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Artificial Potential Fields with Asymptotic Stability Properties

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1 Introduction

An interesting problem in robotics involves the identification in a two- or three-dimensional space a continuous path that allows a robot, or a part of it, to reach its destination without colliding with obstacles that may exist in the space. Sometimes referred to as the *findpath problem*, it is essentially a geometric problem.

In this paper we are proposing that we look at the findpath problem from the point of view of having an obstacle avoidance system that is at least Lyapunov stable. We consider a simple planar obstacle avoidance system that consists of a point-mass being controlled to its destination or target whilst avoiding a fixed object in two-dimensional space. The proposed Lyapunov function for the system produces artificial potential fields both for obstacle avoidance and for target attraction. After establishing Lyapunov stability, we then show that it is possible to identify a region of asymptotic stability in which the target is the only minimum point.

The background of our method lies in the application of the Lyapunov method to qualitative differential games that involve dynamical systems subject to control by one or more players ([2], [1]). Using these differential games concepts, a simple method to solve the findpath problem was proposed in [3]. The methods therein are used in this paper.

2 A Globally Asymptotic Stable Point-Mass System

Consider a point-mass, defined as the disk of radius $r_P \geq 0$, and positioned at $(x(t), y(t)) \in \mathbb{R}^2$ at time $t \geq 0$. That is, the point-mass is

$$P = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - x)^2 + (z_2 - y)^2 \leq r_P^2\}.$$

Its instantaneous velocity is $(v(t), w(t)) := (\dot{x}(t), \dot{y}(t))$. Our general ODE system is therefore of the form

$$\dot{x}(t) = v(x(t), y(t)), \quad \dot{y}(t) = w(x(t), y(t)), \quad (x(0), y(0)) := (x_0, y_0), \qquad (1)$$

and our objective is to steer the point-mass to a goal or target in $\mathbb{R}^2$. The target is defined as the disk with center (τ_1, τ_2) and radius r_T, that is,

$$T = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - \tau_1)^2 + (z_2 - \tau_2)^2 \leq r_T^2\}$$

with $r_T \geq 0$ sufficiently small. Let $\mathbf{e}_0 = (\tau_1, \tau_2)$. We state our first result:

Theorem 1 *Let $v(x, y) = -(x - \tau_1)$ and $w(x, y) = -(y - \tau_2)$. Then the point $\mathbf{e}_0$ is the only equilibrium point of system (1) and is globally asymptotically stable.*

Proof. If $v(x, y) = -(x - \tau_1)$ and $w(x, y) = -(y - \tau_2)$, then it is clear that $\mathbf{e}_0 = (\tau_1, \tau_2)$ is the only equilibrium point of the system. To prove global asymptotic stability, we use the Lyapunov function $V(x, y) = \frac{1}{2}[(x - \tau_1)^2 + (y - \tau_2)^2]$, which is radially unbounded, with $V(\mathbf{e}_0) = 0$. Its time-derivative along a trajectory of system (1) is $\dot{V}_{(1)}(x, y) = -[(x - \tau_1)^2 + (y - \tau_2)^2]$, with $\dot{V}_{(1)}(x, y) < 0$ for all $(x, y) \neq \mathbf{e}_0$, and $\dot{V}_{(1)}(\mathbf{e}_0) = 0$.

3 An Asymptotically Stable Point-Mass System with a fixed Obstacle

We next consider the situation where there is now a fixed obstacle that the point-mass P has to avoid. Precisely, if (o_1, o_2) is the center of the disk, and r_O is the radius of the disk, then the obstacle can be defined as $O = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - o_1)^2 + (z_2 - o_2)^2 \leq r_O^2\}$. Next, we construct an artificial potential field function that guarantees target attraction and collision avoidance.

3.1 Target Attraction and Collision Avoidance

For target attraction, we want to have a measurement, at time $t \geq 0$, of the distance between the position (x, y) of the point-mass P and its target T. A likely function is therefore $G(x, y) = \frac{1}{2}\{(x - \tau_1)^2 + (y - \tau_2)^2\}$. For obstacle-avoidance, we want to have a measurement of the distance between the point-mass P and its obstacle O. For this purpose we shall utilize the following function appropriately:

$$W(x, y) = \frac{1}{2}\{(x - o_1)^2 + (y - o_2)^2 - (r_O + r_P)^2\}. \tag{2}$$

3.2 Lyapunov Function as the the Artificial Potential Field Function

Let $\alpha > 0$ be a constant, and consider as a tentative Lyapunov function for system (1)

$$V(x, y) = G(x, y) + \frac{\alpha G(x, y)}{W(x, y)}. \tag{3}$$

It is clear that V is continuous and locally positive definite on the domain $D(V) = \{(x, y) \in \mathbb{R}^2 : W(x, y) > 0\}$. That is, $V(x, y) > 0$ for all $(x, y) \in D(V) \setminus \{\mathbf{e}_0\}$ and $V(\mathbf{e}_0) = 0$, with $\mathbf{e}_0 \in D(V)$. It is clear that $D(V)$ is a pathwise-connected proper subset of $\mathbb{R}^2$, meaning that for every two points in $D(V)$, there is a path connecting them.

Now, along a trajectory of system (1), we have $\dot{V}_{(1)}(x, y) = f(x, y)v(x, y) + g(x, y)w(x, y)$, where

$$\left.\begin{aligned} f(x, y) &:= \left[1 + \frac{\alpha}{W(x, y)}\right](x - \tau_1) - \frac{\alpha G(x, y)}{W^2(x, y)}(x - o_1), \\ g(x, y) &:= \left[1 + \frac{\alpha}{W(x, y)}\right](y - \tau_2) - \frac{\alpha G(x, y)}{W^2(x, y)}(y - o_2). \end{aligned}\right\} \tag{4}$$

For some arbitrary constant $k > 0$, and define v and ω as

$$v(x, y) = -kf(x, y) \text{ and } w(x, y) = -g(x, y) \tag{5}$$

Then it can easily be verified that

$$\dot{V}_{(1)}(x, y) = -\frac{1}{k}\left[v^2(x, y) + w^2(x, y)\right].$$

This shows that $\dot{V}_{(1)}(x, y) \leq 0$ for all $(x, y) \in D(V)$, with $\dot{V}_{(1)}(\mathbf{e}_0) = 0$. This implies that $\mathbf{e}_0$ is a stable equilibrium point of system (1) if the instantaneous velocities are defined as in (5).

3.3 Main Results

Substituting v and w into (1), we have

$$\left.\begin{array}{rcl} \dot{x} & = & -kf(x,y) = -k\left[1+\dfrac{\alpha}{W(x,y)}\right](x-\tau_1) - \dfrac{\alpha G(x,y)}{W^2(x,y)}(x-o_1), \\ \dot{y} & = & -kg(x,y) = -k\left[1+\dfrac{\alpha}{W(x,y)}\right](y-\tau_2) - \dfrac{\alpha G(x,y)}{W^2(x,y)}(y-o_2) \\ & & (x(0),y(0)) := (x_0,y_0) \end{array}\right\} \tag{6}$$

The functions f_0 and g_0 are continuously differentiable on $D(V)$, and therefore the existence in D(V) of the solutions of system (6) is guaranteed. Moreover, all the solutions that are initiated in $D(V)$ are bounded and lie in $D(V)$ for all $t \geq 0$, given that our discussions above yield the stability of system (6):

Theorem 2 *The point* $\mathbf{e}_0 = (\tau_1, \tau_2)$ *is a stable equilibrium point of system (6).*

We shall endeavor to show next that indeed $\mathbf{e}_0$ is asymptotically stable given a certain set of initial conditions.

We begin by identifying all the equilibrium points of system (6).

3.4 Equilibrium Points of System (6)

In $\mathbb{R}^2$, the line that passes through the center of the obstacle O and the center of the target T is

$$l = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 - \tau_2 = \frac{\tau_2 - o_2}{\tau_1 - o_1}(z_1 - \tau_1)\}.$$

Let $\theta = \tan^{-1}\left(\dfrac{o_2 - \tau_2}{o_1 - \tau_1}\right)$, and let l_b and l_f be proper subsets of l, with

$$l_b = \{(z_1, z_2) \in l : z_1 < o_1 - r_0\cos\theta \text{ and } z_2 < o_2 - r_0 \sin\theta\},$$

and

$$l_f = \{(z_1, z_2) \in l : z_1 > o_1 + r_0\cos\theta \text{ and } z_2 > o_2 + r_0 \sin\theta\}.$$

Definition 1 *The point* (z_1, z_2) *is said to be* behind the obstacle O *if* $(z_1, z_2) \in l_b$, *and is said to be* in front of the obstacle O *if* $(z_1, z_2) \in l_f$.

Clearly, l_b and l_f are segments of l, but in $D(V)$. The point-mass P is behind the obstacle O if it is on the line segment l_b and is in front of O if is on the line segment l_f. The need to have its radius r_P sufficiently small is a practical consideration, because, obviously, we do not want P to overlap with O whose r_O is not restricted.

Lemma 1 *System (6) has a finite number of equilibrium points, of which those with real components are in l.*

Proof. Solving for $\dot{v} = \dot{w} = 0$, we have

$$f(x,y) = \left[1 + \frac{\alpha}{W(x,y)}\right](x - \tau_1) - \frac{\alpha G(x,y)}{W^2(x,y)}(x - o_1) \;=\; 0, \tag{7}$$

$$g(x,y) = \left[1 + \frac{\alpha}{W(x,y)}\right](y - \tau_2) - \frac{\alpha G(x,y)}{W^2(x,y)}(y - o_2) \;=\; 0. \tag{8}$$

Since $1 + \alpha/W \neq 0$ and $\alpha G/W^2 \neq 0$ in $D(V)$, we have, by solving (7) and (8) simultaneously,

$$(x - \tau_1)(y - o_2) = (x - o_1)(y - \tau_2), \tag{9}$$

which corresponds to points in l. However, there are finite number of equilibrium points because f and g involve rational functions, and if they have real components, then they are in l. Indeed, on rewriting $f(x,y) = 0$ and $g(x,y) = 0$, we have

$$\left.\begin{array}{rcl} W(W+\alpha)(x-\tau_1) - \alpha G(x - o_1) & = & 0, \\ W(W+\alpha)(y-\tau_2) - \alpha G(y - o_2) & = & 0. \end{array}\right\} \tag{10}$$

If we substitute (9) into either of the equations in (10), we will have a polynomial of degree 5 in x or in y. To see this, we can assume, without loss of generality, that the line l lies on the x-axis and that the obstacle is centered at the origin. This means that $y = \tau_2 = o_1 = o_2 = 0$. Then the first equation in (10) yields

$$(x - \tau_1)\left\{x^4 - 2\left(r_O + r_P\right)^2 x^2 + 2\alpha\tau_1 x + \left(r_O + r_P\right)^2\left[\left(r_O + r_P\right)^2 - 2\alpha\right]\right\} = 0, \tag{11}$$

which is a polynomial of degree 5 in x. Note that (10) also shows that one of the equilibrium points is (τ_1, τ_2). This ends the proof of Lemma 1.

□

Lemma 2 *Let $\tau_1 > 0$ and $\tau_2 > 0$ be sufficiently large. Then system (6) has exactly one equilibrium point behind the obstacle O and exactly one equilibrium point located at $\mathbf{e}_0 = (\tau_1, \tau_2)$ in front of the obstacle O.*

Proof. We prove this Lemma 2 by showing that of the five roots alluded to by Lemma 1, three are real. One root gives an equilibrium point that is behind the obstacle; another root gives $\mathbf{e}_0$, meaning that the only equilibrium point in front of the obstacle is $\mathbf{e}_0$; and the third root gives an equilibrium point that is always within the obstacle disk. This third root can be ignored.

Assume once again, and without loss of generality, that the line l lies on the x-axis and that the obstacle is centered at the origin, with radius $r_O > 0$. Then we have equation (11), which shows that there are five roots, one of which is $x = \tau_1$. This gives the equilibrium point $\mathbf{e}_0$.

To determine the other roots, let the second term of (11) be $f(x) = a_4x^4 + a_2x^2 + a_1x + a_0$ where $a_4 := 1$, $a_2 := -2\left(r_O + r_P\right)^2$, $a_1 := 2\alpha\tau_1$ and $a_0 := \left(r_O + r_P\right)^2 \left[\left(r_O + r_P\right)^2 - 2\alpha\right]$. Now, a_0 could either be negative, 0 or positive, depending on the values of $\alpha > 0$, $r_O > 0$ and $r_P > 0$. Let us consider each case separately.

1. **Case $a_0 < 0$.** We have

$$f(-r_O) = -r_O^4 - 4r_O^3 r_P - 2r_P^2 r_O^2 - 2\alpha\tau_1 r_O + a_0 < 0. \qquad (12)$$

Now, $f(x) \to +\infty$ as $x \to -\infty$. Thus, by the Intermediate Value Theorem, there is a real root in $(-\infty, -r_O)$. Obviously this root gives an equilibrium point behind the obstacle. Next, we have

$$f(r_O) = -r_O^4 - 4r_O^3 r_P - 2r_P^2 r_O^2 + 2\alpha\tau_1 r_O + a_0 < 2\alpha\tau_1 r_O \qquad (13)$$

It is clear that we can find $\tau_1 > 0$ sufficiently large such that $0 < f(r_0) < 2\alpha\tau_1 r_O$. Moreover, since $f(0) = a_0 < 0$, the Intermediate Value Theorem guarantees the existence of a real root in $[0, r_O)$, giving an equilibrium point within the obstacle disk.

2. **Case a_0=0.** Put $a_0 = 0$ in (12) and (13), respectively, and we obtain the two equilibrium points via the Intermediate Value Theorem; one behind the obstacle and the other within the obstacle disk if $\tau_1 > 0$ is sufficiently large.

3. **Case $a_0 > 0$.** We have

$$f(-r_O) = -r_O^4 - 4r_O^3 r_P - 2r_P^2 r_O^2 - 2\alpha\tau_1 r_O + a_0 < -2\alpha\tau_1 r_O + a_0. \qquad (14)$$

It is clear that we can find $\tau_1 > 0$ sufficiently large such that $f(r_O) < -2\alpha\tau_1 r_O + a_0 < 0$. Since $f(x) \to +\infty$ as $x \to -\infty$, the Intermediate Value Theorem guarantees the existence of a real root that gives an equilibrium point behind the obstacle. Also, since $f(0) = a_0 > 0$, the Intermediate Value Theorem produces a real root that gives an equilibrium point within the obstacle disk.

This completes the proof of Lemma 2.

Next, we show that if a solution of system (6) starts in l_b or in l_f, then it remains in l_b or l_f for all time $t \geq 0$. An implication of this is that it will be necessary to delete the set l_b, which contains the equilibrium point behind the obstacle O, from $D(V)$ to conclude asymptotic stability of $\mathbf{e}_0$.

Lemma 3 *A solution of system (6) that is initiated in l_b or in l_f remains in l_b or l_f, respectively.*

Proof. It is sufficient to show that a solution of system (6) corresponds to the line l. Accordingly, let $x = \eta(t)$ and $y = \zeta(t)$. Put these into the first equation of (6) to get

$$\dot{\eta}(t) = -k \times f_0(\eta(t), \zeta(t)), \quad \eta(0) = x_0. \tag{15}$$

This initial value problem has a unique solution since f_0 is a smooth function on $D(V)$. Next, for ζ, assume the solution of the form

$$y = \zeta(t) = \tau_2 + m(x - \tau_1), \quad m = (o_2 - \tau_2)/(o_1 - \tau_1), \tag{16}$$

which corresponds to the line l. We may also write y as

$$\begin{aligned} y &= \tau_2 + m(x - \tau_1) + mo_1 - mo_1 + o_2 - o_2 \\ &= o_2 + m(x - o_1) + \frac{o_2 - \tau_2}{o_1 - \tau_1}(o_1 - \tau_1) + \tau_2 - o_2 \\ &= o_2 + m(x - o_1). \end{aligned}$$

Substituting g_0, $y - \tau_2 = m(x - \tau_1)$ and $y - o_2 = m(x - o_1)$ into the right-hand side of the second equation of (6), we get

$$-mk_0 \left[\left(1 + \frac{\alpha}{W}\right)(x - \tau_1) - \frac{\alpha G_0}{W^2}(x - o_1) \right] = m\dot{x},$$

and since indeed $\dot{y} = m\dot{x}$, with $y(0) = \tau_2 + m(x(0) - \tau_1) = \tau_2 + m(x_0 - \tau_1) = y_0$, we have that the choice of $y = \zeta(t)$ given in (16) satisfies the second equation of (6) with $y(0) = y_0$. This proves that $(x, y) = (\eta(t), \zeta(t))$ is a solution of (6) with the initial condition (x_0, y_0) in l. This completes the proof of Lemma 3.

Because a solution that starts in l_b remains in l_b, it is necessary to remove the set l_b from $D(V)$ to conclude the asymptotic stability of $\mathbf{e}_0$.

Theorem 3 *Let $\tau_1, \tau_2 > 0$ be sufficiently large. If $(x_0, y_0) \in D(V) \setminus l_b$, then $\mathbf{e}_0$ is an asymptotically stable equilibrium point of system (6).*

Proof. As a consequence of Lemma 2, the set $D(V) \setminus l_b$ has only one equilibrium point and that is $\mathbf{e}_0$. But for all $(x, y) \in D(V) \setminus l_b$, $(x, y) \neq \mathbf{e}_0$, we have that $f_0(x, y) > 0$ and $g_0(x, y) > 0$. Hence, using our Lyapunov function (3), we have $\dot{V}_{(6)}(x, y) < 0$ for $(x, y) \in D(V) \setminus l_b$, $(x, y) \neq \mathbf{e}_0$, and $\dot{V}_{(6)}(\mathbf{e}_0) = 0$. This ends the proof of Theorem 3.

Corollary 1 *A solution that is initiated in l_f follows the line l to $\mathbf{e}_0$.*

Proof. This is a clear consequence of Lemma 2, Lemma 3 and Theorem 3. Lemma 2 shows that there is only one equilibrium point (which is $\mathbf{e}_0$) in front of the obstacle O. Lemma 3 and Theorem 3 imply that if a solution starts in l_f, not only would it remain in l_f but it would also be attracted to $\mathbf{e}_0$. This ends of the proof of Corollary 1.

In the next section, we show the outcome of some computer simulations.

3.5 Example

For our computer simulations, we choose different initial conditions with the same parameters (see Table 1).

Table 1: Simulation Parameters

Point-mass disk radius	$r_P = 0.01$
Obstacle center & disk radius	$(o_1, o_2) = (10, 10)$, $r_O = 3$
Target center & disk radius	$(\tau_1, \tau_2) = (20, 20)$, $r_T = 1$
Parameters	$k = 0.5$, $\alpha = 1$

As Figure 1 shows, all the chosen initial conditions, except that which is directly behind the obstacle, produce trajectories that converge to the target.

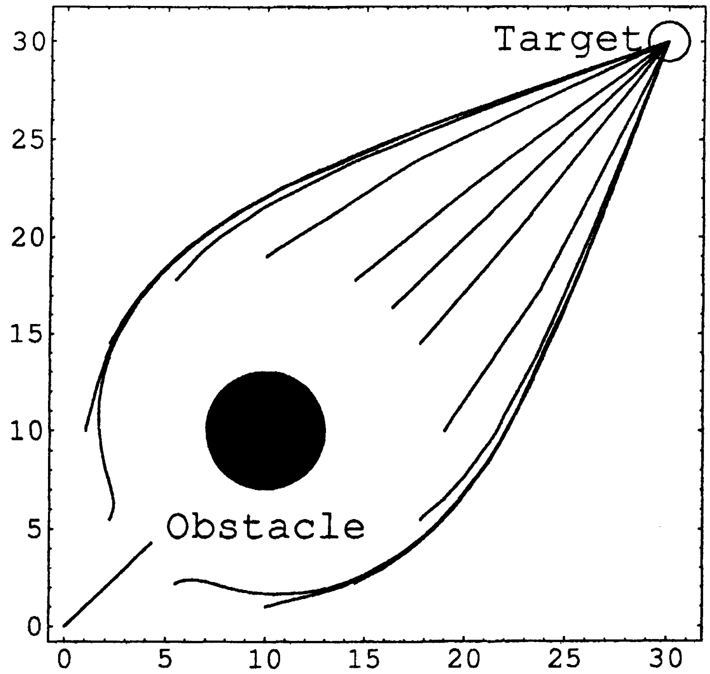

Figure 1: The trajectory which was initiated at $(0,0)$ behind the obstacle converged to the equilibrium point behind the obstacle.

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