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$p(x)$-harmonic functions with isolated singularities

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Introduction

Let Ω be a bounded open set in $\mathbf{R}^N$ $(N \geq 2)$ and let $1 < p \leq N$. Given $a \in \Omega$, $\alpha \in \mathbf{R}$ and $\theta \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, consider the boundary value problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \alpha\delta_a & \text{in } \Omega, \\ u = \theta & \text{on } \partial\Omega. \end{cases} \tag{0.1}$$

In [KV], it is shown that there exists a unique solution u of (0.1) such that $u \in W^{1,p}(\Omega \setminus B(a,R)) \cap C(\Omega \setminus \{a\})$ for small $R > 0$, $|\nabla u|^{p-1} \in L^1(\Omega)$ and

$$u(x) - \alpha^{1/(p-1)}\gamma_p(x-a) \in L^\infty(\Omega),$$

where γ_p is the radial solution of $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \delta_0$. Note that the solution u is p-harmonic in $\Omega \setminus \{a\}$ and $(\operatorname{sgn}\alpha)u$ is p-superharmonic in Ω.

In this paper, we consider a variable exponent $p(x)$ and discuss the boundary value problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \sum_{a\in A}\alpha_a\delta_a & \text{in } \Omega, \\ u = \theta & \text{on } \partial\Omega, \end{cases}$$

where A is a relatively closed isolated set in Ω, $\alpha_a \in \mathbf{R} \setminus \{0\}$ for every $a \in A$ and $\theta \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ (see [KR] for the space $W^{1,p(\cdot)}(\Omega)$). We seek for a solution u which is $p(\cdot)$-harmonic in $\Omega \setminus A$ and $(\operatorname{sgn}\alpha_a)u$ is $p(\cdot)$-superharmonic in a neighborhood of each $a \in A$.

§1. Preliminaries

Throughout this paper, let Ω be a bounded open set in $\mathbf{R}^N$ $(N \geq 2)$. We consider a variable exponent $p(x)$ on Ω such that

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < \infty \tag{1.1}$$

and it is log-Hölder continuous, namely there is a constant $C_p > 0$ such that

$$|p(x) - p(x')| \leq \frac{C_p}{\log(1/|x-x'|)}$$

for x, $x' \in \Omega$ with $|x - x'| \leq 1/2$.

For a set $E \subset \Omega$, let $p_E^+ = \sup_{x\in E} p(x)$ and $p_E^- = \inf_{x\in E} p(x)$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ are defined as in [KR]; in case $p(\cdot)$ satisfies (1.1), we may define

$$L^{p(\cdot)}(\Omega) = \left\{ u \in L^1(\Omega)\,;\ \int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\}$$

and

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega)\,;\ \int_\Omega |\nabla u(x)|^{p(x)}\,dx < \infty \right\}.$$

They are reflexive Banach spaces with respect to the norms

$$\|u\|_{p(\cdot)} = \inf\left\{ \lambda > 0\,;\ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\}$$

on $L^{p(\cdot)}(\Omega)$ and $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ on $W^{1,p(\cdot)}(\Omega)$ (see [KR]). Let $W_0^{1,p(\cdot)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ and let $W_{loc}^{1,p(\cdot)}(\Omega)$ be defined as usual.

Lemma 1.1. *For an open set $G \subset \Omega$, let u be a measurable function on G such that $|u(x)| < \infty$ for a.e. $x \in G$. For $k > 0$, let $T_k(t) = \max(-k, \min(t,k))$, $t \in \mathbf{R}$. If $T_k \circ u \in W_0^{1,p(\cdot)}(G)$ for all $k \ge 1$ and if there exists $M > 0$ independent of $k \ge 1$ such that*

$$\int_G |\nabla(T_k \circ u)|^{p(x)}\,dx \le kM,$$

then

(1) for $r > 0$ such that $r < (p_G^- - 1)N/(N - p_G^-)$ in case $p_G^- < N$ there is a constant $C_0 = C(N, p_G^-, r, G, M) > 0$ (independent of u) such that $\int_G |u|^r\,dx \le C_0$,

(2) for $0 < q < \min\big(p_G^-,\ (p_G^- - 1)N/(N-1)\big)$ there is a constant $C_1 = C(N, p_G^-, q, G, M) > 0$ (independent of u) such that $\int_G |Du|^q\,dx \le C_1$, where, $Du = \lim_{k\to\infty} \nabla(T_k \circ u)$.

Proof. Let $u^+ = \max(u,0)$ and $u^- = -\min(u,0)$. Then $\min(u^\pm, k) \in W_0^{1,p(\cdot)}(G) \subset W_0^{1,p_G^-}(G)$ for $k \ge 1$ and

$$\begin{aligned}\int_G |\nabla \min(u^\pm,\, k)|^{p_G^-}\,dx &\le |G| + \int_G |\nabla \min(u^\pm,\, k)|^{p(x)}\,dx \\ &\le |G| + \int_G |\nabla(T_k \circ u)|^{p(x)}\,dx \le k(|G| + M).\end{aligned}$$

Hence the lemma follows from [HKM; Lemma 7.43].

The $p(\cdot)$-Laplacian $\Delta_{p(\cdot)}$ is given by

$$\Delta_{p(\cdot)} u = \operatorname{div}\big(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u\big).$$

u is called a (weak) solution of $\Delta_{p(\cdot)} u = 0$ in an open set $G \subset \Omega$ if $u \in W_{loc}^{1,p(\cdot)}(G)$ and

$$\int_G p(x)|\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot \nabla\varphi(x)\,dx = 0 \tag{1.2}$$

for all $\varphi \in C_0^\infty(G)$; u is called a supersolution of $\Delta_{p(\cdot)}u = 0$ in $G \subset \Omega$ if $u \in W_{loc}^{1,p(\cdot)}(G)$ and

$$\int_G p(x)|\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot \nabla\varphi(x)\,dx \geq 0 \tag{1.3}$$

for all nonnegative $\varphi \in C_0^\infty(G)$. We may take $\varphi \in W_0^{1,p(\cdot)}(G)$ in (1.2) and (1.3) if $u \in W^{1,p(\cdot)}(G)$.

The following proposition can be shown as in the case of constant exponent (cf. [M; Theorem 2.2], [HKM: Lemma 3.18]: also cf. [HKHLM; Lemma 4] for the case of variable exponent).

Proposition 1.1 (Comparison principle) *Let $u_1,\ u_2 \in W^{1,p(\cdot)}(G)$. If*

$$\int_G p(x)|\nabla u_1|^{p(x)-2}\nabla u_1 \cdot \nabla\varphi\,dx \leq \int_G p(x)|\nabla u_2|^{p(x)-2}\nabla u_2 \cdot \nabla\varphi\,dx$$

for all nonnegative $\varphi \in C_0^\infty(G)$ and $\max(u_1 - u_2,\ 0) \in W_0^{1,p(\cdot)}(G)$, then $u_1 \leq u_2$ a.e. in G.

Corollary 1.1. *If $u \in W^{1,p(\cdot)}(G)$ is a supersolution of $\Delta_{p(\cdot)}u = 0$ in G and if $\min(u - a,\ 0) \in W_0^{1,p(\cdot)}(G)$ for a constant a, then $u \geq a$ a.e. in G.*

It is known (cf. [A]) that every solution of $\Delta_{p(\cdot)}u = 0$ has a locally Hölder continuous representative under our assumptions. A continuous solution of $\Delta_{p(\cdot)}u = 0$ in G is called *$p(\cdot)$-harmonic* in G.

A Harnack inequality for $p(\cdot)$-harmonic functions holds in the following form ([HKL; Theorem 3.17]):

Lemma 1.2. *Given $s > 0$ and $M > 0$, there exists a constant $C > 0$ depending only on N, p^+, p^-, C_p, s and M such that*

$$\sup_{B(x,R)} u \leq C\big(\inf_{B(x,R)} u + R\big)$$

for every $B(x,R)$ such that $B(x,4R) \subset \Omega$ and $p^+_{B(x,4R)} - p^-_{B(x,4R)} < s/N$ and for every nonnegative $p(\cdot)$-harmonic function u on $B(x,4R)$ with $\int_{B(x,4R)} u^s\,dx \leq M$.

Using this Harnack inequality, we obtain (cf. the proof of [HKHLN; Theorem 16] as well as the proof of [S; Theorem 8])

Lemma 1.3. *Let $\mathcal{U}$ be a family of non-negative $p(\cdot)$-harmonic functions in an open set $G \subset \Omega$. If there exists $s > 0$ such that*

$$\left\{\int_V u^s(x)\,dx\right\}_{u\in\mathcal{U}}$$

is bounded for every $V \Subset G$, then $\mathcal{U}$ is locally uniformly bounded and locally equicontinuous in G.

Lemma 1.4. *A locally uniformly bounded sequence of $p(\cdot)$-harmonic functions has a subsequence which converges locally uniformly to a $p(\cdot)$-harmonic function.*

Proof. Let $\{u_n\}$ be a locally uniformly bounded sequence of $p(\cdot)$-harmonic functions in an open set $G \subset \Omega$. Then, by the above lemma, we see that $\{u_n\}$ is locally uniformly bounded and locally equi-continuous on G. Thus, by Ascoli-Arzera's theorem, it has a locally uniformly convergent subsequence. By [HKHLN; Corollary 13], the limit function is also $p(\cdot)$-harmonic in G.

Lemma 1.5. *Let $\{u_n\}$ be a locally uniformly convergent sequence of $p(\cdot)$-harmonic functions in an open set $G \subset \Omega$ and let u be the limit function. Then there exists a subsequence $\{u_{n_j}\}$ such that $\nabla u_{n_j} \to \nabla u$ a,e, in G.*

Outline of the Proof. Let $V \Subset G$ and choose $\eta \in C_0^\infty(G)$ such that $\eta = 1$ on V and $0 \le \eta \le 1$ in G. Then

$$\int_G p(x)|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(u_n \eta^{p^+})\,dx = 0.$$

From this equality, using Young's inequality and the uniform boundedness of $\{u_n\}$, we deduce that $\{\int_V |\nabla u_n(x)|^{p(x)}\,dx\}_n$ is bounded.

Next, from the equalities

$$\int_G p(x)|\nabla u_n|^{p(x)-2}[\nabla u_n \cdot \nabla[(u_n - u))\eta]\,dx = 0$$

and

$$\int_G p(x)|\nabla u|^{p(x)-2}[\nabla u \cdot \nabla[(u_n - u))\eta]\,dx = 0$$

we have

$$\begin{aligned} 0 &\le \int_V p(x)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u) \cdot (\nabla u_n - \nabla u)\,dx \\ &\le p^+\big(\sup_{\mathrm{spt}(\eta)} |u_n - u|\big)(\sup|\nabla\eta|) \int_{\mathrm{spt}(\eta)} (|\nabla u_n|^{p(x)-1} + |\nabla u|^{p(x)-1})\,dx \\ &\to 0 \quad (n \to \infty). \end{aligned}$$

This implies that $\nabla u_{n_j} \to \nabla u$ a.e. in V for some subsequence $\{u_{n_j}\}$. Since this is true for every $V \Subset G$, we obtain the assertion of the lemma.

A $(-\infty, \infty]$-valued function u on G is called $p(\cdot)$-*superharmonic* in G if it is lower semicontinuous, finite a.e. and the following comparison principle holds: if $V \Subset G$ is an open set, $h \in C(\overline{V})$ is $p(\cdot)$-harmonic in V and $h \le u$ on ∂V, then $h \le u$ in V.

The following results are known (see [HKHLM]):

(S1) Every supersolution of $\Delta_{p(\cdot)}u = 0$ has a $p(\cdot)$-superharmonic representative;

(S2) Every locally bounded $p(\cdot)$-superharmonic function is a supersolution of $\Delta_{p(\cdot)}u = 0$.

Also the following properties of $p(\cdot)$-superharmonic functions are easy consequences of the definition as in the case of constant exponent (cf. [HKM; Chap.7]):

(S3) If $\{u_n\}$ is a nondecreasing sequence of $p(\cdot)$-superharmonic functions in G and if $u = \lim_{n\to\infty} u_n$ is finite a.e., then u is $p(\cdot)$-superharmonic in G;

(S4) If $\mathcal{U}$ is a family of $p(\cdot)$-superharmonic functions in G and if it is locally uniformly bounded from below, then the lower semicontinuous regularization of $\inf \mathcal{U}$ is $p(\cdot)$-superharmonic in G.

Proposition 1.2. (cf. [HKHLM; Theorem 25]) *Let u be a $p(\cdot)$-superharmonic function in $G \subset \Omega$ such that $\min(u - \theta, k) \in W_0^{1,p(\cdot)}(G)$ for all $k > 0$ with some $\theta \in W^{1,p(\cdot)}(G) \cap L^\infty(G)$. Let $Du = \lim_{k\to\infty} \nabla \min(u - \theta, k) + \nabla\theta$. Then $u \in L^r(G)$ for $0 < r < (p_G^- - 1)N/(N - p_G^-)$ in case $p_G^- < N$; for any $r > 0$ in case $p_G^- \geq N$ and $|Du| \in L^q(G)$ for $0 < q < \min(p_G^-, (p_G^- - 1)N/(N-1))$.*

Outline of the Proof. By using (S2) and Corollary 1,1, we see that $u \geq \inf_G \theta$. For $k \in \mathbf{N}$, set $E_k = \{x \in G\,;\, k - 1 \leq u(x) - \theta(x) < k\}$ and $F_k = \bigcup_{j=1}^k E_j$. Let

$$w_k = 2\min(u - \theta, k) - \min(u - \theta, k - 1) - \min(u - \theta, k + 1).$$

Then $w_k \in W_0^{1,p(\cdot)}(G)$ and $w_k \geq 0$. Let $k' \geq \max(k - m, 0) + 1$. Since $\min(u, k')$ is a supersolution of $\Delta_{p(\cdot)} u = 0$, we have

$$\begin{aligned} 0 &\leq \int_G p(x)|\nabla \min(u,k')|^{p(x)-2}(\nabla \min(u,k') \cdot \nabla w_k)\,dx \\ &= \int_{E_k} p(x)|Du|^{p(x)-2} Du \cdot (Du - \nabla\theta)\,dx - \int_{E_{k+1}} p(x)|Du|^{p(x)-2}Du \cdot (Du - \nabla\theta)\,dx. \end{aligned}$$

Hence $\left\{\int_{E_k} p(x)|Du|^{p(x)-2}Du \cdot (Du - \nabla\theta)\,dx\right\}_k$ is nonincreasing. Therefore

$$\int_{F_k} p(x)|Du|^{p(x)-2}Du\cdot(Du - \nabla\theta)\,dx \leq k\int_{E_1} p(x)|Du|^{p(x)-2}Du \cdot (Du - \nabla\theta)\,dx.$$

Using Young's inequality, we obtain

$$\begin{aligned} &\int_{F_k} p(x)|Du|^{p(x)}\,dx \\ &\quad \leq 2^{p^+}(1+k)\int_{F_k} p(x)|\nabla\theta|^{p(x)}\,dx + 2^{p^+ + 1}k\int_{E_1} p(x)|Du - \nabla\theta|^{p(x)}\,dx. \end{aligned}$$

Thus, if $k > |m|$ then

$$\begin{aligned} &\int_G |\nabla T_k \circ (u - \theta)|^{p(x)}\,dx \\ &= \int_G |\nabla \min(u - \theta, 0)|^{p(x)}\,dx + \int_{F_k} |Du - \nabla\theta|^{p(x)}\,dx \\ &\leq \int_G |\nabla \min(u-\theta,0)|^{p(x)}\,dx + 2^{p^+-1}\int_{F_k} p(x)|Du|^{p(x)}\,dx + 2^{p^+-1}\int_{F_k} p(x)|\nabla\theta|^{p(x)}\,dx \\ &\leq 2^{2p^+}p^+k\int_{F_k} |\nabla\theta|^{p(x)}\,dx + \int_G |\nabla \min(u-\theta,0)|^{p(x)}\,dx \\ &\quad + 2^{2p^+}k\int_{E_1} p(x)|Du - \nabla\theta|^{p(x)}\,dx \\ &\leq 2^{2p^+}p^+k\left\{\int_G |\nabla\theta|^{p(x)}\,dx + \int_G |\nabla \min(u - \theta, 1)|^{p(x)}\,dx\right\}. \end{aligned}$$

Hence applying Lemma 1.1 to $u-\theta$, we have

$$u-\theta \in L^r(G) \quad \text{and} \quad |Du-\nabla\theta| \in L^q(G)$$

with r and q as in the lemma. Since $\theta \in W^{1,p(\cdot)}(G)\cap L^\infty(G)$, we obtain the assertion of the proposition.

§2. $p(\cdot)$-harmonic functions with isolated singular points.

Lemma 2.1 (cf. [HKHLM; Theorem 26]). *Let $a\in\Omega$ and let V be an open neighborhood of a. If u is $p(\cdot)$-superharmonic in V and is $p(\cdot)$-harmonic in $V\setminus\{a\}$, then*

(1) *$u\in L^r_{loc}(V)$ for $0<r<(p(a)-1)N/(N-p(a))$ in case $p(a)<N$ and for any $r>0$ in case $p(a)\geq N$;*

(2) *$|\nabla u|\in L^q(U)$ for some neighborhood U of a, where*

$$0<q<\min\bigl(p(a),\,(p(a)-1)N/(N-1)\bigr).$$

Proof. Given $r>0$ and $q>0$ as in the lemma, choose a ball $B=B(a,R)\Subset V$ which satisfies the following conditions:

(a) In case $p(a)<N$ or $p(a)=N$ and $p_U^-<N$ for any neighborhood U of a, $r<(p_B^- -1)N/(N-p_B^-)$ and $q<(p_B^- -1)N/(N-1)$;

(b) In case $p_U^-\geq N$ for some neighborhood U of a, $p_B^-\geq N$ and $q<p_B^-$.

Choose $\psi\in C_0^\infty(B)$ which is equal to 1 on $B(a,R/2)$. Then we see that $(1-\psi)u\in W^{1,p(\cdot)}(B)\cap L^\infty(B)$ and $\min(\psi u,\,k)\in W_0^{1,p(\cdot)}(B)$ for $k>0$. Hence, by Proposition 1.2, $u\in L^r(B)$ and $|\nabla u|\in L^q(B)$. Since u is locally bounded on $V\setminus\{a\}$, it follows that $u\in L^r_{loc}(V)$.

Proposition 2.1. (cf. [L; Theorem 4.6]) *Let $a\in\Omega$ and let V be an open neighborhood of a. If u is $p(\cdot)$-superharmonic in V and is $p(\cdot)$-harmonic in $V\setminus\{a\}$, then*

$$|\nabla u|^{p(x)-1}\in L^s_{loc}(V) \quad \text{for} \quad 1\leq s<\min\bigl(N/(N-1),\,p^+/(p^+-1)\bigr)$$

and there exists $\alpha\geq 0$ such that $-\Delta_{p(\cdot)}u=\alpha\delta_a$ in V, namely,

$$\int_V p(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,dx=\alpha\varphi(a)$$

for all $\varphi\in C_0^\infty(V)$.

Proof. Let $1\leq s<\min\bigl(N/(N-1),\,p^+/(p^+-1)\bigr)$. Since $p(a)/(p(a)-1)\geq p^+/(p^+-1)$, in (2) of the above lemma, taking smaller U if necessary, we may assume $s(p_U^+-1)<\min\bigl(p(a)-1)N/(N-1),\,p(a)\bigr)$. Then we can take $q=s(p_U^+-1)$, so that $|\nabla u|^{p(x)-1}\in L^s(U)$. Since $|\nabla u|\in L^{p(\cdot)}_{loc}(V\setminus\{a\})$ and $s<p^+/(p^+-1)\leq p(x)/(p(x)-1)$, it follows that $|\nabla u|^{p(x)-1}\in L^s_{loc}(V)$.

Since $\min(u,\,k)$ is a supersolution of $\Delta_{p(\cdot)}u=0$ for $k>0$, using Lebesgue's convergence theorem we obtain

$$\int_V p(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,dx$$
$$=\lim_{k\to\infty}\int_V p(x)|\nabla\min(u,\,k)|^{p(x)-2}\nabla\min(u,\,k)\cdot\nabla\varphi\,dx\geq 0$$

for all nonnegative $\varphi \in C_0^\infty(V)$. Therefore there exists a nonnegative measure μ on V such that

$$\int_V p(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,dx = \int_V \varphi\,d\mu$$

for all $\varphi \in C_0^\infty(V)$. Since u is $p(\cdot)$-harmonic in $V\setminus\{a\}$, $\operatorname{spt}(\mu)\subset\{a\}$, namely $\mu=\alpha\delta_a$ for some $\alpha\ge 0$.

Combining the above results, we can state

Theorem 2.1. *Let A be a relatively closed isolated set in Ω. If u is a $[-\infty,\infty]$-valued function such that*

(1) u is $p(\cdot)$-harmonic in $\Omega\setminus A$;

(2) for each $a\in A$ there is an open neighborhood V_a in which u is either $p(\cdot)$-superharmonic or $p(\cdot)$-subharmonic (i.e., $-u$ is $p(\cdot)$-superharmonic).

Then $u\in L^r_{loc}(\Omega)$ for $0<r<(p^- -1)N/(N-p^-)$ (any $r>0$ in case $p^-\ge N$), $|\nabla u|^{p(x)-1}\in L^s_{loc}(\Omega)$ for $1\le s<\min(N/(N-1),\, p^+/(p^+-1))$ and $-\Delta_{p(\cdot)}u=\sum_{a\in A}\alpha_a\delta_a$ in Ω, namely

$$\int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,dx = \sum_{a\in A}\alpha_a\varphi(a)$$

for all $\varphi\in C_0^\infty(\Omega)$ with $\alpha_a\in\mathbf{R}$ such that $\alpha_a\ge 0$ if u is $p(\cdot)$-superharmonic in V_a and $\alpha_a\le 0$ if u is $p(\cdot)$-subharmonic in V_a.

Lemma 2.2. *Let $a\in\Omega$ and $B=B(a,R)\subset\Omega$ with $0<R\le 1/2$. If $p(a)\le N$, then there exists a sequence $\{\eta_n\}$ of (Lipschitz continuous) functions in $W_0^{1,p(\cdot)}(B)$ such that $0\le\eta_n\le 1$ on B, $\eta_n=1$ in a neighborhood of a, $\eta_n(x)\to 0$ for all $x\in B\setminus\{a\}$ and*

$$\int_B |\nabla\eta_n|^{p(x)}\,dx\to 0 \quad \text{as } n\to\infty.$$

(This means that the $p(\cdot)$-capacity of $\{a\}$ is zero (cf. [HHKV]).)

Outline of the Proof. Fixing $0<\rho<R$, let

$$\eta_n(x)=\begin{cases} 0 & \text{for } \rho\le|x-a|<R \\ \dfrac{\log(\rho/|x-a|)}{\log n+1} & \text{for } \rho/(en)\le|x-a|<\rho \\ 1 & \text{for } |x-a|\le\rho/(en). \end{cases}$$

Then, using log-Hölder continuity of $p(x)$, elementary computation shows that $\{\eta_n\}$ has the required properties.

Proposition 2.2. (cf. [L; Theorem 4.7]) *Let $a\in\Omega$, V be an open neighborhood of a and let u be a $p(\cdot)$-superharmonic function in V which is $p(\cdot)$-harmonic in $V\setminus\{a\}$.*

(1) If $p(a)\le N$, then $\lim_{x\to a}u(x)=\infty$ unless a is removable for u (i.e., $\alpha=0$ in Proposition 2.1).

(2) If $p(a)>N$, then u is (finite) continuous at a.

Outline of the Proof. (1) Let $p(a)\le N$ and suppose a is not removable for u. We first show that u is unbounded near a. Assume u is bounded near a. Then u is a supersolution of $\Delta_{p(\cdot)}u = 0$ in V, in particular $u \in W^{1,p(\cdot)}_{loc}(V)$. Let $\varphi \in C_0^\infty(V)$ and let $\{\eta_n\}$ be as in Lemma 2.2 with $B = B(a,R) \subset V$. Then $\varphi(1-\eta_n) \in W_0^{1,p(\cdot)}(V \setminus \{a\})$. Since u is $p(\cdot)$-harmonic in $V \setminus \{a\}$,

$$\int_V p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla[\varphi(1-\eta_n)]\,dx = 0.$$

Hence

$$\int_V p(x)|\nabla u|^{p(x)-2}(\nabla u \cdot \nabla\varphi)(1-\eta_n)\,dx = \int_V p(x)|\nabla u|^{p(x)-2}(\nabla u \cdot \nabla\eta_n)\varphi\,dx. \tag{2.1}$$

The left hand side of (2.1) tends to $\int_V p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\,dx$ as $n \to \infty$ by Lebesgue's convergence theorem, while the right hand side of (2.1) tends to 0, since $\int_V |\nabla\eta_n|^{p(x)}\,dx \to 0$. This shows that u is a solution of $\Delta_{p(\cdot)}u = 0$ in V, so that a is removable for u.

Thus, u is unbounded near a, so that there exists x_j, $j = 1, 2, \ldots$ ($x_j \neq a$) such that $x_j \to a$ and $u(x_j) \to \infty$ as $j \to \infty$. Let $\rho_j = |x_j - a|$. By Lemma 2.1 (1), there exists $r > 0$ such that $u \in L^r_{loc}(V)$. Choose $R > 0$ such that $B = B(a,R) \Subset V$ and $p_B^+ - p_B^- < r/N$. We could take x_j so that $\rho_j < R/2$ and $\{\rho_j\}$ is strictly decreasing. Set $m = \inf_{\partial B} u$. Then, $u - m \geq 0$ in B. Applying the Harnack inequality in Lemma 1.2 to $u - m$ on $B(\xi, \rho_j)$ with $\xi \in \partial B(a, \rho_j)$, we see that $k_j := \inf_{\partial B(a,\rho_j)}(u - m) \to \infty$ ($j \to \infty$). Since $u \geq \min(k_j, k_{j+1}) + m$ on $B(a,\rho_j) \setminus B(a, \rho_{j+1})$ by the comparison principle, it follows that $\lim_{x \to a} u(x) = \infty$.

(2) If $p(a) > N$, then by Lemma 2.1 (2), $|\nabla u| \in L^q(U)$ for a neighborhood U of a and $q > N$. Hence by the Sobolev imbedding theorem, u has a continuous representative. Since u is $p(\cdot)$-superharmonic in V, it follows that u is continuous at a.

§3. An existence result

In this section, we prove the following existence theorem:

Theorem 3.1. *Let A be a relatively closed isolated set in Ω. To each $a \in A$ we assign a value $\alpha_a \neq 0$ such that $\sum_{a \in A} |\alpha_a| < \infty$. Let $\theta \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be given. Then there exists a function $u : \Omega \to [-\infty, \infty]$ such that*

(1) *u is $p(\cdot)$-harmonic in $\Omega \setminus A$,*

(2) *u is $p(\cdot)$-superharmonic in a neighborhood of each $a \in A$ with $\alpha_a > 0$ and $p(\cdot)$-subharmonic in a neighborhood of each $a \in A$ with $\alpha_a < 0$,*

(3) $-\Delta_{p(\cdot)}u = \sum_{a \in A} \alpha_a \delta_a$ *in Ω,*

(4) $T_k \circ (u - \theta) \in W_0^{1,p(\cdot)}(\Omega)$ *for every $k > 0$.*

If, in particular, A is a finite set, then we can take u to satisfy the following:

(5) *u is bounded on $\Omega \setminus V$ for any neighborhood V of $A^* = \{a \in A;\, p(a) \leq N\}$.*

(6) *for any $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ in a neighborhood of A^*, $(1-\psi)(u-\theta) \in W_0^{1,p(\cdot)}(\Omega)$,*

To prove this theorem, we need some preparations. First, we note that the following propositon can be shown in a standard way using the theory of monotone operators (cf. [FZ; Theorem 3.1]).

Proposition 3.1. *Let $\theta \in W^{1,p(\cdot)}(\Omega)$ and $\mu \in (W_0^{1,p(\cdot)}(\Omega))^*$ be given. Then there exists a unique $u \in W^{1,p(\cdot)}(\Omega)$ such that $u - \theta \in W_0^{1,p(\cdot)}(\Omega)$ and $-\Delta_{p(\cdot)}u = \mu$ in Ω, namely*

$$\int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \, dx = \mu(v) \tag{3.1}$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$.

Note that the Dirac measure $\delta_a \in (W_0^{1,p(\cdot)}(\Omega))^*$ if and only if $p(a) > N$. In fact, Lemma 2.2 shows that $\delta_a \notin (W_0^{1,p(\cdot)}(\Omega))^*$ if $p(a) \le N$; the Sobolev imbedding theorem implies that $\delta_a \in (W_0^{1,p(\cdot)}(\Omega))^*$ if $p(a) > N$.

Lemma 3.1. *Let μ be a finite signed measure on Ω such that $|\mu| \in (W_0^{1,p(\cdot)}(\Omega))^*$ and let $\theta \in W^{1,p(\cdot)}(\Omega)$. If $u \in W^{1,p(\cdot)}(\Omega)$ is a solution of $-\Delta_{p(\cdot)}u = \mu$ such that $u-\theta \in W_0^{1,p(\cdot)}(\Omega)$, then*

$$\int_{\{l \le |u-\theta| < k\}} |\nabla u|^{p(x)} \, dx \le \int_\Omega |\nabla \theta|^{p(x)} \, dx + (k-l)|\mu|(\Omega) \tag{3.2}$$

for $0 \le l < k$.

Proof. Let $S(t) = T_{k-l}(t - T_l(t))$ and set $v = S \circ (u - \theta)$. Then $v \in W_0^{1,p(\cdot)}(\Omega)$. Hence (3.1) holds with this v. Note that $\nabla v = (\nabla u - \nabla \theta)\chi_{\{l \le |u-\theta| < k\}}$. Since μ is a finite signed measure and $|v| \le k - l$, it follows that

$$\int_{\{l \le |u-\theta| < k\}} p(x)|\nabla u|^{p(x)} \, dx \le \int_{\{l \le |u-\theta| < k\}} p(x)|\nabla u|^{p(x)-1}|\nabla \theta| \, dx + (k-l)|\mu|(\Omega).$$

Using Young's inequality, we obtain (3.2).

Corollary 3.1. *Let μ, θ and u be as in Lemma 3.1. Then*

$$\int_\Omega |\nabla[T_k \circ (u-\theta)]|^{p(x)} \, dx \le 2^{p^+} \int_\Omega |\nabla \theta|^{p(x)} \, dx + 2^{p^+-1}k|\mu|(\Omega)$$

for $k > 0$.

Outline of the Proof of Theorem 3.1. Set $A_+ = \{a \in A\,;\, \alpha_a > 0\}$ and $A_- = \{a \in A\,;\, \alpha_a < 0\}$. For each $a \in A^*$, choose $B_a = B(a, R_a) \Subset \Omega$ $(0 < R_a < 1)$ in such a way that $\overline{B_a} \cap \overline{B_{a'}} = \emptyset$ if $a \ne a'$ $(a,\ a' \in A^*)$ and $B_a \cap (A \setminus A^*) = \emptyset$. Let $\{\Omega_n\}$ be an exhaustion of Ω (i.e., a sequence of open sets such that $\Omega_n \Subset \Omega_{n+1} \Subset \Omega$ for all n and $\bigcup_n \Omega_n = \Omega$). Fix $\eta \in C_0^\infty(\mathbf{R}^N)$ such that $\eta \ge 0$, $\operatorname{spt}(\eta) \subset B(0,1)$ and $\int \eta(x)\, dx = 1$. For $n = 1, 2 \ldots,$ let

$$\mu_n^{(+)} = \sum_{a \in A_+ \cap A^* \cap \Omega_n} \alpha_a \left(\frac{2^n}{R_a}\right)^N \eta\left(\frac{2^n(x-a)}{R_a}\right) dx + \sum_{b \in (A_+ \setminus A^*) \cap \Omega_n} \alpha_b \delta_b,$$

$$\mu_n^{(-)} = \sum_{a' \in A_- \cap A^* \cap \Omega_n} |\alpha_{a'}| \left(\frac{2^n}{R_{a'}}\right)^N \eta\left(\frac{2^n(x-a')}{R_{a'}}\right) dx + \sum_{b' \in (A_- \setminus A^*) \cap \Omega_n} |\alpha_{b'}| \delta_{b'}$$

and $\mu_n = \mu_n^{(+)} - \mu_n^{(-)}$. Then, $\mu_n^{(+)}$ and $\mu_n^{(-)}$ are nonnegative measures and

$$\mu_n^{(+)}(\Omega) \le \sum_{a \in A_+} \alpha_a, \quad \mu_n^{(-)}(\Omega) \le \sum_{a' \in A_-} |\alpha_{a'}|, \quad |\mu_n|(\Omega) \le \sum_{a \in A} |\alpha_a| \tag{3.3}$$

for all n. Since $A \cap \Omega_n$ is a finite set and $\delta_b \in (W_0^{1,p(\cdot)}(\Omega))^*$ for $b \in A \setminus A^*$, all $\mu_n^{(+)}$, $\mu_n^{(-)}$, μ_n belong to $(W_0^{1,p(\cdot)}(\Omega))^*$. Let $u_n^{(+)}$ (resp. $u_n^{(-)}$) be the solution of $-\Delta_{p(\cdot)}u = \mu_n^{(+)}$ (resp. $= \mu_n^{(-)}$) with $u_n^{(\pm)} \in W_0^{1,p(\cdot)}(\Omega)$, and given $\theta \in W^{1,p(\cdot)}(\Omega)$ let u_n be the solutions of $-\Delta_{p(\cdot)}u = \mu_n$ with $u_n - \theta \in W_0^{1,p(\cdot)}(\Omega)$. Existence of such functions are assured by Proposition 3.1. Further, we can take $u_n^{(\pm)}$ to be $p(\cdot)$-superharmonic in Ω and $p(\cdot)$-harmonic in $\Omega \setminus K_n^{(\pm)}$, where

$$K_n^{(\pm)} = \bigcup_{a \in A_\pm \cap A^* \cap \Omega_n} \overline{B(a, R_a/2^n)} \cup (A_\pm \setminus A^*).$$

Also, we can take u_n to be $p(\cdot)$-harmonic in $\Omega \setminus (K_n^{(+)} \cup K_n^{(-)})$ and $p(\cdot)$-superharmonic in a neighborhood of each $a \in A_+ \cap \Omega_n$ and $p(\cdot)$-subharmonic in a neighborhood of each $a' \in A_- \cap \Omega_n$.

By the comparison principle, $u_n^{(\pm)} \geq 0$ and

$$-u_n^{(-)} - \|\theta\|_\infty \leq u_n \leq u_n^{(+)} + \|\theta\|_\infty. \tag{3.4}$$

By Lemma 3.1, (3.3) and Lemma 1.1 (1), we see that $\{\int_\Omega (u_n^{(\pm)})^r \, dx\}_n$ are bounded for some $r > 0$. Hence, by Lemma 1.3, $\{u_n^{(\pm)}\}_{n \geq n_0}$ are locally uniformly bounded in $\Omega \setminus K_{n_0}^{(\pm)}$. In view of (3.4), we also see that $\{u_n\}_{n \geq n_0}$ is locally uniformly bounded in $\Omega \setminus (K_{n_0}^{(+)} \cup K_{n_0}^{(-)})$. Hence by Lemma 1.4, there exists a subsequence $\{u_{n_j}\}$ which locally uniformly converges to a $p(\cdot)$-harmonic function u on $\Omega \setminus A$. By Lemma 1.5, we may assume that $\nabla u_{n_j} \to \nabla u$ a.e. in $\Omega \setminus A$. Further, by using Proposition 1.1, we see that u_{n_j} is uniformly convergent in a neighborhood of each $a \in A \setminus A^*$, so that u is also defined on $A \setminus A^*$ and u is $p(\cdot)$-superharmonic (resp. $p(\cdot)$-subharmonic) in a neighborhood of each $a \in A_+ \setminus A^*$ (resp. $a \in A_- \setminus A^*$).

Let $a \in A_+ \cap A^*$. Since u_n is $p(\cdot)$-superharmonic in B_a, $w_l = \left(\inf_{j \geq l} u_{n_j}\right)^\wedge$ is $p(\cdot)$-superharmonic in B_a by (S4), and hence $w = \lim_{l \to \infty} w_l$ is $p(\cdot)$-superharmonic in B_a by (S3). Since $w = u$ on $B_a \setminus \{a\}$, if we define $u(a) = w(a)$, then u is $p(\cdot)$-superharmonic in B_a. Similarly, for $a \in A_- \cap A^*$, if we define $u(a) = -\lim_{l \to \infty} \left(\inf_{j \geq l}(-u_{n_j})\right)^\wedge(a)$, then u is $p(\cdot)$-subharmonic in B_a. Thus we have obtained a function u on Ω which satisfies (1) and (2) of the theorem.

To prove (3), let $\varphi \in C_0^\infty(\Omega)$. Choose an open set $G \Subset \Omega$ such that $\mathrm{spt}(\varphi) \subset G$. Choosing smaller R_a if necessary, we may assume

$$p_{B_a}^+ - 1 < \frac{N}{N-1}(p_{B_a}^- - 1) \tag{3.5}$$

for each $a \in A^*$. Let $K^* = \bigcup_{a \in A^*} \overline{B(a, R_a/2)}$. As we have seen above, $\{u_{n_j}\}$ is uniformly bounded on $G \setminus K^*$. Then, by Lemma 3.1, we see that $\left\{\int_{G \setminus K^*} |\nabla u_{n_j}|^{p(x)} \, dx\right\}_j$ is bounded. Therefore $\{|\nabla u_{n_j}|^{p(x)-1}\}$ is a bounded sequence in $L^s(G \setminus K^*)$ for $1 < s < p^+/(p^+ - 1)$.

For a fixed $a \in A^*$ choose $\psi_a \in C_0^\infty(B_a)$ such that $\psi_a = 1$ on $B(a, R_a/2)$ and $0 \leq \psi \leq 1$ on B_a. Consider $\gamma_j = u_{n_j}(1 - \psi_a)$ on B_a. Then $\left\{\int_{B_a} |\nabla \gamma_j|^{p(x)} \, dx\right\}_j$ is bounded by the above result. Since u_{n_j} is a solution of

$$-\Delta_{p(\cdot)}u = \alpha_a \left(\frac{2^{n_j}}{R_a}\right)^N \eta\left(\frac{2^{n_j}(x-a)}{R_a}\right) dx$$

in B_a with $u_{n_j} - \gamma_j \in W_0^{1,p(\cdot)}(B_a)$, by Corollary 3.1 and Lemma 1.1 (2),

$$\left\{ \int_{B_a} |\nabla u_{n_j} - \nabla \gamma_j|^q \, dx \right\}_j$$

is bounded for $0 < q < \min\big(p^-_{B_a}, (p^-_{B_a} - 1)N/(N-1)\big)$. Thus $\left\{ \int_{B_a} |\nabla u_{u_j}|^q \, dx \right\}_j$ is bounded for such q. By (3.5), we can take $q > p^+_{B_a} - 1$. Thus there is $s > 1$ such that $s(p(x)-1) \le q$ on B_a. Then $\{|\nabla u_{n_j}|^{p(x)-1}\}$ is a bounded sequence in $L^s(B_a)$.

Therefore together with the above result on $G \setminus K^*$, we see that $\{|\nabla u_{n_j}|^{p(x)-1}\}$ is a bounded sequence in $L^s(G)$ for some $s > 1$. Since $\nabla u_{n_j} \to \nabla u$ a.e., it follows that

$$|\nabla u_{n_j}|^{p(x)-2} \nabla u_{n_j} \to |\nabla u|^{p(x)-2} \nabla u$$

weakly in $L^s(G)^N$. Hence

$$\int_\Omega p(x) |\nabla u_{n_j}|^{p(x)-2} \nabla u_{n_j} \cdot \nabla \varphi \, dx \to \int_\Omega p(x) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx$$

as $j \to \infty$. On the other hand $\mu_{n_j}(\varphi) \to \sum_{a \in A} \alpha_a \varphi(a)$ as $j \to \infty$. Hence (3) of the theorem holds.

By Corollary 3.1, we see that $\{T_k \circ (u_{n_j} - \theta)\}$ is a bounded sequence in $W_0^{1,p(\cdot)}(\Omega)$ for $k > 0$ (cf. [KR; Theorem 3.10]). Since $T_k \circ (u_{n_j} - \theta) \to T_k \circ (u - \theta)$ a.e. in Ω, (4) of the theorem follows.

Next, suppose A is a finite set. If V is a neighborhood of A^*, there is n_0 such that $B(a, R_a/2^{n_0}) \subset V$ for all $a \in A^*$. Let V' be an open neighborhood of $A \setminus A^*$ such that $V' \Subset \Omega \setminus A^*$ and set $U = \bigcup_{a \in A^*} B(a, R_a/2^{n_0}) \cup V'$. Then $\{u_n\}_{n \ge n_0}$ is uniformly bounded on ∂U. Since θ is bounbed, by the comparison principle it is uniformly bounded in $\Omega \setminus U$. Since it is uniformly bounded on V' as we have seen above, it is uniformly bounded on $\Omega \setminus V$. Hence (5) of the theorem holds.

Finally to show (6) of the theorem, take $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ in a neighborhood V of A^*. Then, $(1 - \psi)(u_{n_j} - \theta) \in W_0^{1,p(\cdot)}(\Omega)$ for all j. Since $\{u_{n_j}\}$ is uniformly bounded on $\Omega \setminus V$ and $\left\{ \int_{\Omega \setminus V} |\nabla (u_{n_j} - \theta)|^{p(x)} \, dx \right\}_j$ is bounded,

$$\left\{ \int_\Omega |\nabla [(1 - \psi)(u_{n_j} - \theta)]|^{p(x)} \, dx \right\}_j$$

is bounded. Since $(1 - \psi)(u_{n_j} - \theta) \to (1 - \psi)(u - \theta)$ a.e., it follows that $(1 - \psi)(u - \theta) \in W_0^{1,p(\cdot)}(\Omega)$.

Proposition 3.2. *Let A be a finite set in Ω and let $\alpha_a \neq 0$ be assigned to each $a \in A$. Let $\theta \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. If u satisfies* (1), (2), (3) *and* (6) *of Theorem 3.1, then*

$$\int_{\{|u - \theta| < k\}} |\nabla u|^{p(x)} \, dx \le \int_\Omega |\nabla \theta|^{p(x)} \, dx + k \sum_{a \in A} |\alpha_a|$$

for $k > 0$.

Proof. Let $\varphi = T_k \circ (u - \theta)$. Then, by Proposition 2.2, $\varphi = (\operatorname{sgn} \alpha_a) k$ in a neighborhood V_a of $a \in A^*$. We can take V_a so that $V_a \Subset \Omega$, $\{V_a\}_{a \in A^*}$ is mutually disjoint and

$V_a \cap (A \setminus A^*) = \emptyset$. Choose $\psi_a \in C_0^\infty(\Omega)$ such that $0 \le \psi_a \le 1$ on Ω, $\mathrm{spt}(\psi_a) \subset V_a$ and $\psi_a = 1$ in a neighborhood of a for each $a \in A^*$. Set $\psi = \sum_{a\in A^*} \psi_a$. Then $\psi\varphi = \sum_{a\in A^*}(\mathrm{sgn}\,\alpha_a)k\psi_a \in C_0^\infty(\Omega)$. Hence

$$\int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla(\psi\varphi)\,dx = k\sum_{a\in A^*}|\alpha_a|. \tag{3.6}$$

On the other hand, by property (6), we see that $(1-\psi)\varphi \in W_0^{1,p(\cdot)}(\Omega \setminus A^*)$. Since $\sum_{a\in A\setminus A^*}\alpha_a\delta_a \in (W_0^{1,p(\cdot)}(\Omega\setminus A^*))^*$ and u is a solution of $-\Delta_{p(\cdot)}u = \sum_{a\in A\setminus A^*}\alpha_a\delta_a$ in $\Omega\setminus A^*$,

$$\int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla[(1-\psi)\varphi]\,dx = \sum_{a\in A\setminus A^*}\alpha_a\delta_a(\varphi). \tag{3.7}$$

Combining (3.6) and (3.7), and noting that $\nabla\varphi = (\nabla u - \nabla\theta)\chi_{\{|u-\theta|<k\}}$ and $|\delta_a(\varphi)| \le k$, we obtain the required inequality as in the proof of Lemma 3.1.

§4. Uniqueness results

We can show the uniqueness only in rather restricted cases. In this section, we consider only the case A is a *finite set*. As in the previous section, let $\alpha_a \ne 0$ be assigned to each $a \in A$ and $\theta \in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ be given. Also, let $A^* = \{a \in A\,;\, p(a) \le N\}$ as before. We shall use the notation

$$\mathcal{A}_{p(\cdot)}(\xi_1,\xi_2) = p(x)(|\xi_1|^{p(x)-2}\xi_1 - |\xi_2|^{p(x)-2}\xi_2)$$

for $\xi_1,\ \xi_2 \in \mathbf{R}^N$.

The proof of Propositon 3.2 as well as the proof of the next lemma shows that the function u satisfying (1), (2), (3) and (6) of Theorem 3.1 is a "renormalized solution" in the sense of [DMOP] (also cf. [M]). In fact, we follow arguments in [DMOP; 10.2] to obtain our Theorem 4.1 below.

Lemma 4.1. *Suppose u_1 and u_2 both satisfy* (1), (2), (3) *and* (6) *in Theorem 3.1. For $n > 0$, set $E_n = \{|u_1-\theta| < n\}\cap\{|u_2-\theta|<n\}$. Then*

$$\int_{\{|u_1-u_2|<k\}}\mathcal{A}_{p(\cdot)}(\nabla u_1,\nabla u_2)\cdot(\nabla u_1 - \nabla u_2)\,dx$$
$$\le 2k\liminf_{n\to\infty}\frac{1}{n}\int_{E_n}|\mathcal{A}_{p(\cdot)}(\nabla u_1,\nabla u_2)|\big(|\nabla u_1| + |\nabla u_2| + 2|\nabla\theta|\big)\,dx$$

for $k > 0$.

Proof. For simplicity, let $v_j = u_j - \theta$, $j = 1, 2$. For $n > 0$, let

$$h_n(t) = \max\big(0,\ \min(1,\ 2-2|t|/n)\big)$$

and set $\varphi_n = \big(T_k\circ(u_1-u_2)\big)(h_n\circ v_1)(h_n\circ v_2)$. Since $h_n(t) = 0$ for $|t| \ge n$, $h_n \circ v_j = 0$ in a neighborhood of A^* by Proposition 2.2. Hence $\varphi_n = 0$ in a neighborhood of A^* and

$\varphi_n \in W^{1,p(\cdot)}_{loc}(\Omega)$. Since $|\varphi_n| \le k$, $\varphi_n \in L^{p(\cdot)}(\Omega)$. We have

$$\begin{aligned}\nabla\varphi_n &= (\nabla u_1 - \nabla u_2)\chi_{\{|u_1-u_2|<k\}}(h_n \circ v_1)(h_n \circ v_2)\\ &\quad + \frac{2}{n}\nabla v_1\big(\chi_{\{-n<v_1<-n/2\}} - \chi_{\{n/2<v_1<n\}}\big)(h_n \circ v_2)\big(T_k \circ (u_1-u_2)\big)\\ &\quad + \frac{2}{n}\nabla v_2\big(\chi_{\{-n<v_2<-n/2\}} - \chi_{\{n/2<v_2<n\}}\big)(h_n \circ v_1)\big(T_k \circ (u_1-u_2)\big). \qquad (4.1)\end{aligned}$$

Hence

$$|\nabla\varphi_n| \le \left(1+\frac{2k}{n}\right)\big(|\nabla v_1|\chi_{\{|v_1|<n\}} + |\nabla v_2|\chi_{\{|v_2|<n\}}\big).$$

Thus, by Proposition 3.2, we see that $|\nabla\varphi_n| \in L^{p(\cdot)}(\Omega)$. Therefore, $\varphi_n \in W^{1,p(\cdot)}(\Omega)$. Since $T_n \circ v_j \in W^{1,p(\cdot)}_0(\Omega)$, $j=1,2$, by property (6), it follows that $\varphi_n \in W^{1,p(\cdot)}_0(\Omega)$. Since $\varphi_n = 0$ in a neighborhood of A^*, we also see that $\varphi_n \in W^{1,p(\cdot)}_0(\Omega \setminus A^*)$, so that

$$\int_\Omega p(x)|\nabla u_j|^{p(x)-2}\nabla u_j \cdot \nabla\varphi_n\,dx = \sum_{a\in A\setminus A^*}\alpha_a\delta_a(\varphi_n), \quad j=1,\,2.$$

Hence

$$\int_\Omega \mathcal{A}_{p(\cdot)}(\nabla u_1,\nabla u_2)\cdot\nabla\varphi_n\,dx = 0.$$

Thus, by (4.1)

$$\begin{aligned}\int_{\{|u_1-u_2|<k\}} &\mathcal{A}_{p(\cdot)}(\nabla u_1,\nabla u_2)\cdot(\nabla u_1-\nabla u_2)(h_n\circ v_1)(h_n\circ v_2)\,dx\\ &\le \frac{2k}{n}\int_{E_n}|\mathcal{A}_{p(\cdot)}(\nabla u_1,\nabla u_2)|\big(|\nabla u_1|+|\nabla u_2|+2|\nabla\theta|\big)\,dx.\end{aligned}$$

Since $h_n \to 1$ as $n\to\infty$, we obtain the required inequality.

Corollary 4.1. *Under the same assumptions as in Lemma 4.1,*

$$\big(\mathcal{A}_{p(\cdot)}(\nabla u_1,\nabla u_2)\cdot(\nabla u_1-\nabla u_2)\big)\chi_{\{|u_1-u_2|<k\}} \in L^1(\Omega)$$

for $k>0$.

Proof. First, note that $\mathcal{A}_{p(\cdot)}(\nabla u_1,\nabla u_2)\cdot(\nabla u_1-\nabla u_2)\ge 0$. We have

$$\begin{aligned}&|\mathcal{A}_{p(\cdot)}(\nabla u_1,\nabla u_2)|\big(|\nabla u_1|+|\nabla u_2|+2|\nabla\theta|\big)\\ &\le 4p^+\big(|\nabla u_1|^{p(x)}+|\nabla u_2|^{p(x)}+|\nabla\theta|^{p(x)}\big).\end{aligned}$$

Hence, using the above lemma and Proposition 3.2, we have

$$\int_{\{|u_1-u_2|<k\}}\big(\mathcal{A}_{p(\cdot)}(\nabla u_1,\nabla u_2)\cdot(\nabla u_1-\nabla u_2)\big)\,dx \le 16p^+k\sum_{a\in A}|\alpha_a| < \infty.$$

Proposition 4.1. *Let A be a finite set and let u_1 and u_2 satisfy* (1), (2), (3) *and* (6) *in Theorem 3.1. Let $E_n = \{|u_1 - \theta| < n\} \cap \{|u_2 - \theta| < n\}$. If*

$$\lim_{n\to\infty} \frac{1}{n} \int_{E_n} |\nabla u_1 - \nabla u_2|^{p(x)}\, dx = 0, \tag{4.2}$$

then $u_1 = u_2$.

To prove this proposition, we prepare one more lemma, which is a consequence of Young's inequality:

Lemma 4.2. *For every $\varepsilon > 0$ there exists a constant $C(\varepsilon, p^-, p^+) > 0$ such that*

$$\big||\xi_1|^{q-2}\xi_1 - |\xi_2|^{q-2}\xi_2\big|\,|\eta| \le C(\varepsilon, p^-, p^+)|\xi_1 - \xi_2|^q + \varepsilon(|\xi_1|^q + |\xi_2|^q + |\eta|^q)$$

for any ξ_1, ξ_2, $\eta \in \mathbf{R}^N$ and $p^- \le q \le p^+$.

Proof of Proposition 4.1. Let $\varepsilon > 0$ be arbitrarily given. By the above lemma, there is $C(\varepsilon, p^-, p^+) > 0$ such that

$$\begin{aligned} &\big|\mathcal{A}_{p(\cdot)}(\nabla u_1, \nabla u_2)\big|\big(|\nabla u_1| + |\nabla u_2| + 2|\nabla\theta|\big) \\ &\quad \le C(\varepsilon, p^-, p^+)|\nabla u_1 - \nabla u_2|^{p(x)} + \varepsilon\big\{|\nabla u_1|^{p(x)} + |\nabla u_2|^{p(x)} + |\nabla\theta|^{p(x)}\big\} \end{aligned}$$

for all $x \in \Omega$. Hence, if (4.2) holds, then using Proposition 3.2 again we have

$$\limsup_{n\to\infty} \frac{1}{n} \int_{E_n} \big|\mathcal{A}_{p(\cdot)}(\nabla u_1, \nabla u_2)\big|\big(|\nabla u_1| + |\nabla u_2| + 2|\nabla\theta|\big)\, dx \le 2\varepsilon \sum_{a\in A} |\alpha_a|.$$

Since $\varepsilon > 0$ is arbitrary, from Lemma 4.1 we deduce that

$$\int_{\{|u_1 - u_2| < k\}} \mathcal{A}_{p(\cdot)}(\nabla u_1, \nabla u_2) \cdot (\nabla u_1 - \nabla u_2)\, dx = 0.$$

Therefore

$$\mathcal{A}_{p(\cdot)}(\nabla u_1, \nabla u_2) \cdot (\nabla u_1 - \nabla u_2) = 0$$

a.e. on $\{|u_1 - u_2| < k\}$, and hence $\nabla u_1 = \nabla u_2$ a.e. there. Now, $k > 0$ being arbitrary, $\nabla u_1 = \nabla u_2$ a.e. in Ω. Then, in view of property (6), $u_1 = u_2$ a.e. and in fact everywhere by properties (1) and (2).

Theorem 4.1. *Let A be a finite set. If u_1 and u_2 satisfy* (1), (2), (3) *and* (6) *in Theorem 3.1 and if $u_1 - u_2$ is bounded in a neighborhood of each $a \in A^*$, then $u_1 = u_2$.*

Proof. First note that u_1 and u_2 are bounded outside a neighborhood of A^* by properties (1), (6) and the comparison principle. Hence, $u_1 - u_2$ is bounded on $\Omega \setminus A^*$. Let $|u_1 - u_2| < M$ on $\Omega \setminus A^*$. We shall show that (4.2) holds.

Let $\Omega_1 = \{x \in \Omega\,;\, p(x) \ge 2\}$ and $\Omega_2 = \{x \in \Omega\,;\, p(x) < 2\}$. Since

$$|\xi_1 - \xi_2|^q \le 2^{q-2}\big(|\xi_1|^{q-2}\xi_1 - |\xi_2|^{q-2}\xi_2\big) \cdot (\xi_1 - \xi_2)$$

for $q \geq 2$,

$$\int_{E_n \cap \Omega_1} |\nabla u_1 - \nabla u_2|^{p(x)}\, dx \leq 2^{p^+-1} \int_{E_n \cap \Omega_1} \mathcal{A}_{p(\cdot)}(\nabla u_1, \nabla u_2) \cdot (\nabla u_1 - \nabla u_2)\, dx$$
$$\leq 2^{p^+-1} \int_{\{|u_1-u_2|<M\}} \mathcal{A}_{p(\cdot)}(\nabla u_1, \nabla u_2) \cdot (\nabla u_1 - \nabla u_2)\, dx < \infty$$

by Corollary 4.1. Hence

$$\lim_{n\to\infty} \frac{1}{n} \int_{E_n \cap \Omega_1} |\nabla u_1 - \nabla u_2|^{p(x)}\, dx = 0. \tag{4.3}$$

If $1 < q < 2$, then for $0 < \varepsilon < 1$, we have

$$|\xi_1 - \xi_2|^q \leq \frac{1}{2(q-1)\varepsilon}\left(|\xi_1|^{q-2}\xi_1 - |\xi_2|^{q-2}\xi_2\right) \cdot (\xi_1 - \xi_2) + \varepsilon(|\xi_1| + |\xi_2|)^q.$$

Hence,

$$\int_{E_n \cap \Omega_2} |\nabla u_1 - \nabla u_2|^{p(x)}\, dx$$
$$\leq \frac{1}{(p^- - 1)\varepsilon} \int_{\{|u_1-u_2|<M\}} \mathcal{A}_{p(\cdot)}(\nabla u_1, \nabla u_2) \cdot (\nabla u_1 - \nabla u_2)\, dx$$
$$+ 2^{p^+}\varepsilon \int_{E_n} \left(|\nabla u_1|^{p(x)} + |\nabla u_2|^{p(x)}\right) dx.$$

Thus, by Proposition 3.2 and Corollary 4.1, we see

$$\limsup_{n\to\infty} \frac{1}{n} \int_{E_n \cap \Omega_2} |\nabla u_1 - \nabla u_2|^{p(x)}\, dx \leq 2^{p^+ + 1}\varepsilon \sum_{a \in A} |\alpha_a|.$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \int_{E_n \cap \Omega_2} |\nabla u_1 - \nabla u_2|^{p(x)}\, dx = 0$$

and combining this with (4.3), we see that (4.2) holds.

Theorem 4.2. *Let A be a finite set and assume that $p(x)$ is constant in a neighborhood of a for each $a \in A^*$. Then the function u satisfying* (1), (2), (3) *and* (6) *is unique.*

To prove this theorem, we consider the fundamental solution of $-\Delta_p$ for $1 < p \leq N$:

$$\gamma_p(x) = \begin{cases} C_{p,N}\, |x|^{(p-N)/(p-1)} & \text{if } p < N, \\ C_N \log(1/|x|) & \text{if } p = N, \end{cases}$$

where $C_{p,N}$ and C_N are constants determined to satisfy $-\Delta_p \gamma_p(x) = \delta_0$. The following result follows from [S; Theorem 12] and [KV; Theorem 1.1]:

Lemma 4.3. *Let $1 < p \leq N$ and u be a p-superharmonic function in $B(0, R)$ $(R > 0)$ such that $-\Delta_p u = \alpha\delta_0$ with $\alpha > 0$. Then $u - \alpha^{1/(p-1)}\gamma_p$ is bounded in $B(0, \rho) \setminus \{0\}$ for $0 < \rho < R$.*

By this lemma, Theorem 4.2 immediately follows from Theorem 4.1.

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