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Optimal parameters for damped Sine-Gordon equation

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1 Introduction

In this paper, we study an identification problem for physical parameters α, β and δ appearing in the one-dimensional damped sine-Gordon equation

$$\left.\begin{array}{l} \dfrac{\partial^2 y}{\partial t^2}+\alpha\dfrac{\partial y}{\partial t}-\beta\Delta y+\delta\sin y=g, \quad x\in(0,L), t\in(0,T), \\ y(t,0)=y(t,L)=0, \quad t\in(0,T), \\ y(0,x)=y_0(x) \ \text{ and } \ \dfrac{\partial y}{\partial t}(0,x)=y_1(x), \quad x\in(0,L). \end{array}\right\} \tag{1.1}$$

The identification problem for (1.1) consists in finding the parameters α, β and δ such that the solution of (1.1) exhibits the desired behavior. More precisely, the parameter estimation problem for (1.1) is described as follows. Let $P=\{q=(\alpha,\beta,\delta)\in R^3 \mid \beta>0\}$ be equipped with the Euclidean norm. Let $P_{ad}\subset P$ be an admissible set of parameters and define the cost functional $J(q)$ by

$$J(q)=\int_0^T\int_0^L (y(q;t,x)-z_d(t,x))^2 dxdt, \quad q\in P, \tag{1.2}$$

where z_d is a given function on $(0,T)\times(0,L)$. The data z_d can be thought of as the targeted behavior of (1.1). The parameter identification problem for (1.1) with the objective function (1.2) is to find $q^*=(\alpha^*,\beta^*,\delta^*)\in P_{ad}$ satisfying

$$J(q^*)=\inf_{q\in P_{ad}} J(q), \ P_{ad}\subset P. \tag{1.3}$$

Since q^* is a set of constants, the bang bang control law can be derived from the state system (1.1) and the related adjoint state system. That is, if one chooses P_{ad} to be a closed subset in R^3, then, under certain conditions, q^* is uniquely determined by the extremal values of the parameters in P_{ad}. These results were obtained in [5] and they will be reviewed in Theorem

3.1. It is meaningful to check the conditions on a, b and c which yield the bang bang control law (see Theorem 3.1). Unfortunately, it may be difficult to find q^* numerically by the bang bang control law, since one observes that all the parameter values approach zero.

In this paper we focus on examining the optimal values of a, b and d. The Powell's minimization method is used for the minimization of the cost functional J. The numerical solution of (1.1) is obtained by a Spectral Method [6].

The paper is organized as follows. In Section 2 we review error bounds for the solution of (1.1) and its approximation in a finite dimensional spectral space. In Section 3 we treat the parameter identification problem subject to (1.3) with (1.1). Finally, in Section 4 we present numerical results for the bang bang control law and the parameter estimation problem using the Powell's minimization method.

2 Weak solutions for the damped Sine-Gordon system

Let $I = (0, L)$, $Q = I \times (0, T)$, $H = L^2(I)$, and $H_0^r(I)$ be the Sobolev space on I with the norm $\|v\|_r$. Let the Hilbert space H have the norm $|v|$ and the inner product (u, v). When $r = 1$, we denote the inner product in $H_0^1(I)$ by $((u, u)) = (\nabla u, \nabla u)$, and its norm by $\|u\|$. Let $< u, v >$ denote the duality pairing between $V = H_0^1(I)$ and $V' = H^{-1}(I)$. Then we can define a self-adjoint operator A with the domain $D(A) = H_0^1(I) \cap H^2(I)$ by the relation $< Au, v >= ((u, v))$, and $Au = -\Delta u$ for $u \in D(A)$.

As in [1] the variational formulation for the weak solutions of (1.1) is given by

$$\left.\begin{array}{l} < \frac{\partial^2 y}{\partial t^2}, v > +\alpha(\frac{\partial y}{\partial t}, v) + \beta((y, v)) + \delta(f(y), v) = (g(t), v), \quad v \in V, \quad t \in (0, T), \\ y(0) = y_0 \text{ and } \frac{\partial y}{\partial t}(0) = y_1. \end{array}\right\} \tag{2.1}$$

Here we considered a general nonlinear function $f : V \to H$ instead of $\sin(y)$, having in mind other results involving more general equations, including the ones considered in (1.1). Assume that f is a Lipschitz continuous function with $f(0) = 0$. Problem (2.1) is an initial value problem for a formal abstract second-order differential equation in H:

$$\left.\begin{array}{l} y'' + \alpha y' + \beta A y + \delta f(y) = g, \quad t \in (0, T), \\ y(0) = y_0, \quad y'(0) = y_1, \end{array}\right\} \tag{2.2}$$

where $' = d/dt$ and $'' = d^2/dt^2$. The weak solutions of (2.1) are the solutions of (2.2) sought in the Hilbert space

$$W(0, T) = \{u \mid u \in L^2(0, T; V), u' \in L^2(0, T; H), u'' \in L^2(0, T; V')\}.$$

The existence, uniqueness and regularity results for the weak solutions of (2.2) are summarized in Theorem 2.1, see [4] for the proofs.

Theorem 2.1 Let $\alpha, \delta \in R$, $\beta > 0$ and let us assume that

$$y_0 \in V, \quad y_1 \in H, \quad \text{and} \quad g \in L^2(0,T;H). \tag{2.3}$$

Then there exists a unique weak solution $y \in L^2(0,T;V)$ of (2.2). This solution satisfies $y \in C([0,T];V) \cap W(0,T)$, $y' \in C([0,T];H)$, and

$$\|y(t)\|^2 + |y'(t)|^2 \leq C_1 \left[\|y_0\|^2 + |y_1|^2 + \|g\|^2_{L^2(0,T;H)}\right], \quad \forall t \in [0,T], \tag{2.4}$$

where C_1 is a constant.

Furthermore, if

$$y_0 \in D(A), \quad y_1 \in V \quad \text{and} \quad g' \in L^2(0,T;H), \tag{2.5}$$

then $y \in C([0,T];D(A))$ and $y' \in C([0,T];V)$.

Let N be a positive integer. Now we establish error bounds for finite spectral approximations $y_N(t)$. Let S_N be the subspace of H spanned by the sine functions $\{u_n(x) := \sin(n\pi x/L)\}, n = 1, \cdots, N$. Let $y_N(t) = y_N(\cdot, t) \in S_N$ be the solution of

$$\left.\begin{aligned} &\left(\frac{\partial^2 y_N}{\partial t^2}, v\right) + \alpha\left(\frac{\partial y_N}{\partial t}, v\right) + \beta((y_N, v)) + \delta(f(y_N), v) = (g(t), v), \\ &\qquad\qquad\qquad\qquad\qquad\qquad v \in S_N, \quad t \in (0,T), \\ &((y_N(0) - y(0), v)) = 0, \quad \left(\frac{\partial y_N}{\partial t}(0) - y_1, v\right) = 0, \quad v \in S_N. \end{aligned}\right\} \tag{2.6}$$

We need the following well-known error estimate [6]: for any $s, r \in R$ with $0 \leq s \leq r$,

$$\|P_N u - u\|_s \leq C_0(1 + N^2)^{(s-r)/2}\|u\|_r \quad \text{for} \quad u \in H_0^r(I), \tag{2.7}$$

where $P_N : H \to S_N$ is the projection operator, and C_0 is a constant dependent on L. Using P_N the initial value problem (2.6) can be written in an equivalent form

$$\left.\begin{aligned} &y_N'' + \alpha y_N' + \beta A y_N + \delta P_N f(y_N) = P_N g, \quad t \in (0,T), \\ &y_N(0) = P_N y_0, \quad y_N'(0) = P_N y_1. \end{aligned}\right\} \tag{2.8}$$

The following Theorem for the error estimate is established in [3].

Theorem 2.2 Let $r > 0$. If the solution y of (2.2) satisfy $y \in H_0^r(I)$, then there is a C_1 such that

$$|y(t) - y_N(t)| \leq C_1(1 + N^2)^{-r/2}, \quad \forall t \in [0,T].$$

If the solution y of (2.2) satisfy $y \in H_0^{r+1}(I)$, then there is a constant $C_2 > 0$ such that

$$\|y(t) - y_N(t)\| \leq C_2(1 + N^2)^{-r/2}, \quad \forall t \in [0,T].$$

3 Parameter identification problem

In this section we study a parameter identification problem for the one dimensional damped sine-Gordon equation of the form

$$\left.\begin{array}{l} y'' + \alpha y' + \beta A y + \delta \sin y = g, \quad t \in (0,T), \\ y(0) = y_0, \quad y'(0) = y_1. \end{array}\right\} \tag{3.1}$$

We will always assume that the conditions (2.3) in Theorem 2.1 are satisfied for the initial data y_0, y_1 and the forcing term g. Recall that $P = \{q = (\alpha, \beta, \delta) \in R^3 \mid \beta > 0\}$ with the Euclidean norm. By Theorem 2.1 we have a well-defined solution map from P into $W(0,T) \subset C([0,T]; H)$, denoted by $y(q)$, which is the solution of (3.1).

With the solution $y(q)$ of (3.1) let us define the cost functional by

$$J(q) = \int_0^T |y(q;t) - z_d(t)|^2 dt, \quad z_d \in L^2(Q), \; q \in P. \tag{3.2}$$

The parameter identification problem for (3.1) with the objective function (3.2) is to find $q^* = (\alpha^*, \beta^*, \delta^*) \in P_{ad}$, which is an admissible subset of P, satisfying

$$J(q^*) = \inf_{q \in P_{ad}} J(q). \tag{3.3}$$

The parameter q^* is called an optimal parameter. It is well known that the map $q \to y(q)$ from P into $C([0,T]; H)$ is continuous, see [5]. Hence it is clear that the minimization problem (3.3) has at least one solution, provided P_{ad} is bounded and closed.

The following Theorem and Corollary are proved in [5].

Theorem 3.1 The optimal parameter q^* for (3.3) with (3.1) is characterized by two equations and one constraint

$$\left\{\begin{array}{l} y'' + \alpha^* y' + \beta^* A y + \gamma^* \sin y = g \;\text{ in }\; (0,T), \\ y(0) = y_0, \quad y'(0) = y_1, \end{array}\right. \tag{3.4}$$

$$\left\{\begin{array}{l} w'' - \alpha^* w' + \beta^* A w + \gamma^* \cos(y) w = y - z_d \;\text{ in }\; (0,T), \\ w(T) = 0, \quad w'(T) = 0, \end{array}\right. \tag{3.5}$$

$$\int_0^T ((\alpha^* - \alpha) y' + (\beta^* - \beta) A y + (\gamma^* - \gamma) \sin y + g, w) \, dt \geq 0, \; \forall q \in P_{ad}. \tag{3.6}$$

The constraint (3.6) is known to express the necessary condition for q^*. One can obtain the formula for q^* under the assumptions in Corollary 3.1. This is called the bang bang control law.

Corollary 3.1 Assume that the admissible set is given

$$P_{ad} = [\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \times [\gamma_1, \gamma_2], \quad \beta_1 > 0.$$

Then the optimal parameter $q^* = (\alpha^*, \beta^*, \delta^*)$ subject to (1.2) and (1.1) is determined by the formulas

$$\begin{aligned}
\alpha^* &= \frac{1}{2}\{\text{sign}(a)+1\}\alpha_2 - \frac{1}{2}\{\text{sign}(a)-1\}\alpha_1, \\
\beta^* &= \frac{1}{2}\{\text{sign}(b)+1\}\beta_1 - \frac{1}{2}\{\text{sign}(b)-1\}\beta_1, \\
\gamma^* &= \frac{1}{2}\{\text{sign}(c)+1\}\gamma_2 - \frac{1}{2}\{\text{sign}(c)-1\}\gamma_1
\end{aligned}$$

provided that

$$\begin{aligned}
a &= \int_Q \frac{\partial y}{\partial t}(x,t)w(x,t)\,dxdt \neq 0, \\
b &= \int_Q \nabla y(x,t)\cdot\nabla w(x,t)\,dxdt \neq 0, \\
c &= \int_Q \sin y(t,x)(x,t)w(x,t)\,dxdt \neq 0.
\end{aligned}$$

Now for a numerical analysis let us introduce the cost functional corresponding to (3.2). It can be give by the form

$$J_N(q) = \int_0^T |y_N(q;t) - z_d(t)|^2 dt, \quad q \in P, \tag{3.7}$$

where $y_N(q)$ is the weak solution of (2.6) when $f(y) = \sin y$. Similarly to (3.3), the parameter identification problem for (3.7) is to find $q_N^* \in P_{ad}$ such that

$$J_N(q_N^*) = \min_{q\in P_{ad}} J_N(q). \tag{3.8}$$

As in [5], one can easily prove that the cost functional (3.8) is continuous on P_{ad}. Therefore the minimization problem admits a minimum in P_{ad}.

The following Lemma and Theorem are proved in [3].

Lemma 3.1 There exists $C_3 > 0$ independent on N such that

$$|J_N(q) - J(q)| \leq C_3(1+N^2)^{-r}.$$

Theorem 3.2 Let $\{q_N^*\}$ be a sequence satisfying (3.8) and q^* be its limit point. Then $J(q^*) = \min_{q\in P_{ad}} J(q)$.

4 Numerical results

For our numerical experiments we chose to use a spectral method for the solution of the initial and boundary value problems (3.1) and (3.5), and Powell's minimization method for the minimization of the cost functional. See [6] for a detailed discussion of spectral methods and see [7,2] for the Powell's minimization method.

To accommodate the zero boundary conditions in (3.1) functions $u_n(x) = \sin(\pi n x/L)$, $n = 1, 2, \ldots$ are chosen as a (non-normalized) basis in $H = L_2(I)$. Let P_N be the projection operator onto $S_N = \text{span}\{w_n,\ n = 1, 2, \ldots, N\}$ in H, see (2.6)-(2.8) with $f(y) = \sin y$.

Expanding the functions in (2.6) with $f(y) = \sin y$ into the Fourier sine series, and using $v = w_k$, $k = 1, 2 \cdots N$ there we get

$$\left.\begin{aligned} &Y_k'' + \alpha Y_k' + \beta_k Y_k + \delta S_k(t) = F_k(t),\ t \in (0,T), \\ &Y_k(0) = Y_{k_0},\quad Y_k'(0) = Y_{k_1}. \end{aligned}\right\} \tag{4.1}$$

where $\beta_k = \beta k^2 \pi^2 / L^2$, $S_k(t)$ is the $k-th$ Fourier sine coefficient of $P_N \sin y_N(t)$, and $Y_k(t), F_k(t), Y_{k_0}$, and Y_{k_1} are the Fourier coefficients of the solution $y_N(t)$ and the corresponding functions in (2.6). Finally the approximate solutions $y_N(t) \in S_N$ of (3.4) are given. Similarly one can define the approximate solutions $w_N(t) \in S_N$ of (3.5) by the equations

$$\left.\begin{aligned} &W_k'' - \alpha W_k' + \beta_k W_k + \delta C_k(t) W_k = Y_k(t) - Z_k(t),\ t\ \in (0,T), \\ &W_k(T) = 0,\quad W_k'(T) = 0, \end{aligned}\right\} \tag{4.2}$$

where $C_k(t)$ is the $k - th$ Fourier sine coefficient of $P_N \cos y_N(t)$.

To test the assumptions on a, b, c in Corollary 3.1 and obtain q^* let $z_d(t) = P_N z_d(t) = \sum Z_k(t) w_n$ and introduce the time-discretized cost functional $J_N(q)$ defined by

$$J_N(q) = \frac{L}{2} \sum_{i=1}^{M} \sum_{k=1}^{N} [Y_k(q; t_i) - Z_k(t_i)]^2,\quad q \in P_{ad}, \tag{4.3}$$

where $Y_k(q; t)$ is the solution $Y_k(t)$ of (4.1) for the given values of the parameters $q = (\alpha, \beta, \delta) \in P_{ad}$. Lemma 3.1 and Theorem 3.2 hold for the cost functional (4.3), see [3].

The minimization problem for $J_N(q)$ is solved using a modification of Powell's minimization method. The modified method for solving our problem is described in [3].

To simulate the data let $\hat{q} \in P_{ad}$. Since real data always contain some noise, we set

$$z_d(t, x) = y(\hat{q}; t, x) + \epsilon \eta(x), \tag{4.4}$$

where $\eta(x)$ is a random variable uniformly distributed on interval $[-1, 1]$, and ϵ is a small constant. If $\epsilon = 0$, then $z_d(t) = y(\hat{q}; t)$ for all $t \in [0, T]$. Therefore, in this case one can check the performance of the parameter identification algorithm (i.e. if the algorithm finds the original set of parameters $\hat{q}$) by choosing sufficiently large N and M in (3.7).

We conducted two sets of numerical simulations with $\epsilon = 0$. See [3] for $\epsilon \neq 0$. The problem is to identify three unknown parameters α, β and δ.

In all simulations the initial value problem (4.1) and (4.2) are solved using a Leap-Frog Method with the time step $h = 0.01$ as follows. For example, let $Y_k^j, k = 1, 2, \cdots, N$ be defined by

$$\begin{aligned} &Y_k^{-1} = Y_{k_0} - h Y_{k_1}, \\ &Y_k^{j+1} = \frac{2Y_k^j - [\beta_k Y_k^j - F_k(t_j) + \delta S_k(t_j) h^2] + (1 - \alpha h/2) Y_k^{j-1}}{1 + \alpha h/2}, \end{aligned}$$

表 1: Parameter values for numerical simulations

Time and spatial intervals	$[0,T] \times [0,L] = [0,4] \times [0,\pi]$
Admissible set	$P_{ad} = [0.001,1] \times [0.1,1] \times [0.1,1] \times [0.1,1]$
Initial conditions	$y_0(x) = 0$
	$y_1(x) = \exp[-100(x-\pi/2)^2]$
Forcing function	$f(t,x) = 0.01$
N	16
Observation times	$t_i = (T/M)i, i = 1,2,\cdots,M$

for $j = 0,1,2,\cdots$. Then Y_k^j is an approximation of $Y_k(t)$ at $t = t_j = hj$.

The number of observations M varied in different simulations, but it is fixed as $M = 400$. The results of various observations are in [3].

Finally, let $q_0 \in P_{ad}$ be an arbitrarily chosen set of parameters, and $q_1, q_2, ...$ be the sequence of the sets of parameters iteratively obtained in the Powell's minimization method. The stopping criterion for this iterative process is

$$\frac{|J_N(q_m) - J_N(q_{m-1})|}{|J_N(q_0)|} < 10^{-6}. \tag{4.5}$$

Simulation 4.1 In this simulation let us consider $\hat{q} = (0.02, 0.7, 0.5)$ which is an interior point of P_{ad}, and z_d be computed according to (4.4). Let $q_N^* = q_m$ be the set of parameters attained when the Powell's minimization method was terminated according to the stopping criterion (4.5). The minimizers q_N^* together with the number of iterations m are shown in Tables 1 for the noise level $\epsilon = 0$, and the number of observations M.

Table 2 $\epsilon = 0$

M	m	q_N^*	$J_N(q_N^*)$
400	5	$(0.02000, 0.70000, 0.50001)$	0.000000
a	b	c	
-0.101522×10^{-8}	0.101384×10^{-6}	-0.295462×10^{-9}	

Tables 2 shows the identification algorithm is successful. The excellent simulation results are given in [3] for a small number of observations. As we have mentioned in the Introduction one can observe that all the parameters a, b and c are almost equal to zero.

Simulation 4.2 In this simulation let us consider $\hat{q} = (0.01, 1, 0.1)$ which is a boundary point in P_{ad}. All the procedures are the same as in Simulation 4.1.

Table 3 $\epsilon = 0$

M	m	q_N^*	$J_N(q_N^*)$
400	4	$(0.010040, 0.999992, 0.100026)$	0.000000
a	b	c	
-0.893024×10^{-7}	0.416599×10^{-7}	-0.517080×10^{-7}	

All the parameters a, b and c can be regarded as zeros for the error bound 10^{-6}. Based on the results shown in Tables 2 and 3, one can guess that the assumptions on the parameters a, b, c specified in Corollary 3.1 for finding q^* may be not suitable in these cases.

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