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DIVERGENCE IN DEFORMATION SPACES OF KLEINIAN GROUPS

KEN'ICHI OHSHIKA
DEPARTMENT OF MATHEMATICS, OSAKA UNIVERSITY

This note is based on my talk given in the RIMS on the 4th of December, 2006.

The attention of specialists in the Kleinian group theory is now shifted to the study of the topological structure of deformation spaces after the major problems like Marden's tameness conjecture and the ending lamination conjecture are solved. Although we know, by the resolution of the Bers-Thurston density conjecture ([4]) using the proof of the ending lamination conjecture by Minsky with his collaborators that every finitely generated Kleinian group is an algebraic limit of quasi-conformal deformations of a (minimally parabolic) geometrically finite group, the structure of deformation spaces as topological spaces is far from completely understood.

To understand such a global structure of deformation spaces, the first step would be to give a criterion for sequences in the deformation space to converge or diverge. Let us put it in more concrete terms focusing only on the case of Kleinian groups isomorphic to surface groups. Consider a hyperbolic surface S of finite type and the space of faithful discrete representations of $\pi_1(S)$ to $\mathrm{PSL}_2\mathbb{C}$ preserving the parabolicity modulo conjugacy (both as elements of $\mathrm{PSL}_2\mathbb{C}$ and complex conjugation), which is usually denoted by $AH(S)$. Since the hyperbolic metric of S determines a Fuchsian representation of $\pi_1(S)$ to $\mathrm{PSL}_2\mathbb{R} \subset \mathrm{PSL}_2\mathbb{C}$, as the space of quasi-conformal deformations of this representation, we can consider the space of quasi-Fuchsian representations $QF(S)$ embedded as an open set in $AH(S)$. What we are interested in is the problem to determine in which directions $QF(S)$ has frontier in $AH(S)$ and in which directions it is open-ended. Since by the theory of Ahlfors-Bers, $QF(S)$ is parametrised by $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$, we can describe the directions in $QF(S)$ in terms of the Teichmüller spaces.

The main results in this talk is the following.

Theorem 1. *Let $\{(m_i, n_i)\}$ be a sequence in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ satisfying the following conditions.*

(1) $\{m_i\}$ *converges to a projective lamination* $[\mu^-] \in \mathcal{PML}(S)$ *whereas* $\{n_i\}$ *converges to* $[\mu^+] \in \mathcal{PML}(S)$.
(2) *The supports of* μ^- *and* μ^+ *share a component* μ_0 *which is not a simple closed curve.*

Then the sequence $\{qf(m_i, n_i)\} \subset QF(S)$ *diverges in* $AH(S)$.

Theorem 2. *Let* μ^- *and* μ^+ *be two measured laminations on* S *such that the components shared by* $|\mu^-|$ *and* $|\mu^+|$ *are all simple closed curves, which we denote by* $c_1, \ldots, c_r$.

(1) *Suppose that none of* $c_1, \ldots, c_r$ *lie on the boundary of supporting surfaces of components of* μ^- *or* μ^+. *Then there is a sequence* $\{(m_i, n_i)\}$ *in* $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ *with convergent* $qf(m_i, n_i)$ *such that* m_i *converges* $[\bar{\mu}^-]$ *and* n_i *converges to* $[\bar{\mu}^+]$ *and* $|\bar{\mu}^-| = |\mu^-|, |\bar{\mu}^+| = |\mu^+|$. *Moreover, if* $|\mu^+| = c_1 \cup \cdots \cup c_r$, *we choose* $\{(m_i, n_i)\}$ *so that* $qf(m_i, n_i)$ *converges exotically to a b-group.*
(2) *Otherwise for every* $\{m_i\}$ *converging to* $[\mu^-]$ *and* $\{n_i\}$ *converging to* $[\mu^+]$, *the sequence* $\{qf(m_i, n_i)\} \subset QF(S)$ *diverges in* $AH(S)$.

The proofs of Theorem 1 and Theorem 2 take quite different strategies. For Theorem 1, which is apparently the more complicated case of the two, we can use a rather standard technique of pleated surfaces originally due to Thurston. For Theorem 2, we need to invoke much more sophisticated tool of model manifolds due to Minsky.

In this note we only explain Theorem 1.

1. A sketch of proof of Theorem 1.

Let S be a hyperbolic surface of finite area. Let $\phi_i : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ be a quasi-Fuchsian representation representing $qf(m_i, n_i)$ as was given in Theorem 1. Let G_i be the image of ϕ_i, and M_i the hyperbolic 3-manifold $\mathbb{H}^3/G_i$. Since G_i is a quasi-conformal deformation of the Fuchsian representation of $\pi_1(S)$ associated to the hyperbolic metric on S, there is a natural homeomorphism $\Phi_i : S \times \mathbb{R} \to M_i$ induced by a quasi-conformal homeomorphism, where we regard $S \times \mathbb{R}$ as the hyperbolic 3-manifold containing the hyperbolic surface S in the form of $S \times \{0\}$ as a totally geodesic submanifold. Since G_i is quasi-Fuchsian, the manifold M_i is geometrically finite and has convex core $C(M_i)$, which is homeomorphic to $S \times I$ preserving the parabolicity. We can isotope Φ_i above so that $\Phi_i(S \times [-1, 1]) = C(M_i)$.

Let Σ_i^-, Σ_i^+ be the two frontier components of $C(M_i)$ corresponding to $\Phi_i(S \times \{-1\})$ and $\Phi_i(S \times \{1\})$ respectively. The hyperbolic metric on M_i induces hyperbolic structures on Σ_i^- and Σ_i^+ as length metrics.

We give markings on Σ_i^- and Σ_i^+ by natural homeomorphism between S and $S \times \{-1\}$ and $S \times \{1\}$ obtained by forgetting the second coordinates. It should be noted the orientation given on Σ^+ is different from the ordinary one induced from $C(M_i)$. Let (p_i, q_i) be points in $\mathcal{T}(S)$ determined by these hyperbolic structures on Σ_i^-, Σ_i^+ and markings. Since $(G_i, \phi_i) = qf(m_i, n_i)$ with respect to the Ahlfors-Bers parametrisation, by Bers' inequality, there is a universal bound K between the Teichmüller distances between m_i, p_i and n_i, q_i.

The pleating loci on Σ_i^- and Σ_i^+ give two measured laminations λ_i^-, λ_i^+ on S by pulling back them to S using the inverse of $\Phi_i|S \times \{\pm 1\}$. By passing to a subsequence, we can assume that both $[\lambda_i^-]$ and $[\lambda_i^+]$ converge to projective laminations $[\lambda_\infty^-]$ and $[\lambda_\infty^+]$. We can also assume that the sequences of supports $\{|\lambda_i^-|\}$ and $\{|\lambda_i^+|\}$ converge to geodesic laminations ℓ_∞^- and ℓ_∞^+ in the Hausdorff topology.

We shall prove Theorem 1 by contradiction. Assume that $\{(G_i, [\phi_i])\}$ converges to (Γ, ψ) in $AH(S)$ by taking conjugates and a subsequence. We divide our argument into three cases:

(1) The first case is when either $i(\mu^-, \lambda_\infty^-)$ or $i(\mu^+, \lambda_\infty^+)$is non-zero.
(2) The second case is when both λ_∞^- and λ_∞^+ contain a component shared by $\mu^\mp$ which is not a simple closed curve.
(3) Finally, the third case is when either λ_∞^+ or λ_∞^- is disjoint from any component of μ^+ shared with μ^- that is not a simple closed curve.

In the first case, we assume that $i(\mu^-, \lambda_\infty^-) > 0$. The argument for the case when $i(\mu^+, \lambda^+) > 0$ is completely the same. By the definition of the Thurston compactification of the Teichmüller space (see Fathi-Laudenbach-Poénaru [2]) or the argument in Otal [5], we have $\mathrm{length}_{\Sigma_i^-}(\lambda_i^-) \to \infty$. Since λ_j^- is realised on Σ^-, its length on Σ_i^- with respect to p_i is equal to that in M_i. Therefore we have $\mathrm{length}_{M_i}(\Phi_i(\lambda_i^-)) \to \infty$. On the other hand, by the continuity of the length function (Brock [1]), we have

$$\lim \mathrm{length}_{M_i}(\Phi_i(\lambda_i^-)) = \mathrm{length}_N(\Psi(\lambda_\infty^-))$$

and the right hand side is finite. This is a contradiction, and we have completed the proof of the first case.

Now let us turn to the second case. Let λ_0 be the component shared by $|\lambda_\infty^+|$ and $|\lambda_\infty^-|$, which is not a simple closed curve.

Using the technique of interpolating pleated surfaces due to Thurston, we prove the following.

Proposition 3. *We can take a constant $L > 0$ for which the following holds for large i. There is $t_i \in [0, 1]$ such that $H_i(S(\mu_0), t_i)$ is homotopic* *to $f_i|S(\mu_0)$ by a homotopy staying within the distance L from $f_i(S(\mu_0))$ which keeps the frontier inside the Margulis tubes all the time.*

Then the pleated surface $g_i|S(\mu_0)$ converges to a pleated surface $g_\infty : S(\mu_0) \to M_\infty$ homotopic to f_∞ since the homotopy between $g_i|S(\mu_0)$ and $f_i|S(\mu_0)$ has bounded diameter and converges to a homotopy between g_∞ and $f_\infty|S(\mu_0)$. The limit pleated surface g_∞ realises the limit of the measured laminations $\alpha_i(t_i)|S(\mu_0)$. By taking a subsequence we can assume that $\alpha(t_i)$ converges to a projective lamination on $\alpha([0,1])$, which must have the same support as μ_0 if it is restricted in $S(\mu_0)$. Therefore the limit pleated surface realises μ_0. Since f_∞ is lifted to $f' : S \to N$, the pleated surface g_∞ is also lifted to a pleated surface, which also realises μ_0. This contradicts the fact that μ_0 represents an ending lamination. Thus we have completed the proof of Theorem 1 in this case.

The third case is the most difficult. We need to make an eclectic approach considering Hausdorff limits of the bending loci. The key steps are as follows.

Lemma 4. *Let ℓ be a minimal component of ℓ_∞^- or ℓ_∞^+. Then ℓ does not intersect a component of μ transversely.*

Lemma 5. *Suppose that the Hausdorff limits $\ell_\infty^\mp$ of $|\lambda_i^\mp|$ contain a common component which coincides with the support a component μ_0 of $\mu^\mp$. Then there is an arc $\alpha_i : [0,1] \to \mathcal{PML}(S)$ connecting $[\lambda_i^-]$ with $[\lambda_i^+]$ converging uniformly to an arc α_∞ such that for any sequence $\{t_k\}$ in $[0,1]$ and monotone increasing $\{i_k\}$ for which $|\alpha_{i_k}(t_k)|$ converges in the Hausdorff topology, the limit contains a minimal component which coincides with $|\mu_0|$ except for the case when $t_k = 1/4i_k$ or $t_k = 1-1/4i_k$ for all large k, in which case we have $[\alpha_{i_k}(t_k)] = [\lambda_\infty^+]$ or $[\lambda_\infty^-]$.*

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