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On Separable Extensions of Noncommutative Rings

岡山大学・環境理工学部, 池畑 秀一 (Shûichi IKEHATA)
Fuculty of Environmental Science and Technology
Okyama University

In [2], K. Hirata and K. Sugano generalized the notion of separable algebras defining *separable* extensions of a ring. A ring extension T/S is called a *separable* extension, if the T-T-homomorphism of $T\otimes_{S}T$ onto T defined by $a\otimes b\rightarrow ab$ splits, and T/S is called an H-*separable* extension, if $T\otimes_{S}T$ is T-T-isomorphic to a direct summand of a finite direct sum of copies of T. As is well known every H-separable extension is a separable extension.

Throughout this paper, B will mean a ring with identity 1, ρ an automorphism of B, and Z the center of B. Let $B[X;\rho]$ be the skew polynomial ring in which the multiplication is given by $bX=X\rho(b)$ $(b\in B)$. A monic polynomial f in $B[X;\rho]$ with $fB[X;\rho]=B[X;\rho]f$ is called a separable (resp. H-separable) polynomial if the factor ring $B[X;\rho]/fB[X;\rho]$ is a separable (resp. H-separable) extension of B.

Separable polynomials in skew polynomial rings are extensively studied by Kishimoto, Nagahara, Miyashita, Szeto, Xue and the author (see References). In [21, 22], Kishimoto studied some special type of separable polynomials in skew polynomial rings. In [27], Nagahara gave a thorough investigation of separable polynomial of degree 2. Miyashita [26] studied systematically separable polynomials and Frobenius polynomials. He give a characterization of a separable polynomial.

The following is a theorem of Y. Miyashita which characterizes separability of $X^{n}-u$ in $B[X;\rho]$.

Proposition 1. ([26, Theorem 3.1]) *Let* $f=X^{n}-u$ *be in* $B[X;\rho]$. *Then the following conditions are equivalent:*

(1) f *is a separable polynomial in* $B[X;\rho]$.
(2) (a) $\rho(u)=u$, *and* $\alpha u=u\rho^{n}(\alpha)$ *for all* $\alpha\in B$,
  (b) u *is invertible in* $U(B^{\rho})$, *and there exists an element* $z\in Z$ *such that* $z+\rho(z)+\cdots+\rho^{n-1}(z)=1$.

Remark 0.1. The condition (2)(a) in the Proposition 1 is equivalent to $(X^{n}-u)B[X;\rho]=B[X;\rho](X^{n}-u)$.

In [9, 10, 11], the author has studied H-separable pokynomials in skew polynomial rings. If the coefficient ring is commutative, the existence of H-separable polynomials in skew polynomial rings has been characterized in terms of Azumaya algebras and Galois extensions. In [10], the author proved that $B[X;\rho]$ contains an H-separable polynomial of prime degree if and only if the center Z of B is a Galois extension over

This is an abstract and the details will be published elsewhere.

Z^ρ. In [19], G. Szeto and L. Xue has succeeded in general degree case. The following is their theorem.

Proposition 2. ([19, Theorem 3.6]) *Let $f = X^n - u$ be in $B[X;\rho]$. Then the following conditions are equivalent:*

(1) *f is an H-separable polynomial in $B[X;\rho]$.*
(2) (a) *$\rho(u) = u$, and $\alpha u = u\rho^n(\alpha)$ for all $\alpha \in B$,*
  (b) *u is invertible in $U(B^\rho)$, and Z/Z^ρ is a G-Galois extension, where G is the group generated by $\rho|Z$ of degree n.*

The pourpose of this paper is to generalize these results to the skew polynomial rings in several variables. We need some notations as in K. Kishimoto [21], S. Ikehata [5] and S. A. Amitsur and D. Saltman [1].

Let ρ_i $(1 \leqq i \leqq e)$ be automorphisms of a ring B, and let u_{ij} $(1 \leqq i, j \leqq e)$ be invertible elements in B such that

(i) $u_{ij} = u_{ji}^{-1}$, and $u_{ii} = 1$,
(ii) $\rho_i\rho_j\rho_i^{-1}\rho_j^{-1} = (u_{ij})_\ell(u_{ij}^{-1})_r$,
(iii) $u_{ij}\rho_j(u_{ik})u_{jk} = \rho_i(u_{jk})u_{ik}\rho_k(u_{ij})$.

Then the set of all polynomials in e indeterminates

$$\{\sum X_1^{\nu_1}X_2^{\nu_2}\cdots X_e^{\nu_e}b_{\nu_1\nu_2\cdots\nu_e} \mid b_{\nu_1\nu_2\cdots\nu_e} \in B, \nu_k \geqq 0\}$$

forms an associative ring if we define the multiplication by the distributive law and the rules

$$aX_i = X_i\rho_i(a) \ (a \in B) \text{ and } \quad X_iX_j = X_jX_iu_{ij} \ \ (1 \leqq i, j \leqq e).$$

This ring is denoted by $\boldsymbol{R}_e = B[X_1, X_2, \cdots, X_e; \rho_1, \rho_2, \cdots, \rho_e; \{u_{ij}\}]$ and is called a skew polynomial ring of automorphism type. Moreover, by $\boldsymbol{R}_k$ $(0 \leqq k \leqq e)$, we denote the skew polynomial ring $B[X_1, X_2, \cdots, X_k; \rho_1, \rho_2, \cdots, \rho_k; \{u_{ij}\}]$ which is a subring of $\boldsymbol{R}_e$, where $\boldsymbol{R}_0 = B$.

Remark 0.2. For a permutation π of $\{1, 2, \cdots, k\}$ $(k \leqq e)$, we have a B-ring automorphism $\boldsymbol{R}_k \cong B[X_{\pi(1)}, X_{\pi(2)}, \cdots, X_{\pi(k)}; \rho_{\pi(1)}, \rho_{\pi(2)}, \cdots, \rho_{\pi(k)}; \{u_{\pi(i)\pi(j)}\}]$ which maps X_i to $X_{\pi(i)}$ $(1 \leqq i \leqq k)$

Remark 0.3. ρ_{k+1} can be extended to an automorphism ρ_{k+1}^* of $\boldsymbol{R}_k$ by $\rho_{k+1}^*(X_j) = X_ju_{jk+1}$ $(1 \leqq j \leqq k)$ and $\rho_{k+1}^*|B = \rho_{k+1}$. Moreover there holds $\boldsymbol{R}_{k+1} \cong \boldsymbol{R}_k[X_{k+1}; \rho_{k+1}^*]$.

Now, assume further that there exist elements u_i $(1 \leqq i \leqq e)$ in B such that

(iv) $\quad bu_i = u_i\rho_i^{m_i}(b)\ (b \in B)$

and

(v) $\quad \rho_j(u_i)u_{ji}\rho_i(u_{ji}) \cdots \rho_i^{m_i-1}(u_{ji}) = u_i\ (1 \leqq i \leqq e).$

Then we have,

$$a(X_i^{m_i} - u_i) = (X_i^{m_i} - u_i)\rho_i^{m_i}(a)\ (a \in B)$$

and

$$X_j(X_i^{m_i} - u_i) = (X_i^{m_i} - u_i)X_j u_{ji}\rho_i(u_{ji}) \cdots \rho_i^{m_i-1}(u_{ji})\ (1 \leqq i, j \leqq e).$$

This means $(X_i^{m_i} - u_i)\boldsymbol{R}_k$ is a two-sided ideal of $\boldsymbol{R}_k$ for $i \leqq k \leqq e$. The mapping $\bar{\rho}_i : \boldsymbol{R}_e \to \boldsymbol{R}_e$ defined by

$$\bar{\rho}_i(\sum X_1^{\nu_1}X_2^{\nu_2}\cdots X_e^{\nu_e}b_{\nu_1\nu_2\cdots\nu_e}) = \sum(X_1u_{1i})^{\nu_1}(X_2u_{2i})^{\nu_2}\cdots(X_eu_{ei})^{\nu_e}\rho_i(b_{\nu_1\nu_2\cdots\nu_e})$$

is an automorphism of $\boldsymbol{R}_e$ which is an extension of ρ_i.

We put here

$$\boldsymbol{B}_i = B[X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_e; \rho_1, \cdots, \rho_{i-1}, \rho_{i+1}, \cdots, \rho_e; \{u_{ij}\}].$$

Naturally, we have

$$\boldsymbol{R}_e = \boldsymbol{B}_i[X_i; \bar{\rho}_i], \quad \text{and}$$

$$\beta(X_i^{m_i} - u_i) = (X_i^{m_i} - u_i)\bar{\rho}_i^{m_i}(\beta)\ (\beta \in \boldsymbol{B}_i) \quad \text{and} \quad \bar{\rho}_i(u_i) = u_i,$$

where $\bar{\rho}_i$ means $\bar{\rho}_i|\boldsymbol{B}_i$.

Let $\boldsymbol{M} = (X_1^{m_1} - u_1, X_2^{m_2}, \cdots, X_e^{m_e} - u_e)$ be the two sided ideal of $\boldsymbol{R}_e$ generated by $\{X_1^{m_1} - u_1, X_2^{m_2}, \cdots, X_e^{m_e} - u_e\}$. Then the factor ring $\boldsymbol{R}_e/\boldsymbol{M}$ is a free ring extension over B with a basis

$$\{x_1^{\nu_1}x_2^{\nu_2}\cdots x_e^{\nu_e} \mid 0 \leqq \nu_i < m_i, 1 \leqq i \leqq e\}, \text{ where } x_i = X_i + \boldsymbol{M} \in \boldsymbol{R}_e/\boldsymbol{M}.$$

Under the above notations, we shall state the following theorem which is a generalization of Proposition 1.

Theorem 3. *The following are equivalent.*

(1) $\boldsymbol{R}_e/\boldsymbol{M}$ *is a separable extension of* B.

(2) (a) $u_i \in U(B^{\rho_i}) \quad (1 \leqq i \leqq e)$.

(b) *There exists an element* $z \in Z$ *such that*

$$\sum_{i=1}^{e} \sum_{0 \leqq \nu_i < m_i} \rho_1^{\nu_1}\rho_2^{\nu_2}\cdots\rho_e^{\nu_e}(z) = 1$$

(3) $X_i^{m_i} - u_i$ *is a separable polynomial in* $\boldsymbol{B}_i[X_i; \bar{\rho}_i]$ *for each* i $(1 \leqq i \leqq e)$.

(4) (a) $u_i \in U(B^{\rho_i}) \quad (1 \leqq i \leqq e)$.

(b) *There exist elements* $c_i \in Z^{\rho_1, \rho_2, \cdots, \rho_{i-1}, \rho_{i+1}, \cdots, \rho_e}$ *such that*

$$c_i + \rho_i(c_i) + \cdots + \rho_i^{m_i-1}(c_i) = 1$$

To state the result concerning H-sesparable extensions, we need some more notations.

$$\boldsymbol{S}_0 = B \quad \text{and} \quad \boldsymbol{S}_1 = B[X_1;\rho_1]/(X_1^{m_1} - u_1)B[X_1;\rho_1].$$

For $1 \leqq k \leqq e$, We put here $\boldsymbol{S}_k = \boldsymbol{S}_{k-1}[X_k;\bar{\rho}_k]/(X_k^{m_k} - u_k)\boldsymbol{S}_{k-1}[X_k;\bar{\rho}_k]$.
Naturally, we have

$$\boldsymbol{R}_e/\boldsymbol{M} = \boldsymbol{S}_e \supset \boldsymbol{S}_{e-1} \supset \cdots \supset \boldsymbol{S}_1 \supset \boldsymbol{S}_0 = B.$$

Under the above notations, we have the following:

Theorem 4. *The following are equivalent.*

(1) $\boldsymbol{R}_e/\boldsymbol{M}$ *is an H-separable extension of B, and the centralizers of B in* $\boldsymbol{R}_e/\boldsymbol{M}$, $V_{\boldsymbol{R}_e/\boldsymbol{M}}(B) = Z$.

(2) $X_k^{m_k} - u_k$ *is an H-separable polynomial in* $\boldsymbol{S}_{k-1}[X_k;\bar{\rho}_k]$ *for each* k $(1 \leqq k \leqq e)$.

(3) (a) $u_i \in U(B^{\rho_i})$ $\quad (1 \leqq i \leqq e)$

(b) *The order of* $(\rho_i|Z) = m_i$ $(1 \leqq i \leqq e)$, *the set* $\{\rho_i|Z \mid 1 \leqq i \leqq e\}$ *generates an abelian group* $<\rho_1|Z> \times <\rho_2|Z> \times \cdots \times <\rho_e|Z> = G$, *and* Z/Z^G *is a G-Galois extension.*

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E-mail address: `ikehata@ems.okayama-u.ac.jp`