circ \langle [M]_\Delta, [N]_\Delta \rangle,\\
[\lambda x^\sigma.M]_\Delta &= \mathrm{Cur}([M[x^\sigma := y^\sigma]]_{\Delta, y^\sigma}) && \text{where } y^\sigma \text{ is fresh.}
\end{aligned}
$$

This is regarded as an internal interpretation of M with respect to Δ. It is easy to verify that the axiom schemes considered in the previous section are all satisfied under this manner of interpretation. Thus we can introduce a typed applicative structure $\mathscr{A}_{\mathbf{C},\mathcal{K}} = \langle [\![\,]\!], \mathrm{Fst}, \mathrm{Snd}, \mathrm{Pair}, \mathrm{App}\rangle$ whose components are given by

$$
\begin{aligned}
[\![\sigma]\!] &= \mathbf{C}(1, [\sigma]),\\
\mathrm{Fst}^{\,\sigma,\tau}(s) &= \mathrm{p} \circ s,\\
\mathrm{Snd}^{\,\sigma,\tau}(s) &= \mathrm{q} \circ s,\\
\mathrm{Pair}^{\sigma,\tau}(s,t) &= \langle s, t\rangle,\\
\mathrm{App}^{\sigma,\tau}(s,t) &= \mathrm{ev} \circ \langle s, t\rangle,
\end{aligned}
$$

and it together with the term-interpretation

$$[\![M]\!]_\xi = [M]_\Delta \circ \langle\langle \cdots \langle \mathrm{O}_1, \xi(x_1^{\sigma_1})\rangle \cdots\rangle, \xi(x_m^{\sigma_m})\rangle$$

is shown to be a model of the system $\beta\eta\pi_{\to}$. Under this construction from cartesian closed categories to models, the condition of weak-extensionality is known to be equivalent to the condition of well-pointedness. Some of the known results concerning this categorical construction of models are summarised in the following proposition.

Proposition 1. *For every cartesian closed category* $\mathbf{C}$ *and an interpretation* $\mathcal{K}$ *of* o*, the following hold*:

(1) $\mathscr{A}_{\mathbf{C},\mathcal{K}}$ *gives rise to a model of* $\beta_{\rightarrow}$, $\beta\eta_{\rightarrow}$ *and* $\beta\eta\pi_{\rightarrow}$.
(2) $\mathscr{A}_{\mathbf{C},\mathcal{K}}$ *satisfies weak-extensionality if and only if* $\mathbf{C}$ *is well-pointed.*

3. Extensionality of λ-definable arrows

Even if a given cartesian closed category $\mathbf{C}$ is not well-pointed, we can ensure that the interpretations of closed terms in $\mathscr{A}_{\mathbf{C},\mathcal{K}}$ satisfies the condition of weak-extensionality subject to a certain condition on $\mathbf{C}$. In what follows, we are to elaborate this result.

Our proof is demonstrated by analogy with the observation in [11] and we confine our attention to a restricted form of terms, called Φ-normal forms. Here Φ-normal forms are defined by the following induction:

$$\begin{aligned}&m \in \mathbb{N} \quad \& \\ &M_1, \dots, M_m \in \Lambda_{\rightarrow} \text{ are } \Phi\text{-normal forms of type } \sigma_1 \rightarrow \cdots \rightarrow \sigma_k \rightarrow o \\ &\qquad \Rightarrow \lambda x_1^{\sigma_1} \cdots x_k^{\sigma_k}.x^{\tau_1 \rightarrow \cdots \tau_m \rightarrow o}(M_1 x_1^{\sigma_1} \cdots x_k^{\sigma_k}) \cdots (M_m x_1^{\sigma_1} \cdots x_k^{\sigma_k}) \\ &\qquad\quad \text{is a } \Phi\text{-normal form}\end{aligned}$$

It is easy to verify for every term M that there exists a unique Φ-normal form which equals M under the equality $=_{\beta\eta}$, as is remarked in [3, 11]. Besides, we are able to demonstrate that distinct closed Φ-normal forms can be separated into distinct elements of $\Lambda_{\times,\rightarrow}(X)$ with respect to the equality $=_{\beta\eta\pi}$. To see it, we let X be the set of variables of type o or of type $o \times o \rightarrow o$ and $\Lambda_{\times,\rightarrow}(X)$ the set $\{M \in \Lambda_{\times,\rightarrow} \mid \mathrm{FV}(M) \subseteq X\}$.

Lemma 2. *Suppose that* $M, N \in \Lambda_{\rightarrow}$ *are closed* Φ*-normal forms of type* $\upsilon_1 \rightarrow \cdots \rightarrow \upsilon_l \rightarrow o$. *Then* $M \equiv N$ *holds if and only if* $MQ_1^{\upsilon_1} \cdots Q_l^{\upsilon_l} =_{\beta\eta\pi} NQ_1^{\upsilon_1} \cdots Q_l^{\upsilon_l}$ *for every* $Q_1^{\upsilon_1}, \dots, Q_l^{\upsilon_l} \in \Lambda_{\times,\rightarrow}(X)$.

Proof. To see the if-part, we show the contraposition by induction on the structure of M and N. The cases to be demonstrated are listed below, in which we assume that

$$M \equiv \lambda x_1^{\upsilon_1} \cdots x_l^{\upsilon_l}.x_i^{\rho_1 \rightarrow \cdots \rightarrow \rho_m \rightarrow o}(M_1 x_1^{\upsilon_1} \cdots x_l^{\upsilon_l}) \cdots (M_m x_1^{\upsilon_1} \cdots x_l^{\upsilon_l})$$

and

$$N \equiv \lambda x_1^{\upsilon_1} \cdots x_l^{\upsilon_l}.x_j^{\sigma_1 \rightarrow \cdots \rightarrow \sigma_n \rightarrow o}(N_1 x_1^{\upsilon_1} \cdots x_l^{\upsilon_l}) \cdots (N_n x_1^{\upsilon_1} \cdots x_l^{\upsilon_l}),$$

which do not coincide.

Case 1: Suppose $i \neq j$. Then it suffice to define $Q_i \equiv \lambda y_1^{\rho_1} \cdots y_m^{\rho_m}.y^o$, $Q_j \equiv \lambda z_1^{\sigma_1} \cdots z_n^{\sigma_n}.z^o$ where $y \not\equiv z$ and Q_k to be an arbitrary term of type υ_k for each $k \notin \{i, j\}$. Indeed, we have $MQ_1 \cdots Q_l =_{\beta\eta\pi} y$ and $NQ_1 \cdots Q_l =_{\beta\eta\pi} z$, which never coincide under the equality $=_{\beta\eta\pi}$.

Case 2: Suppose $i = j$, which entails $m = n$ and $\rho_k = \sigma_k$ for every $k \in \{1, \dots, m\}$. Then there exists $k \in \{1, \dots, m\}$ such that $M_k \not\equiv N_k$. Without loss of generality, we may assume that $\rho_k = \upsilon_{l+1} \rightarrow \cdots \rightarrow \upsilon_r \rightarrow o$ for some type $\upsilon_{l+1}, \dots, \upsilon_r$. Then the induction hypothesis allows us to have $Q_1^{\upsilon_1}, \dots, Q_l^{\upsilon_l}, Q_{l+1}^{\upsilon_{l+1}}, \dots, Q_r^{\upsilon_r} \in \Lambda_{\times,\rightarrow}(X)$ such that

$$M_k Q_1^{\upsilon_1} \cdots Q_r^{\upsilon_r} \neq_{\beta\eta\pi} N_k Q_1^{\upsilon_1} \cdots Q_r^{\upsilon_r}.$$

Using them, we define $R_1, \ldots, R_r \in \Lambda_{\times,\to}(X)$ by

$$R_s \equiv \begin{cases} \lambda x_1^{\rho_1} \cdots x_m^{\rho_m}.\, f^{o\times o\to o} \langle x_k^{\rho_k} Q_{l+1} \cdots Q_r, Q_i x_1^{\rho_1} \cdots x_m^{\rho_m} \rangle & \text{if } s = i, \\ Q_s & \text{otherwise,} \end{cases}$$

in which the variable f is fresh. Now let us suppose $MR_1 \cdots R_l =_{\beta\eta\pi} NR_1 \cdots R_l$. Then we have

$$\begin{aligned}
&M_k R_1 \cdots R_r \\
&=_{\beta\eta\pi} (\lambda f.f\,\langle M_k R_1 \cdots R_r\,,\, Q_i(M_1 R_1 \cdots R_l) \cdots (M_m R_1 \cdots R_l)\rangle)\, \lambda y^{o\times o}.\mathrm{Fst}\, y \\
&=_{\beta\eta\pi} (\lambda f.R_i(M_1 R_1 \cdots R_l) \cdots (M_m R_1 \cdots R_l))\, \lambda y^{o\times o}.\mathrm{Fst}\, y \\
&=_{\beta\eta\pi} (\lambda f.MR_1 \cdots R_l)\, \lambda y^{o\times o}.\mathrm{Fst}\, y \\
&=_{\beta\eta\pi} (\lambda f.NR_1 \cdots R_l)\, \lambda y^{o\times o}.\mathrm{Fst}\, y \\
&=_{\beta\eta\pi} (\lambda f.R_i(N_1 R_1 \cdots R_l) \cdots (N_m R_1 \cdots R_l))\, \lambda y^{o\times o}.\mathrm{Fst}\, y \\
&=_{\beta\eta\pi} (\lambda f.f\,\langle N_k R_1 \cdots R_r\,,\, Q_i(N_1 R_1 \cdots R_l) \cdots (N_m R_1 \cdots R_l)\rangle)\, \lambda y^{o\times o}.\mathrm{Fst}\, y \\
&=_{\beta\eta\pi} N_k R_1 \cdots R_r,
\end{aligned}$$

and therefore

$$\begin{aligned}
M_k Q_1 \cdots Q_r &=_{\beta\eta\pi} M_k Q_1 \cdots Q_{i-1}(\lambda x_1 \cdots x_m.Q_i x_1 \cdots x_m) Q_{i+1} \cdots Q_r \\
&=_{\beta\eta\pi} (\lambda f.M_k R_1 \cdots R_r)\, \lambda y^{o\times o}.\mathrm{Snd}\, y \\
&=_{\beta\eta\pi} (\lambda f.N_k R_1 \cdots R_r)\, \lambda y^{o\times o}.\mathrm{Snd}\, y \\
&=_{\beta\eta\pi} N_k Q_1 \cdots Q_{i-1}(\lambda x_1 \cdots x_m.Q_i x_1 \cdots x_m) Q_{i+1} \cdots Q_r \\
&=_{\beta\eta\pi} N_k Q_1 \cdots Q_r.
\end{aligned}$$

This is a contradiction. Hence we obtain $MR_1 \cdots R_l \neq_{\beta\eta\pi} NR_1 \cdots R_l$. □

By this lemma, we know that the equality of the system $\beta\eta_\to$ can be reduced to the equality on a subset of $\Lambda_{\times,\to}(X)$ without loss of any information, an element of which can be regarded as a binary tree characterised by the abstract syntax

$$t ::= x^o \mid f^{o\times o\to o}\langle t, t\rangle$$

in which x varies over the set of term-variables of type o and f over the set of term-variables of type $o \times o \to o$. We use letters s and t as meta-variables to designate terms in this subset. It is easy to verify that for every $M \in \Lambda_{\times,\to}(X)$ there exists a unique term s in the form of a binary tree such that $M =_{\beta\eta\pi} s$. This is why a model constructed from a cartesian closed category induces the same theory as whenever it discriminates distinct binary trees in $\Lambda_{\times,\to}(X)$, that is,

$$s \equiv t \;\Leftrightarrow\; \mathscr{A}_{\mathbf{C},\mathcal{K}} \models s = t \tag{4}$$

is true for every $s, t \in \Lambda_{\times,\to}(X)$ in the form of a binary tree.

Theorem 3. *Suppose* $\mathbf{C}$ *is a cartesian closed category and* $\mathcal{K}$ *an interpretation of* o *such that the model* $\mathscr{A}_{\mathbf{C},\mathcal{K}}$ *satisfies* (4). *Then the following conditions are mutually equivalent for every terms* $M, N \in \Lambda_\to$ *of type* $\tau_1 \to \cdots \to \tau_l \to o$ *:*

(1) $M =_{\beta\eta} N$.

(2) $\mathscr{A}_{\mathbf{C},\mathcal{K}} \models M = N$.
(3) $\mathscr{A}_{\mathbf{C},\mathcal{K}} \models MQ_1^{\tau_1}\cdots Q_l^{\tau_l} = NQ_1^{\tau_1}\cdots Q_l^{\tau_l}$ *for every* $Q_1^{\tau_1},\ldots,Q_l^{\tau_l} \in \Lambda_{\times,\to}(X)$.

Proof. (1) $\Rightarrow$ (2) is by Proposition 1 and (2) $\Rightarrow$ (3) is clear. So we concentrate on the direction (3) $\Rightarrow$ (1). Suppose $M \neq_{\beta\eta} N$ and $\mathrm{FV}(M) \cup \mathrm{FV}(N) = \{x_1^{\sigma_1},\ldots,x_k^{\sigma_k}\}$, and let us denote the Φ-normal forms of $\lambda x_1^{\sigma_1}\cdots x_k^{\sigma_k}.M$ and $\lambda x_1^{\sigma_1}\cdots x_k^{\sigma_k}.N$ by M' and N' respectively. Then $M' \neq_{\beta\eta} N'$ follows, from which we can find $P_1^{\sigma_1},\ldots,P_k^{\sigma_k},Q_1^{\tau_1},\ldots,Q_l^{\tau_l} \in \Lambda_{\times,\to}(X)$ such that

$$M'P_1^{\sigma_1}\cdots P_k^{\sigma_k}Q_1^{\tau_1}\cdots Q_l^{\tau_l} \neq_{\beta\eta} N'P_1^{\sigma_1}\cdots P_k^{\sigma_k}Q_1^{\tau_1}\cdots Q_l^{\tau_l}$$

by Lemma 2. This together with the condition (4) implies

$$\text{(5)} \qquad \mathscr{A}_{\mathbf{C},\mathcal{K}} \not\models M'P_1^{\sigma_1}\cdots P_k^{\sigma_k}Q_1^{\tau_1}\cdots Q_l^{\tau_l} = N'P_1^{\sigma_1}\cdots P_k^{\sigma_k}Q_1^{\tau_1}\cdots Q_l^{\tau_l}.$$

Now suppose $\mathscr{A}_{\mathbf{C},\mathcal{K}} \models MQ_1^{\sigma_1}\cdots Q_l^{\sigma_l} = NQ_1^{\sigma_1}\cdots Q_l^{\sigma_l}$ for every $Q_1^{\sigma_1},\ldots,Q_l^{\sigma_l} \in \Lambda_{\times,\to}(X)$. Then, we obtain

$$\begin{aligned}
[\![M'P_1^{\tau_1}\cdots P_k^{\tau_k}Q_1^{\sigma_1}\cdots Q_l^{\sigma_l}]\!]_\xi &= [\![M'y_1^{\tau_1}\cdots y_k^{\tau_k}Q_1^{\sigma_1}\cdots Q_l^{\sigma_l}]\!]_{\xi'} \\
&= [\![MQ_1^{\sigma_1}\cdots Q_l^{\sigma_l}]\!]_{\xi'} \\
&= [\![NQ_1^{\sigma_1}\cdots Q_l^{\sigma_l}]\!]_{\xi'} \\
&= [\![N'y_1^{\tau_1}\cdots y_k^{\tau_k}Q_1^{\sigma_1}\cdots Q_l^{\sigma_l}]\!]_{\xi'} \\
&= [\![N'P_1^{\tau_1}\cdots P_k^{\tau_k}Q_1^{\sigma_1}\cdots Q_l^{\sigma_l}]\!]_\xi
\end{aligned}$$

for every environment ξ and $P_1^{\tau_1},\ldots,P_k^{\tau_k},Q_1^{\sigma_1},\ldots,Q_l^{\sigma_l} \in \Lambda_{\times,\to}(X)$ where $\xi' = \xi(y_1^{\tau_1} : [\![P_1^{\tau_1}]\!]_\xi)\cdots(y_k^{\tau_k} : [\![P_k^{\tau_k}]\!]_\xi)$. This immediately contradicts to the condition (5). As a result, we can find $Q_1^{\sigma_1},\ldots,Q_l^{\sigma_l} \in \Lambda_{\times,\to}(X)$ such that $\mathscr{A}_{\mathbf{C},\mathcal{K}} \not\models MQ_1^{\sigma_1}\cdots Q_l^{\sigma_l} = NQ_1^{\sigma_1}\cdots Q_l^{\sigma_l}$. $\square$

As a by-product of this theorem, we obtain a version of satisfiability of the rule ξ studied in [2] which turns out to be strictly weaker than the weak-extensionality.

Corollary 4. *Suppose* $\mathbf{C}$ *is a cartesian closed category and* $\mathcal{K}$ *an interpretation of* o *such that the model* $\mathscr{A}_{\mathbf{C},\mathcal{K}}$ *satisfies* (4). *Then* $\mathscr{A}_{\mathbf{C},\mathcal{K}} \models M = N$ *implies* $\mathscr{A}_{\mathbf{C},\mathcal{K}} \models \lambda x^\sigma.M = \lambda x^\sigma.N$ *for every* $M, N \in \Lambda_\to$.

Proof. Suppose $\mathscr{A}_{\mathbf{C},\mathcal{K}} \not\models \lambda x^\sigma.M = \lambda x^\sigma.N$. Then, Theorem 3 ensures existence of a term $P^\sigma \in \Lambda_{\times,\to}(X)$ such that $\mathscr{A}_{\mathbf{C},\mathcal{K}} \not\models (\lambda x^\sigma.M)P^\sigma = (\lambda x^\sigma.N)P^\sigma$. Hence we have

$$\begin{aligned}
[\![M]\!]_{\xi(x^\sigma:[\![P^\sigma]\!]_\xi)} &= [\![(\lambda x^\sigma.M)P^\sigma]\!]_\xi \\
&\neq [\![(\lambda x^\sigma.N)P^\sigma]\!]_\xi \\
&= [\![N]\!]_{\xi(x^\sigma:[\![P^\sigma]\!]_\xi)}
\end{aligned}$$

for some ξ, from which $\mathscr{A}_{\mathbf{C},\mathcal{K}} \not\models M = N$ follows. $\square$

Furthermore, the statement above ensures the extensionality of λ-definable arrows, namely, the arrows represented by the interpretation of a closed term, in a cartesian closed category satisfying (4). In the statement below, obeying the usual abbreviation, for every $f \in \mathbf{C}(1, B^A)$ we write $f^\wr$ for the arrow $\mathrm{ev}\langle f \circ \bigcirc_A, \mathrm{id}_A\rangle \in \mathbf{C}(A, B)$.

Corollary 5. *Suppose* $\mathbf{C}$ *is a cartesian closed category and* $\mathcal{K}$ *an interpretation of* o *such that the model* $\mathscr{A}_{\mathbf{C},\mathcal{K}}$ *satisfies* (4). *Then the following hold for every* λ*-definable arrows* $f, g \in [\![\sigma \to \tau]\!]$:
(1) *If* $\mathrm{App}^{\sigma,\tau}(f, a) = \mathrm{App}^{\sigma,\tau}(g, a)$ *for every* $a \in [\![\sigma]\!]$, *then* $f = g$.
(2) *If* $f^{\wr} \circ a = g^{\wr} \circ a$ *for every* $a \in \mathbf{C}(1, [\![\sigma]\!])$, *then* $f^{\wr} = g^{\wr}$.

Proof. (1) From the assumption of λ-definability, we may assume that $f = [\![\lambda x^\sigma.M]\!]_\xi$ and $g = [\![\lambda x^\sigma.N]\!]_\xi$ for some closed term $\lambda x^\sigma.M, \lambda x^\sigma.N \in \Lambda_\to$. Then we have

$$\begin{aligned}
[\![M]\!]_{\xi(x^\sigma:a)} &= \mathrm{App}^{\sigma,\tau}([\![\lambda x^\sigma.M]\!]_\xi, a) \\
&= \mathrm{App}^{\sigma,\tau}([\![\lambda x^\sigma.N]\!]_\xi, a) \\
&= [\![N]\!]_{\xi(x^\sigma:a)}
\end{aligned}$$

for every $a \in [\![\sigma]\!]$ and ξ, from which $[\![\lambda x^\sigma.M]\!]_\xi = [\![\lambda x^\sigma.N]\!]_\xi$, follows by Corollary 4.
(2) is immediate from (1). □

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