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Free boundary problem for elastic material with linear strain

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1 Introduction

In this paper we consider the dynamics of a one-dimensional elastic material. Let $u = u(t,x)$ be the displacement on the cylindrical domain $Q(T) := (0,T) \times (0,1)$, where $T > 0$. As discussed in [2] we can obtain the following initial boundary value problem which is a mathematical model for the dynamics.

$$u_{tt} + \gamma u_{xxxx} - \mu u_{txx} - \kappa \varepsilon_x = 0 \quad \text{in } Q(T), \tag{1.1}$$

$$\varepsilon = \frac{u_x}{1-u_x} \quad \text{in } Q(T), \tag{1.2}$$

$$u(t,0) = u(t,1) = 0, u_{xx}(t,0) = u_{xx}(t,1) = 0 \quad \text{for } 0 < t < T, \tag{1.3}$$

$$u(0,x) = u_0(x), u_t(0,x) = v_0(x) \quad \text{for } 0 < x < 1, \tag{1.4}$$

where $\gamma > 0$, $\mu > 0$, $\kappa > 0$, and u_0 and v_0 are initial functions. Here, ε indicates the nonlinear strain (see [2]), and the equation (1.1) is the approximation of the kinetic equation $u_{tt} - \kappa\varepsilon_x = 0$. The modelling process for the above problem (P1) := $\{(1.1) \sim (1.4)\}$, is discussed in [2, section 1], precisely.

Moreover, in [2] the following free boundary problem (P2) including the liner strain is proposed as the other mathematical model for the elastic material. The problem (P2) is to find a curve $\ell = \ell(t)$, $t \in [0,T]$, and $u = u(t,x)$ on $Q_\ell(T)$, $T > 0$, where $Q_\ell(T) = \{(t,x) | 0 < x < \ell(t), 0 < t < T\}$, such that

$$u_{tt} + \gamma u_{xxxx} - \mu u_{txx} - \kappa u_{xx} = f \quad \text{in } Q_\ell(T),$$

$$u(t,0) = u_{xx}(t,0) = 0 \quad \text{for } 0 < t < T,$$

$$u(t,\ell(t)) = \ell(t) - \ell_0, u_{xx}(t,\ell(t)) = 0 \quad \text{for } 0 < t < T,$$

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$$u(0) = u_0, u_t(0) = v_0 \text{ on } [0, \ell_0],$$
$$\ell''(t) = g(t) - \kappa u_x(t, \ell(t)),$$
$$\ell(0) = \ell_0, \ell'(0) = \hat{\ell}_0,$$

where f and g are given functions, ℓ_0, $\hat{\ell}_0$, u_0 and v_0 are initial functions.

In the next section we provide the precise definition of a solution of (P1) and a theorem for the well-posedness. Also, in the final section we show a result concerned with the free boundary problem (P2), which will be dealt with in author's forthcoming paper [3].

2 Initial boundary value problem (P1)

In order to give results concerned with (P1) we use the following notations. We put $H = L^2(0,1)$, $X = H_0^1(0,1)$, and write X^* as the dual space of X, $(\cdot,\cdot)$ as the inner product of H and $\langle\cdot,\cdot\rangle$ as the pairing between X^* and X. Moreover, we set $\beta : (-\infty, 1) \to R$, $R := (-\infty, \infty)$, $\beta(r) = \frac{r}{1-r}$ for $r < 1$. Clearly, β is the maximal monotone graph on $R \times R$ and $\partial\hat{\beta}(r) = \beta(r)$, where $\hat{\beta}(r) = -r - \log(1-r)$ if $r < 1$, $= \infty$ otherwise. We note that $\hat{\beta}$ is proper, convex and lower semi-continuous on R. We quote the book by Brezis [4] for definitions and basic properties of convex functions and subdifferentials.

Next, we define a solution of (P1) as follows:

Definition 2.1. Let u be a function on $Q(T)$. We call that u is a weak solution of (P1) on $[0,T]$, $T > 0$, if the conditions (S1) $\sim$ (S4) hold.

(S1) $u \in S_w(T)$, where $S_w(T) := W^{2,2}(0,T;X^*) \cap W^{1,\infty}(0,T;X) \cap L^\infty(0,T;H^3(0,1)) \cap W^{1,2}(0,T;H^2(0,1)) \cap \{u_{xx} \in L^\infty(0,T;X)\}$.

(S2) There exists a positive constant δ such that $1 - u_x \geq \delta$ on $Q(T)$.

(S3) It holds that

$$\langle u_{tt}, \eta\rangle - \gamma(u_{xxx}, \eta_x) - \mu(u_{txx}, \eta) + (\beta(u_x), \eta_x) = 0 \text{ for } \eta \in X \text{ and a.e. } t \in [0,T].$$

(S4) $u(0,\cdot) = u_0$ and $u_t(0,\cdot) = v_0$ on $(0,1)$.

Moreover, if $u \in S_s(T) := S_w(T) \cap W^{2,2}(0,T;H) \cap W^{1,\infty}(0,T;H^2(0,1))$ $\cap L^\infty(0,T;H^4(0,1)) \cap W^{1,2}(0,T;H^3(0,1))$ and (1.1) holds a.e. on $Q(T)$, then we say that u is a strong solution of (P) on $[0,T]$.

The next theorem guarantees the well-posedness of (P1).

Theorem 2.1. *Let $T > 0$, $u_0 \in H^3(0,1) \cap X$ with $u_{0xx} \in X$ and $v_0 \in H_0^1(0,1)$. Then there exists a positive constant c independent of T such that if*

$$|v_0|_H^2 + \frac{\gamma}{2}|u_{0xx}|_H^2 + \int_0^1 \hat{\beta}(u_{0x})dx \le c,$$

then (P1) has a unique weak solution on $[0,T]$ for any $T > 0$. Moreover, if $u_0 \in H^4(0,1)$ and $v_0 \in H^2(0,1)$, then the weak solution is a strong solution.

We can prove the uniqueness under the more general assumptions.

Remark 2.1. If u_1 and u_2 satisfy the following (S1'), (S2'), (S3') and (S4), then $u_1 = u_2$ on $Q(T)$.

(S1') $u, u_x \in L^\infty(Q(T))$;

(S2') $1 - u_x > 0$ a.e. on $Q(T)$ and $\frac{1}{1-u_x} \in L^4(Q(T))$;

(S3') It holds that

$$\int_{Q(T)} u(\eta_{tt} + \gamma\eta_{xxxx} + \mu\eta_{txx})dxdt + \int_{Q(T)} \beta(u_x)\eta_x dxdt = \int_0^1 v_0\eta(0)dx - \int_0^1 u_0\eta_t(0)dx$$

for $\eta \in S_s(T)$ with $\eta(T) = \eta_t(T) = 0$.

We can prove the existence by using Banach's fixed point theorem and standard approximation method. Also, Remark 2.1 is proved by the dual equation method. The dual equation method was already discussed in Chapter 3 of [5] and was applied to one-dimensional shape memory alloy problem called Falk model in [1].

The proof of Theorem 2.1 is quite standard, so we omit it (see [2]). Here, we show the proof of Remark 2.1

Proof of Remark 2.1. We assume that u_1 and u_2 satisfy (S1'), (S2'), (S3') and (S4), and put $u = u_1 - u_2$. Then it holds that

$$\int_{Q(T)} u(\eta_{tt} + \gamma\eta_{xxxx} + \mu\eta_{txx})dxdt + \int_{Q(T)} (\beta(u_{1x}) - \beta(u_{1x}))\eta_x dxdt = 0$$

for $\eta \in S_0(T) := \{\eta \in S_s(T) : \eta(T) = \eta_t(T) = 0\}$. Here, we put

$$F = \frac{1}{1-u_{1x}}\frac{1}{1-u_{2x}} \quad \text{on } Q(T).$$

Immediately, we have $F \in L^2(Q(T))$ because of (S2'). Then (S3') implies

$$\int_{Q(T)} u(\eta_{tt} + \gamma\eta_{xxxx} + \mu\eta_{txx})dxdt + \int_{Q(T)} Fu_x\eta_x dxdt = 0 \quad \text{for } \eta \in S_0(T). \tag{2.1}$$

Here, we can take a sequence $\{F_j\} \subset C_0^\infty(\overline{Q(T)})$ such that $F_j \to F$ in $L^2(Q(T))$ as $j \to \infty$.

Let $z \in C_0^\infty(\overline{Q(T)})$ and for each j consider the following problem.

$$\begin{cases} \eta_{jtt} + \gamma\eta_{jxxxx} + \mu\eta_{jtxx} - (F_j\eta_{jx})_x = z \quad \text{in } Q(T), \\ \eta_j(t,0) = \eta_j(t,1) = \eta_{jxx}(t,0) = \eta_{jxx}(t,1) = 0 \quad \text{for } 0 < t < T, \\ \eta_j(T) = \eta_{jt}(T) = 0. \end{cases}$$

It is easy to show the existence of a unique solution $\eta_j \in S_0(T)$ of the above problem, since F_j and z are smooth.

As the next step we shall obtain some uniform estimates for η_j with respect to j. By putting $\hat{\eta}_j(t,x) = \eta_j(T-t,x)$ for $(t,x) \in Q(T)$ we have

$$\begin{aligned} &\hat{\eta}_{jtt} + \gamma\hat{\eta}_{jxxxx} - \mu\hat{\eta}_{jtxx} - (\hat{F}_j\hat{\eta}_{jx})_x = \hat{z} \quad \text{in } Q(T), \\ &\hat{\eta}_j(t,0) = \hat{\eta}_j(t,1) = \hat{\eta}_{jxx}(t,0) = \hat{\eta}_{jxx}(t,1) = 0 \quad \text{for } 0 < t < T, \\ &\hat{\eta}_j(0) = \hat{\eta}_{jt}(0) = 0, \end{aligned} \tag{2.2}$$

where $\hat{F}(t,x) = F(T-t,x)$ and $\hat{z}(t,x) = z(T-t,x)$ for $(t,x) \in Q(T)$. We multiply (2.2) by $\hat{\eta}_{jt}$ and integrate it over $(0,1)$. Then it yields that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\hat{\eta}_{jt}(t)|_H^2 + \frac{\gamma}{2}\frac{d}{dt}|\hat{\eta}_{jxx}(t)|_H^2 + \mu|\hat{\eta}_{jtx}(t)|_H^2 \\ = \;& (\hat{z}(t), \hat{\eta}_{jt}(t)) - (\hat{F}_j(t)(\hat{\eta}_{jx})(t), \hat{\eta}_{jtx}(t)) \\ \le \;& \frac{1}{2}|\hat{z}(t)|_H^2 + \frac{1}{2}|\hat{\eta}_{jx}(t)|_H^2 + \frac{\mu}{2}|\hat{\eta}_{jtx}(t)|_H^2 + \frac{1}{2\mu}|\hat{F}_j(t)\hat{\eta}_{jx}(t)|_H^2 \\ \le \;& \frac{1}{2}|\hat{z}(t)|_H^2 + \frac{1}{2}|\hat{\eta}_{jx}(t)|_H^2 + \frac{\mu}{2}|\hat{\eta}_{jtx}(t)|_H^2 + \frac{1}{2\mu}|\hat{F}_j(t)|_H^2|\hat{\eta}_{jxx}(t)|_H^2 \quad \text{for a.e. } t \in [0,T]. \end{aligned}$$

Clearly, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\hat{\eta}_{jt}(t)|_H^2 + \frac{\gamma}{2}\frac{d}{dt}|\hat{\eta}_{jxx}(t)|_H^2 + \frac{\mu}{2}|\hat{\eta}_{jtx}(t)|_H^2 \\ \le \;& \frac{1}{2}|\hat{z}(t)|_H^2 + \frac{1}{2}|\hat{\eta}_{jx}(t)|_H^2 + \frac{1}{2\mu}|\hat{F}_j(t)|_H^2|\hat{\eta}_{jxx}(t)|_H^2 \quad \text{for a.e. } t \in [0,T]. \end{aligned} \tag{2.3}$$

Therefore, Gronwall's inequality leads to the following estimate.

$$\begin{aligned} &\frac{1}{2}|\hat{\eta}_{jt}(t)|_H^2 + \frac{\gamma}{2}|\hat{\eta}_{jxx}(t)|_H^2 + \frac{\mu}{2}\int_0^t |\hat{\eta}_{j\tau x}(\tau)|_H^2 d\tau \\ \le \;& \frac{1}{2}\exp\left(\int_0^t (1 + \frac{2}{\gamma}|\hat{F}_j(\tau)|_H^2)d\tau\right)\int_0^t |\hat{z}(\tau)|_H^2 d\tau \quad \text{for } 0 \le t \le T. \end{aligned}$$

Hence, the set $\{\eta_{jx}\}$ is bounded in $L^2(Q(T))$.

Accordingly, by using (2.1) we observe that

$$\begin{aligned}\int_{Q(T)} zudxdt &= \int_{Q(T)} (\eta_{jtt} + \gamma\eta_{jxxxx} + \mu\eta_{jtxx} - (F_j\eta_{jx})_x)udxdt \\ &= -\int_{Q(T)} u_x F\eta_{jx}dxdt + \int_{Q(T)} F_j\eta_{jx}u_x dxdt \quad \text{for each } j.\end{aligned}$$

Therefore, it follows that

$$|\int_{Q(T)} zudxdt| \le |F_j - F|_{L^2(Q(T))}|u_x|_{L^\infty(Q(T))}|\eta_{jx}|_{L^2(Q(T))} \text{ for each } j$$

so that

$$\int_{Q(T)} zudxdt = 0 \quad \text{for any } z \in C_0^\infty(\overline{Q(T)}).$$

This is the assertion of Remark 2.1. ◇

3 Free boundary problem (P2)

The aim of this section is to provide a theorem on the well-posedness for the free boundary problem (P2). Now, in order to consider the problem on the cylindrical domain $Q(T)$ we put $\hat{u}(t,x) = u(t,x) - \frac{x}{\ell(t)}(\ell(t) - \ell_0)$ and $w(t,y) = \hat{u}(t,\ell(t)y)$ for $(t,y) \in Q(T)$. Then we reformulate (P2) as the following problem:

$$w_{tt} + \gamma s^4(t)w_{yyyy} - \mu s^2(t)w_{tyy} = \tilde{f} + \ell_0 y(\frac{\ell\ell'' - 2|\ell'|^2}{\ell^2}) + L(w) \text{ in } Q(T), \tag{3.1}$$

$$\ell''(t) = g(t) - \kappa(s(t)w_y(t,1) + 1 - \frac{\ell_0}{\ell(t)}) \quad \text{for } 0 < t < T, \tag{3.2}$$

$$w(t,0) = w(t,1) = w_{yy}(t,0) = w_{yy}(t,1) = 0 \quad \text{for } 0 < t < T, \tag{3.3}$$

$$w(0,y) = u_0(\ell(0)y), w_t(0,y) = v_0(\ell_0 y) + \ell'(0)yu_{0x}(\ell_0 y) \quad \text{for } 0 < y < 1, \tag{3.4}$$

$$\ell(0) = \ell_0, \ell'(0) = \hat{\ell}_0, \tag{3.5}$$

where $s(t) = 1/\ell(t)$, $L(w) = -s''\ell y w_y + (-|s'\ell y|^2 + \kappa s^2 + 2\mu ss')w_{yy} - s'\ell y w_{ty} + \mu s^2 s'\ell y w_{yyy}$ and $\tilde{f}(t,y) = f(t,\ell(t)y)$.

By using this function w we define a solution of (P2) in the following way.

Definition 3.1. We call $\{u,\ell\}$ is a solution of (P2) on $[0,T]$, $T > 0$, if the following properties (D1) $\sim$ (D4) hold: Let w define as before.

(D1) $w \in W^{1,\infty}(0,T;X) \cap L^\infty(0,T;H^3(0,1)) \cap W^{1,2}(0,T;H^2(0,1)) \cap W^{2,2}(0,T;X^*)$ and $w_{yy}(t) \in X$ for a.e. $t \in [0,T]$.

(D2) (3.4) and (3.5) hold.

(D3) For $\eta \in X$ and a.e. $t \in [0,T]$,

$$\begin{aligned} & \langle w_{tt}(t), \eta\rangle - \gamma s^4(t)(w_{yyy}(t), \eta_y) - \mu s^2(t)(w_{tyy}(t), \eta) \\ = & (\tilde{f}(t) + \ell_0 y(\frac{\ell\ell'' - 2|\ell'|^2}{\ell^2}) + L(w)(t), \eta). \end{aligned}$$

(D4) $\ell \in W^{2,2}(0,T)$ and $\ell > 0$ on $[0,T]$,
$\ell''(t) = g(t) - \kappa(s(t)w_y(t,1) + 1 - \frac{\ell_0}{\ell(t)})$ for a.e. $t \in [0,T]$.

Theorem 3.1. *Assume* $\gamma > 0$, $\mu > 0$, $\kappa > 0$, $f \in L^2(0,T;L^2(0,\infty))$, $g \in L^2(0,T)$, $\ell_0 > 0$, $u_0 \in H^3(0,\ell_0) \cap H_0^1(\ell_0)$, $u_{0xx} \in H_0^1(0,\ell_0)$, $v_0 \in H^1(0,\ell_0)$, $v_0(0) = 0$, $v_0(\ell_0) - \hat{\ell}_0 + \hat{\ell}_0 u_{0x}(\ell_0) = 0$. *Then, (P2) has a unique solution on* $[0,T_0]$ *for some* $T_0 > 0$.

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