数理解析研究所講究録
第 1562 巻 2007 年 10-13

MODULES OVER NON-COMMUTATIVE VALUATION RINGS

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Abstract. A subring R of a division ring D is said to be an *invariant valuation ring* if, for any non-zero element d of D, we have $d \in R$ or $d^{-1} \in R$, and $dRd^{-1} = R$. An R-submodule N of a left R-module M is said to be *relatively divisible* (an *RD-module* for short) if $aN = N \cap aM$ for any $a \in M$. Every finitely generated left R-module M has an RD-composition series with non-decreasing sequence of annihilators. Any two RD-composition series of M is isomorphic and the length of RD-composition series of M is equal to the number of minimal generators of M. [1]

1 Non-commutative valuation rings

Finitely generated modules over commutative valuation rings have been greatly investigated from 1980's (see [FS1], [SZ], [Z]). In this note, we report some results about finitely generated modules over non-commutative valuation rings.

At first, we introduce some non-commutative valuation rings. We refer to [MMU] for details about non-commutative valuation rings.

Let Q be a simple Artinian ring and let R be an order in Q, that is, R is a subring of Q which satisfies the following conditions;

1. any non zero-divisor of R has its inverse in Q, and
2. for any element q of Q, there exist $a,\ b,\ c,\ d \in R$ with $b,\ d$ non zero-divisor, such that $q = ab^{-1} = d^{-1}c$.

An order R in a simple Artinian ring Q is called a *Dubrovin valuation ring* if R is a local Bezout order, that is, if every finitely generated one-sided ideal of R is principal and $R/J(R)$ is simple Artinian, where $J(R)$ is the Jacobson radical of R. There is some characterization of Dubrovin valuation rings (see [MMU, Theorem 5.11]).

[1]This is an abstract and the paper will appear elsewhere.

A *total valuation ring* is an order R in a division ring D which satisfies the following condition;

(T) for any non-zero element $d \in D$, we have $d \in R$ or $d^{-1} \in R$.

If an order R satisfies the condition (T) and the following condition (I), R is called an *invariant valuation ring*;

(I) for any non-zero element d, $dRd^{-1} = R$.

It is clear that an invariant valuation ring is a total valuation ring, and a total valuation ring is a Dubrovin valuation ring (see [MMU, Theorem 5.11]).

Conversely, if a total valuation ring R is integral over its center, then R is an invariant valuation ring (see [MMU, Corollary 8.6]), and a Dubrovin valuation ring R is a total valuation ring if $R/J(R)$ is a division ring (see [MMU, Lemma 8.13]).

2 Modules over non-commutative valuation rings

Throughtout this section, let R be an invariant valuation ring in a division ring D, and we consider finitely generated modules over R.

Let M be a left R-module. An R-submodule N of M is said to be *relatively divisible* (*RD-submodule* for short) if, for any element $a \in R$, we have $aN = N \cap aM$.

Then we have following theorem:

Theorem 2.1 *Let R be an invariant valuation ring and let M be a finitely generated left R-module. Then there exists a sequence*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of R-submodules of M such that

1. *each M_i is an RD-submodule of M, and*
2. *M_i/M_{i-1} is cyclic ($i = 1, 2, \cdots, n$).*

The sequence in Theorem 2.1 is called an *RD-composition series* of M. Two RD-composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ and $0 = N_0 \subset N_1 \subset \cdots \subset N_k = M$ of M are said to be *isomorphic* if $n = k$ and there is some permutation σ of the number $0, 1, \cdots, n-1$ such that $M_i/M_{i-1} \cong N_{\sigma(i)}/N_{\sigma(i)-1}$ ($i = 1, 2, \cdots, n$).

For an RD-composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of M, we set A_i to be the *annihilator* of M_i/M_{i-1}, that is,

$$\begin{aligned} A_i &= \mathrm{Ann}_R(M_i/M_{i-1}) \\ &= \{a \in R \mid a(M_i/M_{i-1}) = 0\}. \end{aligned}$$

If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$, then we say that the annihilator sequence $A_1, A_2, \cdots, A_n$ is *non-decreasing*. Then

Theorem 2.2 *For any RD-composition series of a finitely generated left R-module M, there exists an isomorphic RD-composition series of M with non-decreasing annihilator sequence.*

In some particular case, M is a direct sum of cyclic modules:

Theorem 2.3 *Let* $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ *be an RD-composition series. If there is some* k $(\leq n)$ *such that*

$$Ann_R(M_1) = Ann_R(M_2/M_1) = \cdots = Ann_R(M_k/M_{k-1}),$$

then M_k *is a direct sum of cyclic R-modules. In particular, if all annihilators are equal, then* M *is a direct sum of cyclic R-modules.*

Concerning the lenght of RD-composition series, we have the following:

Theorem 2.4 *The length* $l(M)$ *of an RD-composition series of* M *is equal to the number of minimal generators of* M*.*

We don't know about the relation between the length $l(M)$ of a RD-composition series of M and the Goldie dimension $g(M)$ of M. But, in commutative case, it is proved that $g(M) \leq l(M)$ in general, and that $l(M) = g(M)$ if M is a direct sum of cyclic modules (see [SZ]).

We note that, about modules over total valuation rings or Dubrovin valuation rings, nothing is known yet.

References

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