数理解析研究所講究録
1535 巻 2007 年 11-14

SYMMETRIC OPERATOR WORD EQUATIONS

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ABSTRACT. Symmetric matrix word equations have recently been the topic of active investigation because of their relationship to the Bessis-Moussa-Villani trace conjecture, an open conjecture arising from statistical physics. In this paper we consider infinite dimensional (operator) word equations and show the uniqueness of positive definite solutions of some word equations via the non-positive curvature property of Thompson's part metric on the space of positive definite operators in a Hilbert space.

For a nonempty alphabet $\mathbb{A}$, we consider the concatenation monoid generalized words of the form

$$W = A_1^{p_1} A_2^{p_2} \cdots A_k^{p_k},$$

where each $A_j \in \mathbb{A}$ and each exponent p_j is a real number, subject to the standard exponential laws for adjacent powers with a common base. In particular $A_j^0 = I$, the identity, for each j. The *reversal* W^* of the word W is the word written in reverse order, and the word is *symmetric* (or "palindromic") if it is equal to its reversal. A *symmetric word equation* for $\mathbb{A} = \{X, A, B\}$ is an equation of the form $S(X, A) = B$, where $S(X, A)$ is a symmetric word in X and A; we further assume that the exponents of X are all positive, and other exponents are nonnegative.

Definition 1. *A symmetric word equation $S(X, A) = B$ is called* (uniquely) solvable *if there exists (uniquely) a positive definite solution X of $S(X, A) = B$ for every pair of positive definite operators A and B on a Hilbert space.*

Example 2. *The Riccati matrix equation $XAX = B$ is uniquely solvable. It has a unique positive definite solution, the geometric mean $A^{-1}\#B$ of A^{-1} and B : $A^{-1}\#B = A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2}$. See* ([8]-[10]) *for more general setting.*

Date: January 26, 2007.
Key words and phrases. Operator word equation, positive definite operator, nonlinear operator equation, Thompson's part metric, contraction.

Example 3. *The symmetric word equation $X(AX)^m = B$ appears in [4] which has the unique solution $X = A^{-1}\#_{\frac{1}{m+1}}B$, where $A\#_tB = A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2}$ denotes the t-power mean of A and B.*

In [6], Hillar and Johnson proved that every symmetric word equation is solvable for finite-dimensional case and Armstrong and Hillar [1] have recently showed via the Browner mapping degree that the symmetric word equation of degree 6

$$XBX^2B^3X^2BX = A \tag{1}$$

has multiple positive definite solutions for some 3×3 positive definite matrices A and B solving the conjecture negatively, but it has a unique positive definite solution in 2×2 real positive definite letters remaining the conjecture open for 2×2 positive definite matrices.

Theorem 4 (Armstrong and Hillar, [1]). *The symmetric word equation (1) is not uniquely solvable for 3×3 positive definite letters, but is uniquely solvable for 2×2 real positive definite letters.*

In [14], an explicit form of the unique solution of (1) in 2×2 real positive definite letters is given by

$$X = (sB^{-1} + B^{-3})\#(tI - A)^{-1}$$

where $t > \mathrm{tr}(A)$ and $s > 0$ are uniquely determined by the simultaneous equations

$$\frac{s + \mathrm{tr}(B^2)}{t - \mathrm{tr}(A)} \;=\; \frac{t}{s} = \mathrm{tr}([(sB^{-1} + B^{-3})\#(tI - A)^{-1}]^2B).$$

For a Hilbert space E, let $\mathcal{B}(E)$ denote the set of bounded linear operators, $\mathcal{S}(E) \subseteq \mathcal{B}(E)$ the symmetric operators, and $\Omega \subseteq \mathcal{S}(E)$ the set of positive definite operators on E. We define a closed positive order on $\mathcal{S}(E)$ by $A \leq B$ if $B - A$ is positive semidefinite. The Thompson (or part) metric on Ω given by

$$d(A,B) = \max\{\log M(A/B), \log M(B/A)\}$$

$M(A/B) := \inf\{\lambda > 0 : A \leq \lambda B\}$. A. C. Thompson [17] (cf. [15]-[16]) has shown that Ω is a complete metric space with respect to this metric and the corresponding metric topology on Ω agrees with the relative norm topology. It is easy to see that

$$d(A,B) = d(A^{-1}, B^{-1}) = d(M^*AM, M^*BM)$$

for any invertible operator M. The Löwner-Heinz inequality

$$0 \le A \le B \Longrightarrow A^t \le B^t, \ t \in [0,1]$$

is equivalent to the following nonpositive curvature property of Thompson's part metric (cf. [2], [3], [12], [13]).

Theorem 5. *For $A, B, C \in \Omega$,*

$$d(A\#_t B, A\#_t C) \le t d(B,C), \ t \in [0,1].$$

Theorem 6. *Symmetric word equations through degree 5 is uniquely solvable.*

Proof. Let $S(X,A) = B$ be a symmetric word equation with degree $n \le 5$. By the Riccati Lemma (Example 2), we assume that $n \ge 3$. If $n = 3$, then the only non-trivial equation is $XAXAX = B$, which has the unique positive definite solution $X = A^{-1}\#_{1/3}B$ from $XAXAX = X(AX)^2 = B$ if and only if $X = A^{-1}\#_{1/3}B$ (Example 3). If $n = 4$, then up to equivalence it is enough to consider the equations $XAXA^mXAX = B$ and $XAX^2AX = B$. By the Riccati Lemma,

$$XAXA^mXAX = B \iff XAX = B\#A^{-m} \iff X = (B\#A^{-m})\#A^{-1},$$
$$XAX^2AX = B \iff (XAX)^2 = B \iff XAX = B^{1/2} \iff X = A^{-1}\#B^{1/2}.$$

Let $n = 5$. In this case, non-trivial equations (up to equivalence) are $XAX^3AX = B$ and $XAXA^mXA^mXAX = B$. Let

$$f(X) = (B\#X^{-1})\#A^{-1}, \ \ g(X) = [B\#(A^mXA^m)^{-1}]\#A^{-1}.$$

By the invariance properties and non-positive curvature property, f and g are strict contractions for the Thompson metric and hence by completeness of the metric have unique fixed points, respectively. Then the proof follows from $(XAX)X(XAX) = B$ if and only if $XAX = B\#X^{-1}$ if and only if $X = (B\#X^{-1})\#A^{-1}$ if and only if $X = f(X)$, and $(XAX)(A^mXA^m)(XAX) = B$ if and only if $XAX = B\#(A^mXA^m)^{-1}$ if and only if $X = [B\#(A^mXA^m)^{-1}]\#A^{-1}$ if and only if $X = g(X)$. □

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