

SMOOTH SHOCK PROFILES FOR HYPERBOLIC SYSTEMS OF BALANCE LAWS

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ABSTRACT. In this report, we review some existence results on traveling wave solutions for hyperbolic systems of balance laws satisfying a structural stability condition. As an application, we show the existence of smooth shock profiles for discrete-ordinate systems of equations for radiation hydrodynamics.

1. INTRODUCTION

The goal of this report is to review some existence results on smooth shock profiles for general hyperbolic systems of balance laws:

$$(1.1) \quad U_t + \sum_{j=1}^d F_j(U)_{x_j} = Q(U).$$

Here U is the unknown n -vector valued function of $(x, t) \equiv (x_1, x_2, \dots, x_d, t) \in \mathbf{R}^d \times [0, +\infty)$, taking values in an open subset \mathcal{G} of \mathbf{R}^n (called *state space*); $Q(U)$ and $F_j(U)$ ($j = 1, 2, \dots, d$) are given n -vector valued smooth functions of $U \in \mathcal{G}$.

Balance laws of the form (1.1) describe various non-equilibrium phenomena in physics. Important examples occur in inviscid gas dynamics with relaxation, kinetic theories (moment closure systems and discrete-velocity models), radiation hydrodynamics, chemically reactive flows, traffic flows, nonlinear optics, the numerical solution of conservation laws with relaxation schemes, and so on. See [18] and references cited therein.

In the aforementioned applications, the source term $Q(U)$ has, or can be transformed with a linear transformation into, the form

$$Q(U) = \begin{pmatrix} 0 \\ q(U) \end{pmatrix}$$

with $q(U) \in \mathbf{R}^r$ consisting of r linearly independent functions of $U \in \mathcal{G}$. Accordingly, we often write

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad F_j(U) = \begin{pmatrix} f_j(u, v) \\ g_j(u, v) \end{pmatrix}, \quad Q(U) = \begin{pmatrix} 0 \\ q(u, v) \end{pmatrix}.$$

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Then (1.1) can be rewritten as

$$(1.2) \quad \begin{aligned} u_t + \sum_{j=1}^d f_j(u, v)_{x_j} &= 0, \\ v_t + \sum_{j=1}^d g_j(u, v)_{x_j} &= q(u, v). \end{aligned}$$

Throughout this paper, we assume that $q(u, v) = 0$ uniquely determines v in terms of u , say $v = h(u)$.

For the above equations, our interest here is traveling wave solutions of the form

$$U(x, t) = \phi(\omega \cdot x - s_* t)$$

with $\omega = (\omega_1, \dots, \omega_d) \in \mathbf{R}^d \setminus \{0\}$, $\omega \cdot x = \sum_{j=1}^d \omega_j x_j$, $s_* \in \mathbf{R}$ and $\phi(\pm\infty) = U_{\pm}$ appropriately given. As a solution to (1.1), $\phi = \phi(\sigma)$ solves

$$(1.3) \quad \sum_{j=1}^d \omega_j F_j(\phi)_{\sigma} - s_* \phi_{\sigma} = Q(\phi).$$

Therefore, our task is to show the existence of a trajectory $\phi = \phi(\sigma)$ connecting U_- on the left to U_+ on the right.

Note that system (1.3) together with $\phi(\pm\infty) = U_{\pm}$ suggests $Q(U_{\pm}) = 0$. According to our assumption following (1.2), such U_{\pm} can be written as

$$U_{\pm} = \begin{pmatrix} u_{\pm} \\ h(u_{\pm}) \end{pmatrix}$$

with $u_{\pm} \in \mathbf{R}^{n-r}$. Since the first $(n-r)$ components of $Q(U)$ vanish identically, it is easy to see from the first $(n-r)$ equations in (1.3) that the quantities u_{\pm} , s_* and ω satisfy the following relation

$$(1.4) \quad s_*(u_+ - u_-) = \sum_{j=1}^d \omega_j [f_j(u_+, h(u_+)) - f_j(u_-, h(u_-))].$$

This relation particularly determines s_* as a function of (u_-, u_+, ω) . Note that as u_+ tends to u_- , $s_* = s_*(u_-, u_+, \omega)$ converges to an eigenvalue, say $\lambda_k(u_-, \omega)$, of $(n-r) \times (n-r)$ -matrix $\sum_j \omega_j \partial_u (f_j(u, h(u)))|_{u=u_-}$.

Because of (1.4), it is well known that piecewise-constant function

$$(1.5) \quad u(x, t) = \begin{cases} u_- , & \omega \cdot x < s_* t \\ u_+ , & \omega \cdot x > s_* t \end{cases}$$

is a weak solution of the following conservation laws

$$(1.6) \quad u_t + \sum_{j=1}^d f_j(u, h(u))_{x_j} = 0.$$

This is the so-called *equilibrium system* corresponding to the balance laws (1.2). It governs the dynamics of (1.2) in the *zero relaxation limit* (see, e.g. [18]). The piecewise-constant solution (1.5) is called a *shock front* if (1.6) is (strictly) hyperbolic and the solution satisfies the entropy condition [6].

The above discussion relates the traveling wave solutions to shock waves (1.5). Therefore, the former are often called *shock profiles* and their existence is of physical interest [1, 10, 14, 16]. The analogous existence problem was first posed by Gelfand [4] for viscosity approximations of hyperbolic conservation laws and was solved by Majda and Pego [8].

On the other hand, it was observed in [17, 18] that many equations of the form (1.1) from mathematical physics satisfy a structural stability condition proposed in [17]. Under the stability condition, the existence of smooth shock profiles was established in [19] under the additional assumption that the matrix $A(U_-)$ is invertible, where

$$(1.7) \quad A(U) := \sum_{j=1}^d \omega_j F_{jU}(U) - \lambda_k(u_-, \omega) \mathcal{I}_n$$

with \mathcal{I}_n the unit matrix of order n . The result is stated as Theorem 2.1 in Section 2.

When $A(U_-)$ is not invertible, it was speculated that smooth shock profiles might not exist, that is, the profiles might contain subshocks. The speculation was denied in [16] with numerical evidences for a number of specific systems of the form (1.1) from extended thermodynamics [10].

In [3], we singled out a set of sufficient conditions which ensure the existence of smooth shock profiles, even if $A(U_-)$ is not invertible. One of the conditions is of Kawashima-like [15] and another one is that the matrix $A(U)$ has only one eigenvalue tending to 0 as U goes to U_- . We notice that for 2×2 systems (1.2) with $r = 1$, the stability condition and the Kawashima-like condition together guarantee the invertibility of $A(U_-)$. Let us mention that for 2×2 systems with $r = 1$, the above existence problem was satisfactorily solved by Liu [7].

Recently in [13], the above existence results are applied successfully to discrete-ordinate systems of equations for radiation hydrodynamics [9, 12]. In particular, we show that for the discrete-ordinate systems, the matrix $A(U)$ has only one eigenvalue tending to 0 as U goes to U_- .

This paper is organized as follows. In Section 2 we review two existence results from [19, 3]. The application to radiation hydrodynamics is sketched in Section 3.

2. EXISTENCE RESULTS

In this section we present two existence theorems. To begin with, we refer to (1.2) and recall the assumption that $q(u, v) = 0$ if and only if $v = h(u)$.

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Then we have

$$(2.1) \quad \begin{aligned} \mathcal{E} &:= \{U \in \mathcal{G} \mid Q(U) = 0\} \\ &= \left\{ U = \begin{pmatrix} u \\ h(u) \end{pmatrix} \in \mathcal{G} \quad \forall u \in \mathbf{R}^{n-r} \right\}. \end{aligned}$$

With \mathcal{E} (the so-called *equilibrium manifold*) defined thus, we fix

$$U_- \equiv \begin{pmatrix} u_- \\ h(u_-) \end{pmatrix} \in \mathcal{E}$$

throughout this paper. Moreover, we fix $\omega = (\omega_1, \omega_2, \dots, \omega_d) \in \mathbf{R}^d \setminus \{0\}$.

Our next assumption on the balance laws (1.1) is the structural stability condition previously proposed in [17]. Referring to the Appendix of [3], we only need the following three consequences of the condition:

$$(h1). \quad Q_U(U_-) = \text{diag}(0, S) \quad \text{with } S \in \mathbf{R}^{r \times r},$$

$$(h2). \quad \sum_{j=1}^d \omega_j F_{jU}(U_-) \text{ is a symmetric matrix,}$$

$$(h3). \quad S + S^* \leq -\mathcal{I}_r.$$

Here the subscript U refers to the usual partial derivatives with respect to U , the superscript $*$ denotes the transpose operator acting on matrices or vectors, and \mathcal{I}_r is the unit matrix of order r .

The structural stability condition and thereby (h1)–(h3) was shown in [17, 18] to be fulfilled by many systems of balance laws (1.1) from mathematical physics. On the other hand, it was proved in [17] that the stability condition implies the (strong) hyperbolicity of the equilibrium system (1.6): for each $\tilde{\omega} = (\tilde{\omega}_1, \dots, \tilde{\omega}_d) \in \mathbf{R}^d$, the characteristic matrix

$$(2.2) \quad a(\tilde{\omega}) := \sum_{j=1}^d \tilde{\omega}_j \partial_u (f_j(u, h(u)))|_{u=u_-}$$

has only real eigenvalues with a complete set of eigenvectors.

As in [19], we further assume that the matrix $a(\omega)$ has an isolated eigenvalue λ_k . Then we know from [6] that the Rankine-Hugoniot relation

$$(2.3) \quad s(u - u_-) = \sum_{j=1}^d \omega_j (f_j(u, h(u)) - f_j(u_-, h(u_-)))$$

defines a smooth curve $(u(\delta), s(\delta))$, for $\delta \in (-\bar{\delta}, \bar{\delta})$ with $\bar{\delta} > 0$ sufficiently small, satisfying

$$u(0) = u_- \quad \text{and} \quad s(0) = \lambda_k.$$

According to Liu [6], the piecewise-constant solution (1.5) is called a *shock-front* if there is a $\delta_* \in (-\bar{\delta}, \bar{\delta})$ such that $(u_+, s_*) = (u(\delta_*), s(\delta_*))$ and

$$s_* \leq s(\delta)$$

for all δ between 0 and δ_* .

Now we are in a position to state the first existence theorem established in [19].

Theorem 2.1. ([19]) *Suppose Q and F_j are smooth functions of U , the system (1.1) admits the structural conditions (h1)–(h3), λ_k is an isolated eigenvalue of the characteristic matrix $a(\omega)$ defined in (2.2), and the matrix $A(U_-)$ defined in (1.7) is invertible.*

Then there exists $\bar{\delta} > 0$ such that for any δ_ satisfying $|\delta_*| < \bar{\delta}$ and $s_* := s(\delta_*) < \lambda_k$, the connection problem (1.9) has a unique smooth solution $\phi = \phi(\sigma)$ with $\phi(+\infty) = U_+ := \begin{pmatrix} u(\delta_*) \\ h(u(\delta_*)) \end{pmatrix}$, which decays exponentially to U_{\pm} as σ goes to $\pm\infty$.*

The proof of this theorem was motivated by the work of Majda and Pego [8]. Indeed, when $r > 1$, (1.3) is a genuinely multidimensional connection problem for which we do not know a general existence theorem. As in [8] for viscosity problems, our strategy is to use the classical center manifold theorem [5] to reduce (1.3) to a one-dimensional problem, which is possible under the assumptions of Theorem 2.1.

Theorem 2.1 requires that $A(U_-)$ be invertible. In [3], we try to weaken the requirement and obtain the following result.

Theorem 2.2. ([3]) *Assume $A(U_-)$ is not invertible. If*

$$(h_4). \quad \text{Ker}(Q_U(U_-)) \cap \text{Ker}(A(U_-)) = \{0\}$$

and $A(U)$ has only one eigenvalue tending to 0 as U goes to U_- , then the conclusion of Theorem 2.1 still holds. In (h₄), $\text{Ker}(M)$ denotes the null space of matrix M .

About this theorem, we make two remarks.

Remark 2.1. If $A(U_-)$ is invertible, then $\text{Ker}(A(U_-)) = \{0\}$ and therefore (h₄) is implied by the assumptions of Theorem 2.1. Moreover, we notice that (h₄) appears necessary for the existence of the smooth profiles. To see this, we consider the *decoupled* system

$$\begin{aligned} u_t + f(u)_x &= 0, \\ v_t &= -v. \end{aligned}$$

This system violates (h₄) and has no smooth shock profiles.

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In addition, (h4) is very similar to but much weaker than the Kawashima condition [15]:

$$\text{Ker}(Q_U(U_-)) \cap \text{Ker}\left(\sum_{j=1}^d \tilde{\omega}_j F_{jU}(U_-) - \lambda \mathcal{I}_n\right) = \{0\}$$

for any $\lambda \in \mathbf{R}$ and any $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_d) \in \mathbf{R}^d \setminus \{0\}$.

Remark 2.2. For $n = 2$ and $r = 1$, (h4) implies that $A(U_-)$ is invertible. In fact, it follows from (h1) that

$$\text{Ker}(Q_U(U_-)) = \mathbf{R}^{n-r} \times \{0\} \subset \mathbf{R}^n.$$

On the other hand, it is obvious for $n - r = 1$ that the upper-left corner of $A(U_-)$ is zero. Thus, if (h4) holds true, then the 2×2 symmetric matrix $A(U_-)$ is not diagonal. Therefore, $A(U_-)$ is invertible.

For the proof of Theorem 2.2, we notice that the equations (1.3) have a singularity when (ϕ, s_*) is close to $(U_-, \lambda_k(u_-, \omega))$. To deal with the singularity, the underlying idea is to view it as an extremely large scale in the equations (1.3). With this understanding, we first reduce the existence problem to a parametrized one without singularity by using the center manifold theorem [5]. The main technical issue in [3] is to show that the structural conditions ensure the effectiveness of the reduction. Then, we modify the argument in [19] to show the existence of solutions for the parametrized problem. In this way, we get the existence of a trajectory for the multidimensional connection problem (1.3) when u_+ is close to u_- and $s_* < \lambda_k(u_-, \omega)$. See [3] for details.

Let us also remark that the Jacobian matrix of the left-hand side of the traveling wave equations (1.3) does not necessarily have a null space of constant dimension. So our singularity is more general than that treated by Pego [11] for physical viscosity problems.

3. AN APPLICATION

In this section, we apply the existence theory developed in [19, 3] to radiation hydrodynamics. The motion of a radiating fluid is governed by compressible Euler equations coupled via an integral-type source term to a family of radiation transport equations [9, 12]:

$$\begin{aligned} \rho_t + \text{div}(\rho \mathbf{v}) &= 0, \\ (\rho \mathbf{v})_t + \text{div}(\rho \mathbf{v} \otimes \mathbf{v} + p \mathcal{I}_3) &= 0, \\ (3.1) \quad E_t + \text{div}(E \mathbf{v} + p \mathbf{v}) &= \rho \oint_{S^2} (I - B(\theta)) d\xi, \\ I_t + c \text{div}(\xi I) &= c \rho (B(\theta) - I). \end{aligned}$$

Here the unknowns are the density $\rho = \rho(x, t)$ of the fluid, the velocity $\mathbf{v} = \mathbf{v}(x, t) \in \mathbf{R}^3$, the temperature $\theta = \theta(x, t)$, and the radiation intensity $I = I(x, t, \xi)$ for $(x, t) \in \mathbf{R}^3 \times [0, \infty)$ and $\xi \in S^2$ (the 2-sphere). In (3.1), div is

the usual divergence operator with respect to the spatial variable, \otimes stands for the tensor product, $p = p(\rho, \theta)$ is a given function denoting the pressure,

$$E = \rho\epsilon + \frac{\rho}{2}|\mathbf{v}|^2$$

with $\epsilon = \epsilon(\rho, \theta)$ a given function denoting the specific energy, $B(\theta)$ is the Planck function, and c is the speed of light.

Since the basic assumptions of radiation hydrodynamics are not valid at low temperatures, we restrict the temperature domain to $[\theta_0, \infty)$ with $\theta_0 > 0$ a constant. In this domain, the standard Planck function $B(\theta) = \theta^4$ is positive and strictly increasing. Thus, we assume that

$$(3.2) \quad B(\theta) > 0, \quad B'(\theta) > 0$$

for $\theta \geq \theta_0$.

In (3.1), the energy equation has an integral-type source term. In practice (computation) [2], the integral is often replaced with a finite sum of the form

$$\sum_{l=1}^L c_l (I(x, t, \xi_l) - B(\theta))$$

with $c_l > 0$ and $\xi_l \in \mathcal{S}^2$ for each l . By setting

$$(3.3) \quad I_l = c_l I(x, t, \xi_l) \quad \text{and} \quad B_l(\theta) = c_l B(\theta),$$

we obtain the following discrete-ordinate system of $(5 + L)$ equations:

$$(3.4) \quad \begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{v}) &= 0, \\ (\rho \mathbf{v})_t + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} + p \mathcal{I}_3) &= 0, \\ E_t + \operatorname{div}(E \mathbf{v} + p \mathbf{v}) &= \rho \sum_{l=1}^L (I_l - B_l(\theta)), \\ I_{lt} + c \operatorname{div}(\xi_l I_l) &= c \rho (B_l(\theta) - I_l). \end{aligned}$$

This system can be rewritten as (1.1) with

$$U = \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ E \\ I_1 \\ \vdots \\ I_L \end{pmatrix}, \quad F_j(U) = \begin{pmatrix} \rho v_j \\ \rho v_1 v_j + \delta_{1j} p \\ \rho v_2 v_j + \delta_{2j} p \\ \rho v_3 v_j + \delta_{3j} p \\ E v_j + p v_j \\ c \xi_{1j} I_1 \\ \vdots \\ c \xi_{Lj} I_L \end{pmatrix}, \quad Q(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \rho \sum_{l=1}^L (I_l - B_l(\theta)) \\ -c \rho (I_1 - B_1(\theta)) \\ \vdots \\ -c \rho (I_L - B_L(\theta)) \end{pmatrix}.$$

Here v_j is the j^{th} component of velocity \mathbf{v} , $\xi_l = (\xi_{l1}, \xi_{l2}, \xi_{l3})$, and δ_{ij} is the standard Kronecker delta. The state space \mathcal{G} is $(0, \infty) \times \mathbf{R}^3 \times (0, \infty)^{L+1}$.

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In [13], it was shown that system (3.4) admits the structural stability condition proposed in [17], under the standard thermodynamic assumption

$$(3.5) \quad p_\rho > 0, \quad \epsilon_\theta > 0.$$

Note that $B_l(\theta)$ defined in (3.3) preserves the monotonicity of $B(\theta)$ in (3.2).

Regarding the dissipative structure of (3.4), we proved in [13]:

Proposition 3.1. *Assume*

$$(3.6) \quad \text{Rank}[\xi_2 - \xi_1, \xi_3 - \xi_1, \dots, \xi_L - \xi_1] = 3.$$

If there is a $\tilde{\omega} \in \mathcal{S}^2$ such that $\text{Ker}(Q_U(U_-))$ contains an eigenvector of the characteristic matrix $\sum_{j=1}^3 \tilde{\omega}_j F_{jU}(U_-)$ associated with eigenvalue λ , then $\lambda = \mathbf{v}_- \cdot \tilde{\omega}$.

To prove the existence of smooth shock profiles for the discrete-ordinate system (3.4), we refer to (3.3) and assume

$$(3.7) \quad \sum_{l=1}^L \xi_l c_l = 0$$

for simplicity. Recall that

$$\sum_{l=1}^L c_l = |\mathcal{S}^2| \equiv c\kappa$$

with $\kappa = O(c^{-1})$ a constant. Under the assumption (3.7), the equilibrium system is

$$(3.8) \quad \begin{aligned} \rho_t + \text{div}(\rho \mathbf{v}) &= 0, \\ (\rho \mathbf{v})_t + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p &= 0, \\ (E + \kappa B(\theta))_t + \text{div}(E \mathbf{v} + p \mathbf{v}) &= 0. \end{aligned}$$

This system is different from the classical Euler equations, which are (3.8) with $\kappa = 0$.

In order to analyse its characteristic fields, we rewrite (3.8) in terms of $(\rho, \mathbf{v}, \theta)$:

$$(3.9) \quad \begin{aligned} \rho_t + \mathbf{v} \cdot \nabla \rho + \rho \text{div} \mathbf{v} &= 0, \\ \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p / \rho &= 0, \\ \theta_t + (1 + \alpha) \mathbf{v} \cdot \nabla \theta + \beta(\kappa) \text{div} \mathbf{v} &= 0. \end{aligned}$$

Here $\alpha = -\frac{\kappa B'(\theta)}{\rho \epsilon_\theta + \kappa B'(\theta)}$ and $\beta(\kappa) = \frac{\theta p_\theta}{\rho \epsilon_\theta + \kappa B'(\theta)}$. Thus, the characteristic matrix is

$$(\mathbf{v} \cdot \tilde{\omega}) \mathcal{I}_5 + \frac{1}{\rho} \begin{pmatrix} 0 & \rho^2 \tilde{\omega}^T & 0 \\ p_\rho \tilde{\omega} & 0 & p_\theta \tilde{\omega} \\ 0 & \beta(\kappa) \rho \tilde{\omega}^T & \alpha \rho \mathbf{v} \cdot \tilde{\omega} \end{pmatrix},$$

which is symmetrizable since $p_\theta \beta(\kappa) > 0$ and has $\mathbf{v} \cdot \tilde{\omega}$ as its eigenvalue of multiplicity two. The rest three eigenvalues are $\mathbf{v} \cdot \tilde{\omega} + \lambda$ where λ solves

$$(3.10) \quad \rho(p_\rho - \lambda^2)(\lambda - \alpha \mathbf{v} \cdot \tilde{\omega}) + p_\theta \beta(\kappa) \lambda = 0.$$

In case $\mathbf{v} \cdot \tilde{\omega} = 0$, $\lambda = 0$ is a solution (eigenvalue) to (3.10) and the characteristic equation for the other two eigenvalues becomes

$$\rho\lambda^2 = \rho p_\rho + p_\theta \beta(\kappa).$$

Thus these two non-zero eigenvalues are isolated for the system of conservation laws (3.8). In addition, by using (3.10) it is easy to verify that the corresponding characteristic fields are both genuinely nonlinear near $\mathbf{v} = 0$. We write the two isolated eigenvalues as $\lambda_k(U)$ with $k \in \{1, 5\}$, which are well defined in a neighborhood of $\mathbf{v} = 0$.

Now we solve the Rankine-Hugoniot relation (2.3) for the system of conservation laws (3.8). In doing so, we obtain two shock curves $(s_k(\delta), u_k(\delta))$ defined for δ close to zero. Moreover, $s_k(\delta)$ is strictly increasing (entropy inequality). Thus, for $\delta < 0$, the system of conservation laws (3.8) has two families of shock wave solutions of the form:

$$u(x, t) = \begin{cases} u_-, & \omega \cdot x < s_k(\delta)t, \\ u_k(\delta), & \omega \cdot x > s_k(\delta)t. \end{cases}$$

If $(\mathbf{v}_- \cdot \omega + \lambda_k(U_-))$ is not an eigenvalue of the discrete-ordinate system (3.4), we know from Theorem 2.1 that the structural stability condition ensures the existence of a smooth profile for all δ close to and less than zero.

Next we assume that $\mathbf{v}_- = 0$ and $\lambda_k = \lambda_k(U_-)$ is an eigenvalue of the discrete-ordinate system (3.4). Then it follows from $\lambda_k \neq 0$ and Proposition 3.1 that

$$\text{Ker}(Q_U(U_-)) \cap \text{Ker}\left(\sum_{j=1}^3 \omega_j F_{jU}(U_-) - \lambda_k \mathcal{I}_5\right) = \{0\}.$$

Moreover, since

$$\rho\lambda_k^2 = \rho p_\rho + p_\theta \beta(\kappa) \neq \rho p_\rho + p_\theta \beta(0),$$

λ_k can only be some of the constant eigenvalues $c\xi_i \cdot \omega$ for the discrete-ordinate system. Let $\lambda(U)$ be any eigenvalue for the system, which is equal to λ_k at $U = U_-$. Then $\lambda(U)$ is constant for U close to U_- . Thus, condition (h4) is verified.

In conclusion, we deduce the following result from the general theory developed in [19, 3].

Theorem 3.2. *Under the physical assumptions (3.2) and (3.5), under the discretization conditions (3.6) and (3.7), let $\mathbf{v}_- = 0$. Then for each $k \in \{1, 5\}$ and for each δ close to and less than zero, the discrete-ordinate system (3.4) has a smooth traveling wave solution $\phi(x - s_k(\delta)t)$ connecting U_- on the left to $U_+ := \begin{pmatrix} u_k(\delta) \\ h(u_k(\delta)) \end{pmatrix}$ on the right. Moreover, $\phi(\sigma)$ converges exponentially to its end-states as σ goes to $\pm\infty$.*

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