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3D Conway's Solitaire

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We give a concrete process to send a peg to the seventh level in Conway's Solitaire game in a three dimensional space.

1 Conway's Solitaire

Conway's Solitaire is a game played by one person on a board. Pegs are placed at lattice points of the plane. One can move a peg jumping over another peg horizontally or vertically into an empty position. After the move the second peg is removed from the plane:

$$\odot\ \odot\ \bigcirc \quad \Rightarrow \quad \bigcirc\ \bigcirc\ \odot \tag{1}$$

An initial configuration consists of finite number of pegs placed at positions in the lower half plane (below the line $y = 1$). The goal of the game is to send a peg as high as possible on the plane. The problem is to find a configuration and a sequence of moves which send a peg at the possibly highest position. For example, starting with the T-configuration, we can go three steps up:

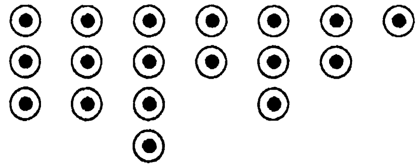
$$\begin{array}{ccccc} \odot & \odot & \odot & \odot & \odot \\ & & \odot & & \\ & & \odot & & \\ & & \odot & & \end{array} \tag{2}$$

The following two configurations also can be used to send a peg to the third level:

$$\begin{array}{ccccc} \odot & \odot & \odot & \odot & \odot \\ \odot & \odot & \odot & & \end{array} \tag{3}$$

$$\begin{array}{ccc} \odot & \odot & \odot \\ \odot & \odot & \odot \\ \odot & \odot & \odot \\ & & \odot \end{array} \tag{4}$$

It is known that starting with the configuration given as (5), one can reach the fourth line above the initial line:



(5)

On the other hand, it is impossible to reach the fifth line starting with any configuration. This last assertion can be proved in the following way (see [1, 2]).

Let $\omega = \frac{-1+\sqrt{2}}{2}$, the positive real number satisfying $\omega^2 + \omega = 1$. Give the value $\omega^{|i|-j}$ at a position (i, j) on the plane. A legal move does not increases the sum of the values of positions where pegs are placed. The infinite sum of all the value of the positions on the lower half line is equal to ω^{-5}, which is equal to the value of the position $(0, 5)$. Therefore, starting with a configuration with a finite number of pegs, one can never reach $(0, 5)$.

2 3D Conway's Solitaire

The three-dimensional version of Conway's Solitaire is played in a space of dimension 3. A legal move is similar to dimension 2. A peg can jump in one of the three directions. From a finite number of pegs initially placed in the lower half space, one tries to send a peg as high as possible.

A similar argument to the ordinary case in Section 1 shows that it is impossible to get the 8th level. On the other hand it is not so difficult to make a configuration which gives the sixth level. In this note we give a concrete process to attain seven. Below we give a sketch of our process.

For an integer n, let $L_n = L(x, n, z)$ be the xz-plane containing the point $(0, n, 0)$ in the space.

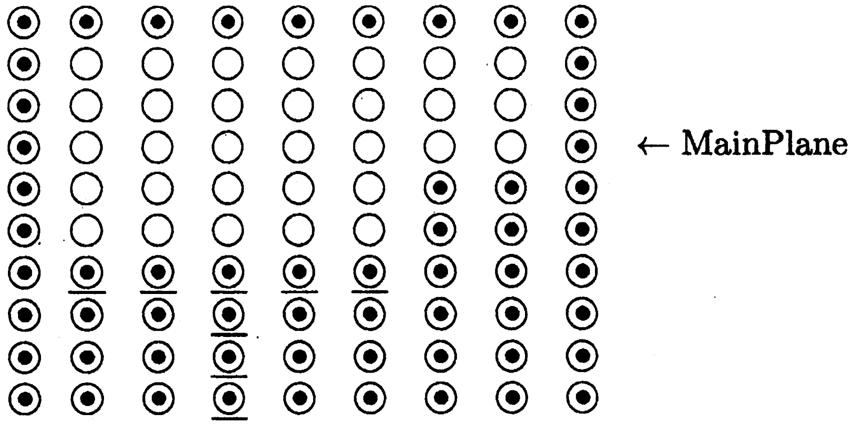
Suppose that at every position on the lower half space a peg is placed beforehand. So, we start with an infinite number of pegs filling the lower half space.

On the main plane $L_0 = L(x, 0, z)$, we can send a peg up to the fourth level using the pegs arranged in the configuration (5) in Section 1. Next, we fill all the empty holes in the lower half of the main plane using pegs on the planes L_1 and L_2. Then, using the T-configuration (2) in L_0, we send another peg up to the third level. Using this peg and the previous one just above it, we can bring a peg to the fifth level.

Fill all the empty holes in the lower half of the main plane using pegs on the planes L_{-1} and L_{-2}. Again, using the pegs in the configuration (5) in the main plane, we send a peg to the fourth level. Using this peg and the one already sent just above it, we can bring a peg to the sixth level.

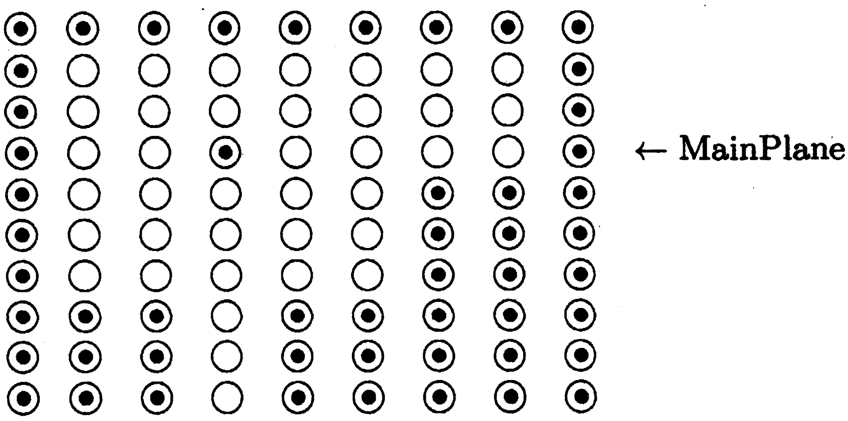
Next, we are going to fill the empty holes remained in the shape of the configuration (5) on the main plane.

First, we fill the hole at (0, 0, 0) using the pegs in the T-configuration (underlined in (6)) on the horizontal xy-plane $L(x, y, 0)$ of level 0. Cosequently, the plane looks like (7).



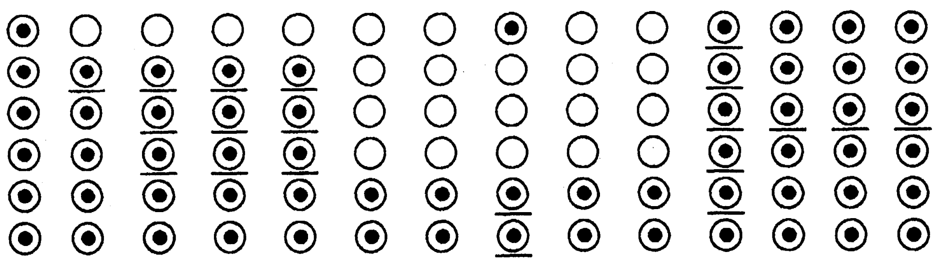
(6)

$\Downarrow$



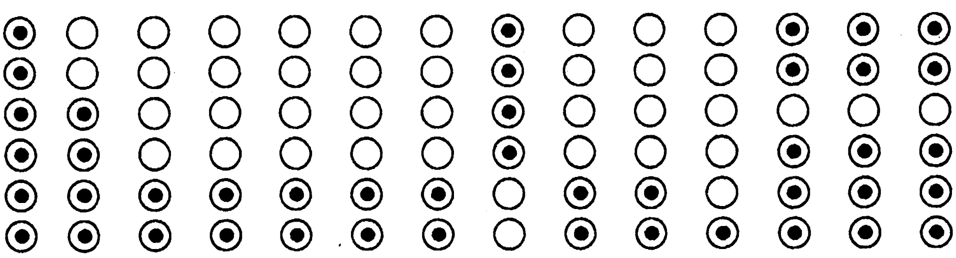
(7)

Next, we fill the holes at $(0, 0, -1)$, $(0, 0, -2)$ and $(0, 0, -3)$ in the main plane inside the yz-plane $L(0, y, z)$ using (modified) T-configurations underlined in (8) to get (9).



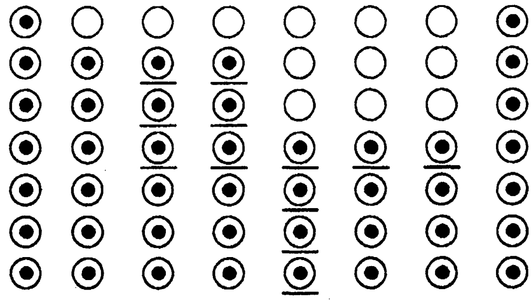
(8)

$\Downarrow$



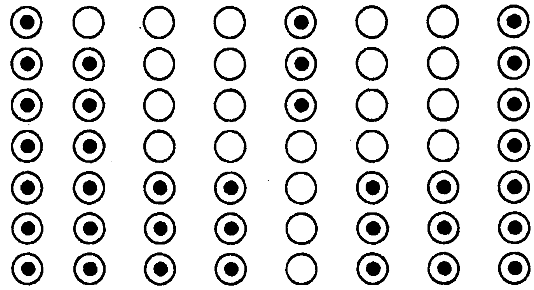
(9)

Then, we play in the plane $L(1, y, z)$ to fill the holes $(1, 0, 0)$, $(1, 0, -1)$ and $(1, 0, -2)$. They are filled as shown in (10) and (11).



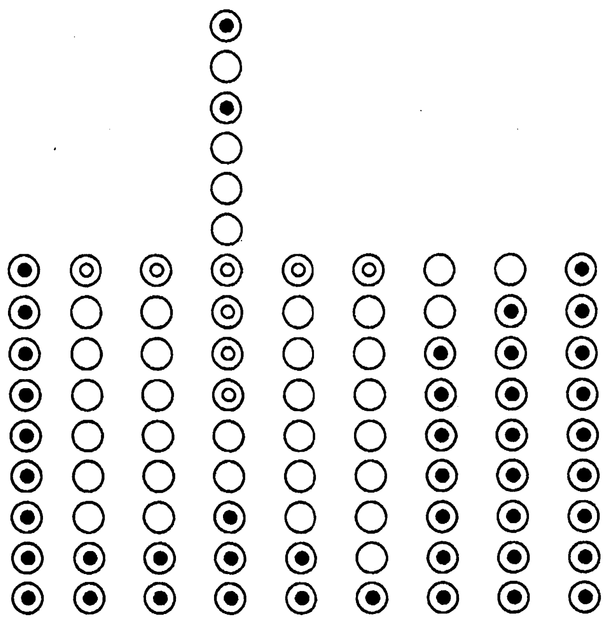
(10)

$$\Downarrow$$



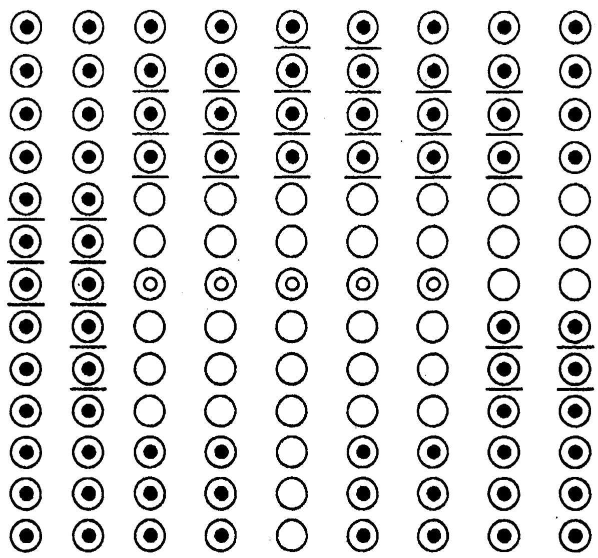
(11)

In a similar manner, we fill the necessary holes in the planes $L(2, y, z)$, $L(3, y, z)$, $L(4, y, z)$, $L(-1, y, z)$ and $L(-2, y, z)$. In this way we reconstruct the configuration (5) in the main plane. With this configuration we send a peg to the fourth floor. After this process, the pegs are placed like (12) in the main plane.



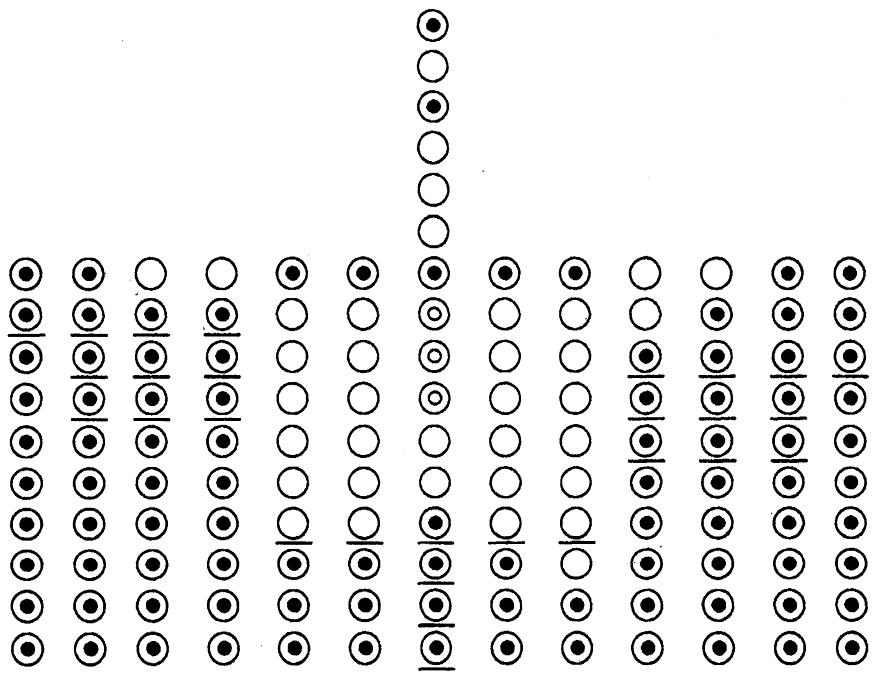
(12)

To attain seven, we have to reconstruct a T-configuration marked as ◎ in (12). Again in the horizontal plane $L(x, y, 0)$, we fill the holes in the main plane marked ◎ using configurations of types (3) and (4) in (13).



(13)

The holes at $(0,0,-1)$ and $(0,0,-2)$ marked ◎in (14) can be filled in the main plane using configurations of type (4). Using pegs remained in the main plane, we can arrange a T-configuration underlined three steps below the hole at $(0,0,-3)$ marked ◎in (14). Thus, this last hole can be filled.



(14)

In this way, we can fill all the holes marked ◎in (12), and with this T-configuration, we can send a peg on the third level. Now, using the pegs at $(0,0,6)$, $(0,0,4)$ and $(0,0,3)$ that we have already sent, we finally climb up to the seventh level.

3 General dimension

We can play Conway's Solitaire game in an n-dimensional space. Pegs are initially placed in a half space bordered by an $(n-1)$-dimensional hyperplane H. A peg can jump over another peg in one of n directions parallel with the space axis. A similar argument to Section 1 would show that $3n-2$ is an upper bound, that is, one can not send a peg higher than the $(3n-2)$-th level above H (see [3]).

On the other hand, if we can attain $\ell(n)$ in the n-dimensional space, we can attain $\ell(n)+2$ in the $(n+1)$-dimensional space. In fact, a similar process used in Section 2 sending a peg to the sixth level in the 3-dimensional space is available. So, by induction starting from the results on 3-dimensional game in Section 2, we can show

Theorem 3.1. *Let $n \geq 3$. In n-dimensional Conway's Solitaire game, we can send a peg to the $(2n+1)$-th level.*

References

[1] J. Akiyama and G. Nakamura, *Mathematics behind games*, Morikita Publ., Tokyo, 1998 (Japanese).

[2] E. R. Berlekamp, J. H. Conway and R. K. Guy, *Winning ways I, II*, Academic, London, 1982.

[3] F. R. Niculescu and R. S. Niculescu, *Solitaire army and related games*, MATH REPORTS 8 (2006), 179-217.