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The Navier–Stokes Equation with Slip Boundary Conditions

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Abstract

The paper presents an alternative to the no-slip Dirichlet or the slip Navier boundary conditions in the mathematical theory of a viscous incompressible fluid. The text is mostly based on our previous papers [1], [13], [14] and [15].

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1 Introduction

We deal with the Navier–Stokes system

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \tag{1.2}$$

in $\Omega\times(0,T)$, where Ω is a domain in $\mathbb{R}^3$ and $T>0$. The first equation represents the balance of momentum in the motion of a viscous incompressible fluid with a constant density and the second equation expresses the condition of incompressibility. We denote by $\boldsymbol{u}$ the velocity, by p the pressure, by $\boldsymbol{f}$ the specific body force and by ν the kinematic coefficient of viscosity. Since the equation (1.1) is non-steady, we also add the initial condition

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{1.3}$$

in Ω, at time $t=0$.

The system (1.1), (1.2), (1.3) is mostly considered with the homogeneous Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \tag{1.4}$$

on $\partial\Omega\times(0,T)$ in the case when $\partial\Omega$ is a fixed wall. This condition was suggested by G. G. Stokes in 1845, see [17], and it is equivalent to three scalar conditions

$$\text{(a)}\quad \boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \text{(b)}\quad \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \text{(c)}\quad \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} = 0 \tag{1.5}$$

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where $\boldsymbol{n}$ denotes the outer normal vector on $\partial\Omega$. (This simple assertion was proved, for $\boldsymbol{u} \in W^{1,2}(\Omega)^3$ such that $\operatorname{div} \boldsymbol{u} = 0$ a.e. in Ω, in [13].) The condition (1.5a) says that the normal component of $\boldsymbol{u}$ equals zero on $\partial\Omega$, which is a *natural requirement if $\partial\Omega$ is impermeable.* The third condition (1.5c) says that the normal component of the viscous stress, acting on the boundary, is zero. It means that $\boldsymbol{n} \cdot \mathbb{T}(\boldsymbol{u}) \cdot \boldsymbol{n} = 0$, where $\mathbb{T}$ is the viscous stress tensor. (For the incompressible isotropic Newtonian fluid, $\mathbb{T}$ satisfies $\mathbb{T}(\boldsymbol{u}) = 2\nu\mathbb{D}$ where $\mathbb{D}$ denotes the symmetrized gradient of $\boldsymbol{u}$.) The conditions (1.5b) and (1.5c) together guarantee that the tangential component of $\boldsymbol{u}$ is also equal to zero, which means that the fluid cannot slip on the boundary.

Mathematical arguments supporting the correctness of the no–slip boundary condition on a rugose wall can be found e.g. in the paper [3] by J. Casano–Díaz, E. Fernández–Cara and J. Simon. On the other hand, it is still a matter of discussions whether it is realistic to assume that $\boldsymbol{u}$ *cannot slip on $\partial\Omega$,* indeed, and this holds especially in situations when $\partial\Omega$ is smooth not only in the usual mathematical sense, but also in microscopic scales whose size is comparable with the size of fluid particles.

The boundary condition proposed by C. L. Navier in 1823, see [12], says that the velocity on the boundary should be proportional to the tangential component of the stress. This can be expressed by the equations

$$\text{(a)} \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \text{(b)} \quad (\mathbb{T}(\boldsymbol{u}) \cdot \boldsymbol{n})_\tau + k\boldsymbol{u} = \boldsymbol{0}, \tag{1.6}$$

on $\partial\Omega \times (0,T)$, where $(\mathbb{T}(\boldsymbol{u}) \cdot \boldsymbol{n})_\tau$ denotes the tangential component of $\mathbb{T}(\boldsymbol{u}) \cdot \boldsymbol{n}$ and k is the coefficient of proportionality. (Note that $(\mathbb{T}(\boldsymbol{u}) \cdot \boldsymbol{n})_\tau = \boldsymbol{n} \times [\mathbb{T}(\boldsymbol{u}) \cdot \boldsymbol{n}] \times \boldsymbol{n}$.) The condition (1.6b) naturally follows from the weak formulation of the problem (1.1), (1.2): *Given $\boldsymbol{u}_0 \in L^2_\sigma(\Omega)^3$ and $\boldsymbol{f}$ in $L^2(0,T;\, W^{-1,2}_\sigma(\Omega)^3)$. We search for $\boldsymbol{u} \in L^2(0,T;\, W^{1,2}_\sigma(\Omega)^3) \cap L^\infty(0,T;\, L^2_\sigma(\Omega)^3)$ such that*

$$\begin{aligned} &\int_0^T \int_\Omega [-\boldsymbol{u} \cdot \partial_t \boldsymbol{\phi} + \mathbb{T}(\boldsymbol{u}) \cdot \nabla\boldsymbol{\phi} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{\phi}] \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} k\boldsymbol{u} \cdot \boldsymbol{\phi} \,\mathrm{d}S\,\mathrm{d}t \\ &\quad = \int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{\phi}(0)\,\mathrm{d}\boldsymbol{x} + \int_0^T \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle_\Omega \,\mathrm{d}t \end{aligned} \tag{1.7}$$

for all $\boldsymbol{\phi} \in C^\infty(0,T;\, W^{1,2}_\sigma(\Omega)^3)$ such that $\boldsymbol{\phi}(T) = \boldsymbol{0}$. Here, $L^2_\sigma(\Omega)^3$ is the space of all divergence–free (in the sense of distributions) vector functions in $L^2(\Omega)^3$ whose normal component on $\partial\Omega$ equals zero (in the sense of traces). $W^{1,2}_\sigma(\Omega)^3$ denotes the intersection $W^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)^3$ and $\langle\, .\,,\, .\, \rangle_\Omega$ denotes the duality between $W^{-1,2}_\sigma(\Omega)^3$ and $W^{1,2}_\sigma(\Omega)^3$. Indeed, if $\boldsymbol{u}$ is a "smooth" solution of this problem then, considering at first the test functions $\boldsymbol{\phi}$ with a compact support in $\Omega \times [0,T)$, we show that there exists a pressure p such that $(\boldsymbol{u}, p)$ satisfies the equation (1.1) a.e. in $\Omega \times (0,T)$. Then, considering all possible test functions $\boldsymbol{\phi} \in C^\infty(0,T;\, W^{1,2}_\sigma(\Omega)^3)$ and integrating by parts in (1.7), we arrive at the identity

$$\int_0^T \int_{\partial\Omega} [\mathbb{T}(\boldsymbol{u}) \cdot \boldsymbol{n} + k\boldsymbol{u}] \cdot \boldsymbol{\phi} \,\mathrm{d}S\,\mathrm{d}t = 0$$

which implies (1.6b).

Navier's boundary conditions have been studied and applied in many papers, let us e.g. mention W. Jäger and A. Mikelič [9] and W. Zajaczkowski [20]. *They admit the fluid to slip on the boundary.* Indeed, under the assumptions that the velocity on the boundary satisfies the condition (1.6a), the law of the conservation of momentum holds "up to the boundary" and the friction between the fluid and the wall is proportional to $-\boldsymbol{u}$ (with the positive coefficient of proportionality k), one can derive (1.6b). However, the physical analysis shows that k depends on pressure, which complicates the analysis as well as numerical solution of the model.

If we compare Dirichlet's boundary condition (1.4) with Navier's boundary conditions (1.6), we observe that while in the first case we put the strong requirement on velocity $\boldsymbol{u}$ on the boundary (i.e. that it equals zero), in the second case the only actual geometrical condition we impose is (1.6a). (We have mentioned that (1.6b) follows from physical considerations.

This situation motivated us to study other boundary conditions which also admit the fluid to slip on the boundary, whose requirements on the behavior of velocity on the boundary are in a certain sense "between (1.4) and (1.6)" and which enable us to create a relatively consistent theory of the Navier–Stokes equation, similarly as e.g. in the case of the boundary condition (1.4). We have shown in several papers (see [1], [13], [14]) that the boundary conditions

$$\text{(a)}\quad \boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \text{(b)}\quad \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \text{(c)}\quad \mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n} = 0 \tag{1.8}$$

on $\partial\Omega\times(0,T)$, which we call the *generalized impermeability boundary conditions,* have all these properties.

We observe that the conditions (1.8) differ from (1.5) only in the third condition (1.8c). The condition (1.8c) can also be written in the form $\operatorname{Div}\mathbb{T}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, which says that the normal component of the rate of production of the viscous stress on the boundary equals zero.

It is not usual in the theory of partial differential equations to prescribe a boundary condition which involves partial derivatives of the same order as is the order of the equation. However, in our case, this is possible due to the fact that the vector function $\mathbf{curl}^2\boldsymbol{u}$ is divergence–free. Thus, on the level of strong solutions, $\mathbf{curl}^2\boldsymbol{u}\in L^2(\Omega)^3$ for a.a. time instants t and so it makes sense to speak on the normal component of $\mathbf{curl}^2\boldsymbol{u}$ on $\partial\Omega$ as on an element of $W^{-1/2,2}(\partial\Omega)$. (See e.g. [8], p. 27.) On the other hand, as we shall see in Section 4, the condition (1.8c) does not explicitly appear in the weak formulation of the problem (1.1)–(1.3) with the boundary conditions (1.8). However, if the solution is sufficiently smooth then it automatically satisfies (1.8c) as a natural boundary condition.

If we formally apply the operator **curl** to the equation (1.1) and denote $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u}$ then we obtain the well known equation for the vorticity $\boldsymbol{\omega}$:

$$\partial_t\boldsymbol{\omega} - \nu\Delta\boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} \;=\; \mathbf{curl}\,\boldsymbol{f}. \tag{1.9}$$

If we assume that $\boldsymbol{u}$ satisfies Dirichlet's boundary condition (1.4) or Navier's boundary conditions (1.6) and we wish to formulate a well–posed problem for $\boldsymbol{\omega}$ based on the equation (1.9) then there arises a serious problem, i.e. what boundary conditions satisfies $\boldsymbol{\omega}$? We shall show in Section 3 that this problem does not appear if we assume that $\boldsymbol{u}$ fulfills the boundary conditions (1.8). In other words: the boundary conditions (1.8) naturally induce boundary conditions for vorticity.

We give a brief survey of main definitions and properties of solutions of the problem (1.1)–(1.3), (1.8) in next sections. We also deal with an inhomogeneous form of the boundary conditions (1.8) in Section 3 and Section 4. We do not solve the question which boundary conditions of (1.4), (1.6) or (1.8) are more or less appropriate in concrete situations. This should be, in our opinion, a subject of physical rather than mathematical considerations and an important role should also play comparisons of numerical results with experiments. We only focus on the analytical part of the problem in this paper.

2 Notation and auxiliary results

We suppose that Ω is a bounded simply connected domain in $\mathbb{R}^3$ with a $C^{2,1}$–boundary $\partial\Omega$. We are actually preparing another paper where we intend to show that many of the results we mention in this article are also valid (eventually after certain modification) in a general domain. However, on the other hand, the assumptions on Ω formulated above enable us to present the main ideas in a simple way.

We list some notation and auxiliary results from [1] and [14]:

- $\|\,.\,\|_r$, respectively $\|\,.\,\|_{m,r}$, is the norm of a scalar– or vector– or tensor–valued function with components in $L^r(\Omega)$, respectively in $W^{m,r}(\Omega)$.
- $\|\,.\,\|_{r;\partial\Omega}$ or $\|\,.\,\|_{m,r;\partial\Omega}$, is the norm of a scalar– or vector– or tensor–valued function with the components in $L^r(\partial\Omega)$ or in $W^{m,r}(\partial\Omega)$. Similarly, $\|\,.\,\|_{r;\Omega'}$ or $\|\,.\,\|_{m,r;\Omega'}$ denote the norms of functions in $L^r(\Omega')$ or in $W^{m,r}(\Omega')$ in the case when $\Omega' \neq \Omega$.
- We have already defined the space $L^2_\sigma(\Omega)^3$ in Section 1. The equivalent definition is: $L^2_\sigma(\Omega)^3$ is the closure of $\{\boldsymbol{u} \in C_0^\infty(\Omega)^3;\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega\}$ in $L^2(\Omega)^3$. The orthogonal complement to $L^2_\sigma(\Omega)^3$ in $L^2(\Omega)^3$ consists of functions of the type ∇q for $q \in W^{1,2}(\Omega)$.
- $W^{1,2}_{0,\sigma}(\Omega)^3$ denotes the space of all divergence–free vector functions in $W^{1,2}_0(\Omega)^3$. It is a subspace of $W^{1,2}_0(\Omega)^3$ and of $W^{1,2}_\sigma(\Omega)^3$.
- $D^1 := \{\boldsymbol{u} \in W^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)^3;\ \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0 \text{ in the sense of traces}\} =$ $\{\boldsymbol{u} = \boldsymbol{u}_0 + \nabla\varphi;\ \boldsymbol{u}_0 \in W^{1,2}_0(\Omega)^3,\ \Delta\varphi = -\nabla\cdot\boldsymbol{u}_0 \text{ in } \Omega \text{ and } \partial\varphi/\partial\boldsymbol{n}|_{\partial\Omega} = 0\} = P_\sigma W^{1,2}_0(\Omega)^3$ (where P_σ is the orthogonal projection of $L^2(\Omega)^3$ onto $L^2_\sigma(\Omega)^3$).
- $\mathcal{R} := \mathbf{curl}\,|_{D^1}$ (We use the letter $\mathcal{R}$ because it denotes the operator of rotation, restricted to the space D^1.)
- The equation $\mathcal{R}\boldsymbol{u} = \boldsymbol{f}$ (for $\boldsymbol{f} \in L^2_\sigma(\Omega)^3$) has a unique solution $\boldsymbol{u} \in D^1$ such that
$$\|\boldsymbol{u}\|_{1,2} \le c_1\,\|\boldsymbol{f}\|_2 \tag{2.1}$$
where constant c_1 is independent of $\boldsymbol{f}$. (See O. A. Ladyzhenskaya, V. A. Solonnikov [11].)
- There exist constants c_2, $c_3 > 0$ such that
$$c_2\,\|\mathcal{R}\boldsymbol{u}\|_2 \le \|\boldsymbol{u}\|_{1,2} \le c_3\,\|\mathcal{R}\boldsymbol{u}\|_2 \quad \text{for all } \boldsymbol{u} \in D^1. \tag{2.2}$$
- $D^2 = D(\mathcal{R}^2) = \{\boldsymbol{u} \in W^{2,2}(\Omega)^3 \cap D^1;\ (\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n})|_{\partial\Omega} = 0 \text{ in the sense of traces}\}$
- There exist constants c_4, $c_5 > 0$ such that
$$c_4\,\|\mathcal{R}^2\boldsymbol{u}\|_2 \le \|\boldsymbol{u}\|_{2,2} \le c_5\,\|\mathcal{R}^2\boldsymbol{u}\|_2 \quad \text{for all } \boldsymbol{u} \in D^2. \tag{2.3}$$
- $\mathcal{R}$ is a self–adjoint operator in $L^2_\sigma(\Omega)^3$ (see Z. Yosida and Y. Giga [19] or R. Picard [16]) and the resolvent operator $(\lambda I - \mathcal{R})^{-1}$ is compact in $L^2_\sigma(\Omega)^3$ for all λ from the resolvent set of $\mathcal{R}$.
- The spectrum $\mathrm{Sp}(\mathcal{R})$ consists of isolated real eigenvalues $\{\lambda_i;\ i \in \mathbb{Z}^*\}$ (where $\mathbb{Z}^* := \mathbb{Z} - \{0\}$). Each eigenvalue has the same finite algebraic and geometric multiplicity. The eigenvalues can be ordered so that $\lambda_i < 0$ if $i < 0$, $\lambda_i > 0$ if $i > 0$ and $\lambda_i \le \lambda_j$ if $i < j$. The corresponding eigenfunctions form a complete orthonormal system in $L^2_\sigma(\Omega)^3$.

3 The local in time existence of a strong solution and related results

3.1 The case of the homogeneous boundary conditions (1.8)

The next theorem provides the information on the local in time solvability, in a strong sense, of the initial–boundary value problem (1.1)–(1.3), (1.8). Similar theorems on the Navier–Stokes equation with the no–slip boundary condition (1.4) are well known, see e.g. the book by O. A. Ladyzhenskaya [10] or the survey paper by G. P. Galdi [6].

Theorem 3.1 *Let $\boldsymbol{u}_0 \in D^1$ and $\boldsymbol{f} \in L^2(0,T;\, L^2(\Omega)^3)$. Then there exists $T_1 \in (0,T]$ such that the initial-boundary value problem, given by the equation*

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + P_\sigma(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \;=\; P_\sigma \boldsymbol{f}, \tag{3.1}$$

by the initial condition (1.3) and by the boundary conditions (1.8), has a unique strong solution $\boldsymbol{u}$ on the time interval $(0,T_1)$. The solution satisfies the inclusions $\boldsymbol{u} \in C(0,T_1;\, D^1)$ and $\mathcal{R}^2\boldsymbol{u},\ \partial_t\boldsymbol{v} \in L^2(0,T_1;\, L^2_\sigma(\Omega)^3)$.

The proof, in the case $\boldsymbol{f} = \boldsymbol{0}$, can be found in [14]. The used method, based on the construction of Galerkin approximations as linear combinations of eigenfunctions of the operator $\mathcal{R}$, can also be used in the situation when $\boldsymbol{f} \neq \boldsymbol{0}$.

The equation (3.1) formally follows from (1.1) by applying the projection P_σ to (1.1). The projection P_σ can be omitted in front of $\Delta\boldsymbol{u}$ because $P_\sigma\Delta\boldsymbol{u} = -P_\sigma\mathbf{curl}^2\boldsymbol{u} = -\mathbf{curl}^2\boldsymbol{u} = \Delta\boldsymbol{u}$.

3.2 The Neumann boundary condition for pressure

If $\boldsymbol{u}$ is the solution given by Theorem 3.1 then, obviously, $\boldsymbol{u}$ also satisfies the equation of continuity (1.2) because $\boldsymbol{u}(t)$ is an element of D^1 for a.a. $t \in (0,T)$. If we choose an associated pressure p so that $\nabla p = (I - P_\sigma)\,[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f}]$ then the pair $\boldsymbol{u}$, p satisfies the Navier–Stokes equation (1.1) a.e. in $\Omega \times (0,T_1)$. The pressure p is thus given uniquely up to an additive function of t and it can be chosen so that $p \in L^2(0,T_1;\, W^{1,2}(\Omega))$. In fact, in order to construct p, one has to solve the Poisson equation $\Delta p = -\partial_i\partial_j(u_iu_j) + \operatorname{div}\boldsymbol{f}$ which arises from equation (1.1) if we apply the operator div to both its sides. A possible boundary condition for p directly follows from the boundary conditions (1.8) and from the equation (1.1) if we multiply both the sides by the normal vector $\boldsymbol{n}$ on $\partial\Omega$:

$$[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \boldsymbol{f}]\cdot\boldsymbol{n} \;=\; 0.$$

The term on the left hand side is an element of $L^2(0,T_1;\, W^{-1/2,2}(\partial\Omega))$ because the expression in the brackets belongs to $L^2(0,T_1;\, L^2_\sigma(\Omega)^3)$. If we formally multiply each term in the brackets separately by $\boldsymbol{n}$, we obtain the Neumann boundary condition for p:

$$\frac{\partial p}{\partial \boldsymbol{n}} \;=\; -(\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}\cdot\boldsymbol{n}) + \boldsymbol{u}\cdot\nabla\boldsymbol{n}\cdot\boldsymbol{u} + \boldsymbol{f}\cdot\boldsymbol{n} \;=\; \boldsymbol{u}\cdot\nabla\boldsymbol{n}\cdot\boldsymbol{u} + \boldsymbol{f}\cdot\boldsymbol{n}. \tag{3.2}$$

(The term $(\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}\cdot\boldsymbol{n})$ is zero because $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ and its derivative in any tangential direction to $\partial\Omega$ equals zero.) The right hand side depends on the curvature of $\partial\Omega$ and it equals only $\boldsymbol{f}\cdot\boldsymbol{n}$ on those parts where $\partial\Omega$ coincides with a plane.

Note that (3.2) is simpler than the Neumann boundary condition for pressure obtained if $\boldsymbol{u}$ is supposed to satisfy the no–slip boundary condition (1.4). Then $\nabla p = (I - P_\sigma)[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} - \boldsymbol{f}]$ where the right hand side contains the additional term $(I - P_\sigma)\,\nu\Delta\boldsymbol{u}$ which, in the case of boundary conditions (1.8), equals zero.

3.3 The boundary conditions for vorticity

Suppose that $\boldsymbol{u}_0$ and $\boldsymbol{f}$ satisfy stronger requirements than in the assumptions of Theorem 3.1, i.e. that $\boldsymbol{u}_0 \in D^2$ and $\boldsymbol{f} \in L^2(0,T;\, W_\sigma^{1,2}(\Omega)^3)$. Then the same procedure as in the proof of Theorem 3.1 enables us to obtain more information on solution $\boldsymbol{u}$ than what is provided by Theorem 3.1: i.e. that $\mathcal{R}^2\boldsymbol{u}$, $\partial_t\boldsymbol{u} \in L^2(0,T_1;\, W_\sigma^{1,2}(\Omega)^3)$. (See [14].) Consider the equation (1.9) in $\Omega\times(0,T_1)$. The boundary conditions (1.8b), (1.8c) imply that $\boldsymbol{\omega}\cdot\boldsymbol{n} = \mathbf{curl}\,\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$ on $\partial\Omega\times(0,T_1)$. The expression $(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}$ in the equation (1.9) equals $\mathbf{curl}\,(\boldsymbol{\omega}\times\boldsymbol{u})$. Its normal component on the boundary equals zero because $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are tangent to $\partial\Omega$, their cross product is therefore normal and consequently, its **curl** is again tangent. Hence $[(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{n} = 0$ on the boundary. Since $\partial_t\boldsymbol{\omega}\cdot\boldsymbol{n}$ is also zero, the equation (1.9) implies that $\nu\,\mathbf{curl}^2\boldsymbol{\omega} = -\nu\Delta\boldsymbol{\omega}\cdot\boldsymbol{n} = \mathbf{curl}\,\boldsymbol{f}\cdot\boldsymbol{n}$ on $\partial\Omega\times(0,T_1)$. Thus, the boundary conditions (1.8) and the equation (1.9) imply the series of the boundary conditions

$$\text{(a)}\quad \boldsymbol{\omega}\cdot\boldsymbol{n}\big|_{\partial\Omega} = 0, \qquad \text{(b)}\quad \mathbf{curl}\,\boldsymbol{\omega}\cdot\boldsymbol{n}\big|_{\partial\Omega} = 0, \qquad \text{(c)}\quad \mathbf{curl}^2\boldsymbol{\omega}\cdot\boldsymbol{n}\big|_{\partial\Omega} = \frac{1}{\nu}\,\mathbf{curl}\,\boldsymbol{f}\cdot\boldsymbol{n}\big|_{\partial\Omega}$$

on $\partial\Omega\times(0,T_1)$. These boundary conditions, although inhomogeneous, are of the same nature as (1.8). The fact that the boundary conditions (1.8) for velocity naturally induce the complete set of boundary conditions for vorticity is an advantage in comparison to (1.4) or (1.6).

3.4 The case of the inhomogeneous boundary conditions of the type (1.8)

Here we deal with the same problem as in the part 3.1, we only consider the inhomogeneous version of the boundary conditions (1.8):

$$\text{(a)}\quad \boldsymbol{u}\cdot\boldsymbol{n} = \alpha_0, \qquad \text{(b)}\quad \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n} = \alpha_1, \qquad \text{(c)}\quad \mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n} = \alpha_2 \tag{3.3}$$

on $\partial\Omega\times(0,T)$. Let us suppose, for simplicity, that α_0, α_1 and α_2 do not depend on time. The next lemma says that under some assumptions on α_0, α_1 and α_2, there exists a divergence-free vector function $\boldsymbol{a} \in W^{2,2}(\Omega)^3$ which satisfies the conditions (3.3). We suppose that $\partial\Omega$ has the components $\Gamma_0, \Gamma_1, \dots, \Gamma_N$ such that

$$\Omega = \mathrm{Int}(\Gamma_0 \cap \mathrm{Ext}(\Gamma_1) \cap \dots \cap \mathrm{Ext}(\Gamma_N).$$

Lemma 3.1 *Given $\alpha_0 \in W^{3/2,2}(\partial\Omega)$, $\alpha_1 \in W^{1/2,2}(\partial\Omega)$ and $\alpha_2 \in W^{-1/2,2}(\partial\Omega)$ such that*

$$\int_{\partial\Omega} \alpha_0\,\mathrm{d}S = 0, \qquad \int_{\Gamma_i} \alpha_1\,\mathrm{d}S = \langle\alpha_2, 1\rangle_{\Gamma_i} = 0 \quad (i = 0,1,\dots,N),$$

there exists a vector function $\boldsymbol{a} \in W^{2,2}(\Omega)^3$ such that $\operatorname{div}\boldsymbol{a} = 0$ *a.e. in Ω and*

$$\text{(a)}\quad \boldsymbol{a}\cdot\boldsymbol{n}\big|_{\partial\Omega} = \alpha_0, \qquad \text{(b)}\quad \mathbf{curl}\,\boldsymbol{a}\cdot\boldsymbol{n}\big|_{\partial\Omega} = \alpha_1, \qquad \text{(c)}\quad \mathbf{curl}^2\boldsymbol{a}\cdot\boldsymbol{n}\big|_{\partial\Omega} = \alpha_2. \tag{3.4}$$

Moreover, there exists a constant $c_6 > 0$, independent of α_0, α_1 and α_2, such that

$$\|\boldsymbol{a}\|_{2,2} \le c_6\left(\|\alpha_0\|_{3/2,2;\partial\Omega} + \|\alpha_1\|_{1/2,2;\partial\Omega} + \|\alpha_2\|_{-1/2,2;\partial\Omega}\right). \tag{3.5}$$

Proof. (i) At first we solve the Neumann problem

$$\Delta\psi_2 = 0 \quad \text{in } \Omega, \qquad \frac{\partial\psi_2}{\partial\boldsymbol{n}}\bigg|_{\partial\Omega} = \alpha_2. \tag{3.6}$$

There exists a unique (up to an additive constant) weak solution $\psi_2 \in W^{1,2}(\Omega)$.

(ii) Next we consider the problem

$$\mathbf{curl}\,\boldsymbol{\varphi}_1 = \nabla\psi_2 \quad \text{in } \Omega, \qquad \boldsymbol{\varphi}_1\big|_{\partial\Omega} = \mathbf{0}. \tag{3.7}$$

Since $\langle\alpha_2, 1\rangle_{\Gamma_i} = 0$ $(i = 0, 1, \ldots, N)$, the flux of $\nabla\psi_2$ through each component of $\partial\Omega$ equals zero. Thus, due to [2], Theorem 2.1, the problem (3.7) is solvable in $\boldsymbol{W}_0^{1,2}(\Omega)$.

(iii) Now we solve the Neumann problem

$$\Delta\psi_1 = -\mathrm{div}\,\boldsymbol{\varphi}_1 \quad \text{in } \Omega, \qquad \frac{\partial\psi_1}{\partial\boldsymbol{n}}\bigg|_{\partial\Omega} = \alpha_1. \tag{3.8}$$

This problem has a unique (up to an additive constant) solution $\psi_1 \in W^{2,2}(\Omega)$.

(iv) Next we solve the problem

$$\mathbf{curl}\,\boldsymbol{\varphi}_0 = \nabla\psi_1 + \boldsymbol{\varphi}_1 \quad \text{in } \Omega, \qquad \boldsymbol{\varphi}_0\big|_{\partial\Omega} = \mathbf{0}. \tag{3.9}$$

Since $\int_{\Gamma_i} \alpha_1 \, \mathrm{d}S = 0$ $(i = 0, 1, \ldots, N)$, the flux of $\nabla\psi_1 + \boldsymbol{\varphi}_1$ through each component of $\partial\Omega$ equals zero. Thus, the problem (3.9) is solvable in $W^{2,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3$.

(v) Finally we solve the Neumann problem

$$\Delta\psi_0 = -\mathrm{div}\,\boldsymbol{\varphi}_0 \quad \text{in } \Omega, \qquad \frac{\partial\psi_0}{\partial\boldsymbol{n}}\bigg|_{\partial\Omega} = \alpha_0. \tag{3.10}$$

This problem has a unique (up to an additive constant) solution $\psi_0 \in W^{3,2}(\Omega)$.

Now we put $\boldsymbol{a} := \nabla\psi_0 + \boldsymbol{\varphi}_0$. The function $\boldsymbol{a}$ is divergence–free because ψ_0 satisfies the equation (3.10). The normal component of $\boldsymbol{a}$ on $\partial\Omega$ equals α_0 because $\boldsymbol{a}\cdot\boldsymbol{n} = \nabla\psi_0\cdot\boldsymbol{n} = \alpha_0$ on $\partial\Omega$. Since $\mathbf{curl}\,\boldsymbol{a} = \mathbf{curl}\,\boldsymbol{\varphi}_0 = \nabla\psi_1$ and consequently, $\mathbf{curl}\,\boldsymbol{a}\cdot\boldsymbol{n} = \nabla\psi_1\cdot\boldsymbol{n} = \alpha_1$ on $\partial\Omega$, the function $\boldsymbol{a}$ also satisfies (3.4b). We can similarly verify that $\boldsymbol{a}$ also satisfies (3.4c). The solutions of all the problems in paragraphs (i)–(v) depend continuously on the given data and their norms can be estimated by means of appropriate norms of the data. Summing all these estimates, we can arrive at (3.5). □

Now we can search for a solution of the problem (1.1)–(1.3), (3.3) in the form $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{v}$ where $\boldsymbol{a}$ is the function given by Lemma 3.1. Substituting it into (1.1), (1.2), we obtain the equations

$$\partial_t\boldsymbol{v} - \nu\Delta\boldsymbol{v} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{a} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = \boldsymbol{g} \tag{3.11}$$

$$\mathrm{div}\,\boldsymbol{v} = 0 \tag{3.12}$$

in $\Omega \times (0, T)$, where $\boldsymbol{g} = \boldsymbol{f} - \nu\Delta\boldsymbol{a} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{a}$. The initial condition (1.3) implies that

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \tag{3.13}$$

in Ω, where $\boldsymbol{v}_0 = \boldsymbol{u}_0 - \boldsymbol{a}$. Further, (3.3), (3.4) imply that $\boldsymbol{v}$ should satisfy the homogeneous boundary conditions (1.8). In order to prove the local in time existence of a strong solution of this problem, we can use the same approach as in [14] (the proof of Theorem 1). The presence of the function $\boldsymbol{a}$ does not negatively influence the possibility of deriving necessary estimates of the approximations and we can thus prove the theorem analogous to Theorem 3.1:

Theorem 3.2 *Let $\boldsymbol{v}_0 \in D^1$ and $\boldsymbol{g} \in L^2(0, T; L^2(\Omega)^3)$. Then there exists $T_1 \in (0, T]$ such that the initial–boundary value problem (3.11), (3.12), (3.13) with the homogeneous boundary conditions (1.8) has a unique strong solution $\boldsymbol{v}$ on the time interval $(0, T_1)$. The solution satisfies the inclusions $\boldsymbol{v} \in C(0, T_1; D^1)$ and $\mathcal{R}^2\boldsymbol{v}$, $\partial_t\boldsymbol{v} \in L^2(0, T_1; L^2_\sigma(\Omega)^3)$.*

4 The weak formulation of the problem (1.1)–(1.3), (1.8)

4.1 The case of the homogeneous boundary conditions (1.8)

The equation (1.1) can be written in the equivalent form

$$\partial_t \boldsymbol{u} + \mathbf{curl}^2 \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} + \nabla q \;=\; \boldsymbol{f}, \tag{4.1}$$

where $q = p + \frac{1}{2}|\boldsymbol{u}|^2$. The following weak formulation of the problem (1.1)–(1.3), (1.8) is based on this form of the equation (1.1). Recall that $\mathcal{R}\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{u}$ for $\boldsymbol{u} \in D^1$.

Definition 4.1 *Let $T > 0$, $\boldsymbol{f} \in L^2(0,T;\, D^{-1})$ and $\boldsymbol{u}_0 \in L^2_\sigma(\Omega)^3$. We call a function $\boldsymbol{u} \in L^\infty(0,T;\, L^2_\sigma(\Omega)^3) \cap L^2(0,T;\, D^1)$ a weak solution of the problem (4.1), (1.2), (1.3), (1.8) if*

$$\begin{aligned} &\int_0^T \int_\Omega \big[-\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + \nu\,\mathcal{R}\boldsymbol{u}\cdot\mathcal{R}\boldsymbol{\phi} + (\mathcal{R}\boldsymbol{u}\times\boldsymbol{u})\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\phi}(0)\,\mathrm{d}\boldsymbol{x} \\ &\quad = \int_0^T \langle \boldsymbol{f},\, \boldsymbol{\phi}\rangle_\Omega\,\mathrm{d}t \end{aligned} \tag{4.2}$$

for all $\boldsymbol{\phi} \in C^\infty([0,T];\, D^1)$ such that $\boldsymbol{\phi}(T) = \mathbf{0}$.

Here $\langle\,.\,,\,.\,\rangle_\Omega$ denotes the duality between D^{-1} and D^1.

The weak solution $\boldsymbol{u}$ satisfies the first two boundary conditions in (1.8) in the sense of traces for a.a. $t \in (0,T)$ because $\boldsymbol{u}(t) \in D^1$ for a.a. $t \in (0,T)$. Note that these conditions are identical with (1.5a) and (1.5b). A natural question is in which sense the weak solution also satisfies the boundary condition (1.8c) which says that $\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega \times (0,T)$. If $\boldsymbol{u}$ is a solution which, in addition to the assumptions in Definition 4.1, belongs to $L^2(0,T;\, W^{2,2}(\Omega)^3)$ and $\partial_t\boldsymbol{u} \in L^2(0,T;\, L^2_\sigma(\Omega)^3)$ then considering at first the test functions in (4.2) with a compact support in $\Omega \times [0,T)$ and integrating by parts in (4.2), we show that there exists a scalar function q such that $\nabla q \in L^2(Q_T)^3$ and $\boldsymbol{u}$, q satisfy the equation (4.1) a.e. in $\Omega\times(0,T)$. Then, using this information and applying again the integration by parts to the terms containing $\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi}$ and $\mathcal{R}\boldsymbol{u}\cdot\mathcal{R}\boldsymbol{\phi}$ in (4.2), this time with all acceptable test functions $\boldsymbol{\phi}$, we obtain:

$$\int_0^T \int_{\partial\Omega} \mathcal{R}\boldsymbol{u}\cdot(\boldsymbol{\phi}\times\boldsymbol{n})\,\mathrm{d}S\,\mathrm{d}t \;=\; 0. \tag{4.3}$$

Due to the characterization of D^1, see Section 2, the test function $\boldsymbol{\phi}(t)$ can be decomposed to the sum $\boldsymbol{\phi}_0(t) + \nabla\varphi(t)$ where $\boldsymbol{\phi}_0(t) \in W^{1,2}_0(\Omega)^3$ and $\varphi(t) \in W^{2,2}(\Omega)$ for all $t \in [0,T]$. Hence (4.3) implies that

$$\begin{aligned} 0 \;&=\; \int_0^T \int_{\partial\Omega} \mathbf{curl}\,\boldsymbol{u}\cdot(\nabla\varphi\times\boldsymbol{n})\,\mathrm{d}S\,\mathrm{d}t \;=\; -\int_0^T \int_\Omega \mathrm{div}\,(\nabla\varphi\times\mathbf{curl}\,\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &=\; \int_0^T \int_\Omega \nabla\varphi\cdot\mathbf{curl}^2\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \;=\; \int_0^T \langle(\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n}),\, \varphi\rangle_{\partial\Omega}\,\mathrm{d}t \end{aligned}$$

where $\langle\,.\,,\,.\,\rangle_{\partial\Omega}$ denotes the duality between elements of $W^{-1/2,2}(\partial\Omega)$ and $W^{1/2,2}(\partial\Omega)$. The set of traces on $\partial\Omega$ of all the functions φ is dense in $W^{1/2,2}(\partial\Omega)$ for each $t \in (0,T)$. Thus, the condition $\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n} = 0$ is satisfied in the sense of equality in $W^{-1/2,2}(\partial\Omega)$ for a.a. $t \in (0,T)$.

We have proved in [1] that if $\boldsymbol{v}_0 \in D^1$ then a weak solution of the problem (4.1), (1.2), (1.3), (1.8) can be constructed so that $\int_0^T \|\boldsymbol{v}(t)\|_{2,2}^{2/3}\,\mathrm{d}t < +\infty$. (The same result was earlier obtained by G. F. D. Duff in [4] in the case of the Dirichlet boundary condition (1.4).) This result enables us to give another explanation of the sense in which the weak solution satisfies the third boundary condition in (1.8). The integrability of $\|\boldsymbol{v}\|_{2,2}^{2/3}$ on $(0,T)$ implies that $\mathbf{curl}^2\boldsymbol{v}(t) \in L^2(\Omega)^3$ for a.a. $t \in (0,T)$. As a divergence–free vector function, $\mathbf{curl}^2\boldsymbol{v}(t)$ has the normal component on the boundary in $W^{-1/2,2}(\partial\Omega)$ in the sense of traces. (See e.g. [8], p. 27.) Now the condition $\mathbf{curl}^2\boldsymbol{v}(t)\cdot\boldsymbol{n} = 0$ is satisfied as an equality in $W^{-1/2,2}(\partial\Omega)$.

The existence of the weak solution introduced in Definition 4.1 can be proved by the Galerkin method in the usual way. The Galerkin approximations can be constructed in the form of linear combinations of the eigenfunctions of operator $\mathcal{R}$, as for Theorem 3.1.

4.2 The case of the inhomogeneous boundary conditions (3.3)

Here we consider the same problem as in part 4.1, however with the inhomogeneous boundary conditions (3.3). We have seen in Definition 4.1 that the term $\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n}$ does not explicitly appear in the weak formulation of the initial–boundary value problem. This is why we use a slightly modified approach than in the part 3.4 and instead of function $\boldsymbol{a}$ satisfying (3.4), we shall use a function which satisfies only the first two conditions in (3.4), but on the other hand, it is harmonic in Ω. Its existence is given by the next lemma. We assume again for simplicity, as in the part 3.4, that α_0 and α_1 do not depend on time.

Lemma 4.1 *Given $\alpha_0 \in W^{1/2,2}(\partial\Omega)$ and $\alpha_1 \in W^{-1/2,2}(\partial\Omega)$ such that*

$$\int_{\partial\Omega} \alpha_0 \,\mathrm{d}S = 0, \qquad \langle\alpha_1, 1\rangle_{\Gamma_i} = 0 \quad (i = 0, 1, \ldots, N),$$

there exists a vector function $\boldsymbol{a} \in W^{1,2}(\Omega)^3$ such that $\operatorname{div}\boldsymbol{a} = 0$ *a.e. in Ω, $\boldsymbol{a}$ is harmonic (in the sense of distributions) in Ω and*

$$\text{(a)} \quad \boldsymbol{a}\cdot\boldsymbol{n}\big|_{\partial\Omega} = \alpha_0, \qquad \text{(b)} \quad \mathbf{curl}\,\boldsymbol{a}\cdot\boldsymbol{n}\big|_{\partial\Omega} = \alpha_1. \tag{4.4}$$

Moreover, there exists a constant $c_7 > 0$, independent of α_0 and α_1, such that

$$\|\boldsymbol{a}\|_{1,2} \;\le\; c_7\left(\|\alpha_0\|_{1/2,2;\,\partial\Omega} + \|\alpha_1\|_{-1/2,2;\,\partial\Omega}\right). \tag{4.5}$$

The lemma is proved in [15]. The proof is analogous to the proof of Lemma 3.1. Suppose further that α_0 and α_1 satisfy the assumptions of Lemma 4.1 and $\boldsymbol{a}$ is a given by this lemma. The weak solution $\boldsymbol{u}$ of the problem (4.1), (1.2), (1.3) and (3.3) can be constructed in the form $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{v}$ where $\boldsymbol{v}$ satisfies in a weak sense the equations

$$\partial_t\boldsymbol{v} + \nu\,\mathbf{curl}^2\boldsymbol{v} + \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{v} + \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{a} + \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{v} + \nabla q \;=\; \boldsymbol{g} \tag{4.6}$$

$$\operatorname{div}\boldsymbol{v} \;=\; 0 \tag{4.7}$$

(where $\boldsymbol{g} = \boldsymbol{f} - \nu\,\mathbf{curl}^2\boldsymbol{a} - \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{a}$) in $\Omega\times(0,T)$, the initial condition

$$\boldsymbol{v}(0) \;=\; \boldsymbol{v}_0 \tag{4.8}$$

(where $\boldsymbol{v}_0 = \boldsymbol{u}_0 - \boldsymbol{a}$) and the homogeneous boundary conditions (1.8a), (1.8b) on $\partial\Omega\times(0,T)$. This guaranties that $\boldsymbol{u}$ satisfies the conditions (3.3a) and (3.3b) on $\partial\Omega\times(0,T)$, but it does not solve the question of validity of (3.3c).

The reason why the condition (3.3c) cannot be treated in the same way as (3.3a) and (3.3b) is that (3.3c) involves the second derivatives of $\boldsymbol{u}$ and the required smoothness of the weak solution $\boldsymbol{u}$ in does not directly provide an opportunity to control $\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n}$ on $\partial\Omega\times(0,T)$. Thus, the boundary condition (3.3c) enters the weak formulation through a certain linear functional $\boldsymbol{b}$ which, in the case when the weak solution is "smooth", causes that it satisfies (3.3c) as a "natural boundary condition".

The weak formulation of the problem (4.6)–(4.8), (1.8) is:

Definition 4.2 *Let $T>0$, $\boldsymbol{g}\in L^2(0,T;\,D^{-1})$, $\boldsymbol{v}_0\in L^2_\sigma(\Omega)^3$ and $\boldsymbol{b}\in \boldsymbol{W}^{-1/2,2}(\partial\Omega)$. We call a function $\boldsymbol{v}\in L^\infty(0,T;\,L^2_\sigma(\Omega)^3)\cap L^2(0,T;\,D^1)$ a weak solution of the problem (4.6)–(4.8), (1.8) if*

$$\int_0^T\int_\Omega\big[-\boldsymbol{v}\cdot\partial_t\boldsymbol{\phi}+\nu\mathcal{R}\boldsymbol{v}\cdot\mathcal{R}\boldsymbol{\phi}+\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{v}\cdot\boldsymbol{\phi}+\mathcal{R}\boldsymbol{v}\times\boldsymbol{a}\cdot\boldsymbol{\phi}+(\mathcal{R}\boldsymbol{u}\times\boldsymbol{u})\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$-\int_\Omega\boldsymbol{v}_0\cdot\boldsymbol{\phi}(0)\,\mathrm{d}\boldsymbol{x} \;=\; \int_0^T\langle\boldsymbol{g},\boldsymbol{\phi}\rangle_\Omega\,\mathrm{d}t+\int_0^T\langle\boldsymbol{b},\boldsymbol{\phi}\rangle_{\partial\Omega}\,\mathrm{d}t \tag{4.9}$$

for all $\boldsymbol{\phi}\in C^\infty([0,T];\,D^1)$ such that $\boldsymbol{\phi}(T)=\mathbf{0}$.

By analogy with Definition 4.1, we write $\mathcal{R}\boldsymbol{v}$ and $\mathcal{R}\boldsymbol{\phi}$ instead of $\mathbf{curl}\,\boldsymbol{v}$ and $\mathbf{curl}\,\boldsymbol{\phi}$.

The existence of a solution $\boldsymbol{v}$ of the problem formulated in Definition 4.2 can be proved in a similar way as in the case of the problem with the homogeneous boundary conditions, formulated in Definition 4.1. The procedure is standard and it does not substantially differ from the classical proof of the existence of a weak solution of the Navier–Stokes initial–boundary value problem with the homogeneous Dirichlet boundary condition,see e.g. [5], [6], [10] and [18].

Let us now explain how the weak problem formulated in Definition 4.2 involves the boundary condition (3.3c). Given $\boldsymbol{b}\in\boldsymbol{W}^{-1/2,2}(\partial\Omega)$, we define $\alpha_2\in W^{-3/2,2}(\partial\Omega)$ by the equation

$$\nu\,\langle\alpha_2,\varphi\rangle^*_{\partial\Omega} \;=\; \langle\boldsymbol{b},\nabla\varphi\rangle_{\partial\Omega} \tag{4.10}$$

for all $\varphi\in W^{2,2}(\Omega)$. Here $\langle\,.\,,\,.\,\rangle^*_{\partial\Omega}$ denotes the duality between $W^{-3/2,2}(\partial\Omega)$ and $W^{3/2,2}(\partial\Omega)$.

If $\boldsymbol{g}\in L^2(0,T;\,L^2_\sigma(\Omega)^3)$ and $\boldsymbol{v}$ is a solution of (4.9) that belongs to $L^2(0,T;\,W^{2,2}(\Omega)^3)$, then we can at first consider the test functions $\boldsymbol{\phi}$ with a compact support in $\Omega\times[0,T)$ and show that there exists a scalar function q such that $\boldsymbol{v}$, q satisfy the equations (4.6), (4.7) a.e. in $\Omega\times(0,T)$. Then, following the standard procedure, we consider all acceptable test functions from $C^\infty([0,T];\,D^1)$ and show, by means of the integration by parts in (4.9), that $\boldsymbol{v}$ satisfies

$$\int_0^T\int_{\partial\Omega}\nu\,\mathbf{curl}\,\boldsymbol{v}\cdot(\boldsymbol{n}\times\boldsymbol{\phi})\,\mathrm{d}S\,\mathrm{d}t \;=\; \int_0^T\langle\boldsymbol{b},\boldsymbol{\phi}\rangle_{\partial\Omega}\,\mathrm{d}t. \tag{4.11}$$

For each $t\in[0,T]$, function $\boldsymbol{\phi}(t)$ is an element of D^1. Hence it can be written in the form

$$\boldsymbol{\phi}(t) \;=\; \boldsymbol{\phi}_0(t)+\nabla\varphi(t) \tag{4.12}$$

where $\boldsymbol{\phi}_0(t)\in\boldsymbol{W}^{1,2}_0(\Omega)$ and $\varphi(t)\in W^{2,2}(\Omega)$, see [1]. Recall that $\boldsymbol{\phi}_0(t)$ is a solution of the boundary–value problem

$$\mathbf{curl}\,\boldsymbol{\phi}_0(t) \;=\; \mathbf{curl}\,\boldsymbol{\phi}(t)\quad\text{in }\Omega,\qquad \boldsymbol{\phi}_0(t)\big|_{\partial\Omega} \;=\; \mathbf{0}. \tag{4.13}$$

Substituting $\phi(t)$ in the form (4.12) into the left hand side of (4.11), we obtain:

$$\begin{aligned}
\int_0^T \int_{\partial\Omega} \mathbf{curl}\, \boldsymbol{v} \cdot (\boldsymbol{n} \times \boldsymbol{\phi})\, \mathrm{d}S\, \mathrm{d}t \;&=\; -\int_0^T \int_{\partial\Omega} \boldsymbol{n} \cdot (\mathbf{curl}\, \boldsymbol{v} \times \nabla\varphi)\, \mathrm{d}S\, \mathrm{d}t \\
=\; -\int_0^T \int_{\Omega} \mathrm{div}\, (\mathbf{curl}\, \boldsymbol{v} \times \nabla\varphi)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \;&=\; -\int_0^T \int_{\Omega} \mathbf{curl}^2 \boldsymbol{v} \cdot \nabla\varphi\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \\
=\; -\int_0^T \big\langle \mathbf{curl}^2 \boldsymbol{v} \cdot \boldsymbol{n},\, \varphi \big\rangle_{\partial\Omega}\, \mathrm{d}t.&
\end{aligned}$$

The duality in the last term can also be expressed as $\langle \mathbf{curl}^2 \boldsymbol{v} \cdot \boldsymbol{n},\, \varphi\rangle^*_{\partial\Omega}$. Thus, (4.10) and (4.11) yield

$$\int_0^T \nu \left\langle \alpha_2 - \mathbf{curl}^2 \boldsymbol{v}(t) \cdot \boldsymbol{n},\, \varphi \right\rangle^*_{\partial\Omega} \mathrm{d}t \;=\; 0. \tag{4.14}$$

This equation shows that $\boldsymbol{v}$ satisfies the boundary condition $\mathbf{curl}^2 \boldsymbol{v}(t) \cdot \boldsymbol{n} = \alpha_2$ in the sense of the equality in $W^{-3/2,2}(\partial\Omega)$ for a.a. $t \in (0,T)$. Since $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{v}$ and $\mathbf{curl}^2 \boldsymbol{a} = 0$ in the sense of distributions, $\boldsymbol{u}(t)$ fulfills the boundary condition (3.3c) as an equality in $W^{-3/2,2}(\partial\Omega)$ for a.a. $t \in (0,T)$.

Concluding remark. The results presented in this paper show that the homogeneous boundary conditions (1.8) or the inhomogeneous boundary conditions (3.3) represent an alternative to Dirichlet's boundary condition (1.4) and Navier's boundary conditions (1.6) (or to their inhomogeneous versions) which enables the fluid to slip on the boundary, is not in contradiction with physical laws, enables us to develop the mathematical theory of the Navier–Stokes equation as e.g. in the case of the boundary condition (1.4), and has some mathematical advantages in comparison with (1.4). (E.g. that the projection P_σ commutes with the Laplace operator Δ, see part 3.1, or that the conditions (1.8) induce the complete analogous set of boundary conditions for the vorticity, see part 3.3.)

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