数理解析研究所講究録
第 1576 巻 2008 年 67-81

Characteristic classes relating to quantization

宮崎　直哉*
慶應義塾大学日吉数学研究室
Naoya MIYAZAKI
Department of Mathematics,
Hiyoshi Campus, Keio University,
Yokohama, 223-8521, JAPAN

Keywords: Infinite-dimensional group, quantization, Fedosov connection.
Mathematics Subject Classification (2000): Primary 58B25; Secondary 53D55

Abstract It is well-known that there are so many examples of infinite-dimensional groups. We study infinite-dimensional groups which have non-trivial homotopy types. Especially we are interested in symplectic diffeomorphism groups and their quantization. Several characteristic forms are useful to construct non-trivial cycles and cocycles.

1 Introduction

As well-known, the concept of infinite-dimensional group has a long history. It originated from Sophus Lie who initiated the systematic investigation of group germs of continuous transformations. It is also known that he seemed to be motivated by the followings:

- To construct a theory for differential equation similar to Galois theory.

*This research is partially supported by Grant-in-Aid for Scientific Research, Ministry of Education, Culture, Sports, Science and Technology, Japan. It is also supported by Keio Gijuku Academic Funds.

- To investigate groups such as continuous transformations that leave various geometrical structure invariant.

The most striking impression of theory of infinite-dimensional groups is how different in treating from theory of finite-dimensional case. When first encountered, we are perplexed for lack of techniques, other than the implicit function theorem and the Frobenius theorem, to handle the geometrical, topological problems relating to them. Under the situation above, in the present article, we are concerned with examples of infinite-dimensional groups which have non-trivial homotopy types. Moving on the argument for main objects, we give several examples of infinite-dimensional groups.

1. $\mathrm{U}(\mathcal{H}) = \{u :$ unitary operators on a Hilbert space $\mathcal{H}\}$.
2. $\mathrm{U}(\infty) =$ infinite unitary group.
3. $\mathrm{LG} =$ loop group. (See [25] for details).
4. $\mathrm{Diff}(M) = \{$diffeomorphisms on a smooth manifold $M\}$.
5. $\mathrm{Diff}(M, \mathrm{vol}) = \{$volume preserving diffeomorphisms on a smooth manifold $M\}$.
6. $\mathrm{Diff}(M, \omega) = \{$ symplectic diffeomorphisms on a symplectic manifold $M\}$.
7. $\mathrm{Diff}(S^*N, \theta) = \{$contact diffeomorphisms on a contact manifold $S^*N\}$.
8. $\mathrm{GFIO}(N)$= invertible Fourier integral operators on a smooth manifold N with appropriate amplitude functions.
9. $\mathrm{G\Psi DO}(N)$= invertible pseudo-differential operators on a smooth manifold N with appropriate symbol functions.
10. $\mathrm{Aut}(M, *)$= automorphisms of a $*$-product (star product) on a symplectic manifold M.
11. $\underline{\mathrm{Aut}}(M, *)$= $\{\Psi \in \mathrm{Aut}(M, *) | \Psi$ induces the identity map on the base manifold$\}$.

Relating to these examples, we give miscellaneous remarks. First note that

$$(1) \qquad 1 \to \mathrm{G\Psi DO}(N) \to \mathrm{GFIO}(N) \to \mathrm{Diff}(S^*N, \theta) \to 1$$

is exact [1, 8, 20]. As for topology of $\mathrm{U}(\mathcal{H})$, according to Kuiper's theorem, it is known that it is contractible. On the other hand, thanks to Bott's periodicity,

$$\pi_k(\mathrm{U}(\infty)) \cong \begin{cases} \mathbb{Z} & (k = \text{odd}), \\ 0 & (k = \text{even}). \end{cases} \tag{2}$$

As for the diffeomorphism groups, we know that the inclusion maps

$$\mathrm{SO}(2) \subset \mathrm{Diff}_+(S^1),\ \mathrm{SO}(3) \subset \mathrm{Diff}_+(S^2),\ \mathrm{SO}(4) \subset \mathrm{Diff}_+(S^3),\ T^2 \subset \mathrm{Diff}_0(T^2)$$

give homotopy equivalences. On the contrary, if $m \geq 2$, $\mathrm{SO}(2m)$ and $\mathrm{Diff}_+(S^{2m+1})$ are not homotopy equivalent. Except the diffeomorphism groups of Riemann surfaces [1], it seems very difficult to determine homotopy types of the diffeomorphism groups as far as I know.

Although determination of homotopy types of all the examples above is far beyond the scope of the current article, we try to show non-triviality of homotopy types of the groups of symplectic diffeomorphisms and their quantization. The main results of this article are as follows:

Theorem 1.1 *Homotopy type of the symplectic diffeomorphism group* $\mathrm{Diff}(M, \omega)$ *is not trivial in general.*

Quantizing the argument employed to show Theorem 1.1, we have

Theorem 1.2 *Homotopy type of the automorphism group* $\mathrm{Aut}(M, *)$ *of* $*$*-product is not trivial in general.*

While preparing the paper [17], which the current article is based on, Professor Akira Yoshioka informed me that there are mistakes relating to the arguments of the preliminary version of [17] which is concerned with construction of lifts of symplectic diffeomorphisms as $*$-automorphisms. In the revised version of [17] and this article, I used completely different arguments based on Fedosov quantization to construct lifts.

2 Deformation quantization

2.1 $*$-product, Moyal product

The concept of quantization as deformation theory seems to have been introduced by Weyl, who constructed a map from classical observables (functions

[1]These exceptional phenomena support the development of theory of surface bundles with the mapping class groups and the Miller-Morita-Mumford classes.

on the phase space) to quantum obsevables (operators on Hilbert space). The inverse map was constructed by Wigner by interpreting functions (classical observables) as symbols of operators. It is known that the exponent of the bidifferential operator (Poisson bivector) coincides with the product formula of Weyl type symbol calculus developed by Hörmander who established theory of pseudo-differential operators and used them to study partial differential equations (cf. [14] and [19]).

In the 1970s, supported by the mathematical developments above, Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [3] considered quantization as a deformation of the usual commutative product of classical observables into a noncommutative associative product which is parametrized by the Planck constant $\hbar$ and satisfies the correspondence principle. Nowadays deformation quantization, or more precisely, $*$-product has gained support from geometricians and mathematical physicists. In fact, it plays an important role to give passage from Poisson algebras of classical observables to noncommutative associative algebras of quantum observables. In the approach above, the precise definition of the space of quantum observables and $*$-product is given in the following way(cf. [3]):

Definition 2.1 *A* $*$-product *of Poisson manifold* (M, π) *is a product, denoted by* $*$, *on the space* $C^\infty(M)[[\hbar]]$ *of formal power series of parameter* $\hbar$ *with coefficients in* $C^\infty(M)$, *defined by*

$$f * g = fg + \hbar\pi_1(f,g) + \cdots + \hbar^n\pi_n(f,g) + \cdots, \quad \forall f, g \in C^\infty(M)[[\hbar]]$$

satisfying
(a) $*$ *is associative,*
(b) $\pi_1(f,g) = \frac{1}{2\sqrt{-1}}\{f,g\}$,
(c) *each* π_n $(n \geq 1)$ *is a* $\mathbb{C}[[\hbar]]$*-bilinear and bidifferential operator,*
where $\{,\}$ *is the Poisson bracket defined by the Poisson structure* π.

A deformed algebra (resp. a deformed algebra structure) is called a *star algebra* (resp. a $*$*-product*). Note that on a symplectic vector space $\mathbb{R}^{2n}$, there exists the "canonical" deformation quantization, the so-called Moyal product:

$$(3) \qquad \begin{aligned} f * g &= f \exp[\frac{\nu}{2} \overleftarrow{\partial_x} \wedge \overrightarrow{\partial_y}]g \\ &= \sum_{\alpha\beta} \frac{(\nu/2)^{|\alpha|+|\beta|}}{\alpha!\beta!} \partial_x^\alpha \partial_y^\beta f \; \partial_x^\beta(-\partial_y)^\alpha g, \end{aligned}$$

where f, g are smooth functions of a Darboux coordinate (x, y) on $\mathbb{R}^{2n}$ and $\nu = i\hbar$.

Because of its physical origin and motivation, the problems of deformation quntization was first considered for symplectic manifolds, however, the problem of deformation quantization is naturally formulated for the Poisson manifolds as well. The existence and classification problems of $*$-products have been solved by succesive steps from special classes of symplectic manifolds to general Poisson manifolds. Finally, Kontsevich showed that deformation quantization exists on any Poisson manifold. (cf. [3], [4], [5], [6], [7], [9], [12], [21], [22], [26] and [27]).

2.2 Fedosov quantization

In this subsection, we recall the fundamentals for Fedosov quantization.

Definition 2.2 *Let (M, ω) be a symplectic manifold.*

1. *Set $W_M := (\odot T^*M) \otimes \mathbb{R}[[\nu]]$, where $\odot$ means symmetric tensor product. In order to distinguish between a symmetric tensor and an anti-symmetric tensor dz^k, we denote a generator of symmetric tensor by Z^k.*

2. *For elements $a(z, Z, \nu)$, $b(z, Z, \nu) \in W_M$ (where z denotes a point in the base manifold), set*
$$\begin{aligned} & a(z, Z, \nu) *_M b(z, Z, \nu) \\ = \; & a(z, Z, \nu) e^{\left[\frac{\nu}{2}\partial_{Z^i}\Lambda^{ij}\partial_{Z^{j\prime}}\right]} b(z, Z', \nu)\,|_{Z'=Z} \end{aligned}$$
This gives a fiberwise Moyal product. From this product, we naturally obtain a product $\hat{*}$ on the space of smooth sections. Hence we obtain a noncommutative associateve Fréchet algebra. This is called a Weyl algebra bundle.

3. *For any element $a = \nu^l Z^\alpha dz^\beta \in \Gamma(W_M \otimes \Lambda_M)$, we define several operators by*
$$\begin{aligned} a_0 &= a(z, Z, dz, \nu)|_{Z=0}, \qquad a_{00} = a(z, Z, dz, \nu)|_{Z=0, dz=0}, \\ \sigma(a) &= a_0 = a_{00} \; (a \in \Gamma(W_M)), \\ \delta a &= dz^k \wedge \frac{\partial a}{\partial Z^k}, \qquad \delta^{-1} a = \begin{cases} \frac{1}{|\alpha|+|\beta|} Z^k \iota_{\partial_{z^k}}, & (|\alpha| + |\beta| \neq 0), \\ 0 \quad (|\alpha| + |\beta| = 0), \end{cases} \\ W\ell &- deg(a) = |\alpha| + |\beta| + 2l. \end{aligned}$$

Under the notations above, we see

Proposition 2.3 *1. The definitions of δ and δ^{-1} does not depend on the choice of Darboux coordinate.*

2. **Hodge decomposition** $a = \delta\delta^{-1}a + \delta^{-1}\delta a + a_{00}$.

3. $\delta a = -\frac{1}{\nu}[\omega_{ij}Z^i dz^j, a]$.

Let ∇ be a symplectic connection. We define a conncetion D by

$$\begin{aligned} D &= \nabla - \delta + \frac{1}{\nu}[\gamma, \bullet] \\ &= d + [\frac{1}{2\nu}\sum \Gamma_{ijk}z^i z^j dz^k, \bullet] + [\frac{1}{\nu}\omega_{ij}Z^i dz^j, \bullet] + [\frac{1}{\nu}\gamma, \bullet]. \end{aligned}$$

where $\gamma \in \Gamma(W_M \otimes \Lambda_M)$. We would like to find γ such that $D^2 = 0$.

Theorem 2.4 ([9, 10]) *There exists an element γ satisfying the above condition, which is unique under the following conditions.*

$$deg\,\gamma \geq 2, \qquad \delta^{-1}\gamma = 0. \tag{4}$$

The connection obtained as above is called a **Fedosov connection**. Relating to this connection, we obtain the following.

Proposition 2.5 ([9, 10]) *Let D be the Fedosov connection defined as above. Then there exists a linear isomorphism σ between the space $\Gamma^F(W_M)$ of parallel sections with respect to the Fedosov connection and $C^\infty(M)[[\nu]]$. We can also construct the inverse map $\tau = \sigma^{-1}$ explicitly. In fact, for a function $f \in C^\infty(M)$, we define a parallel section $\tau(f)$ by solving the following equation.*

$$\begin{aligned} &\tau(f)_0 = f, \\ &\tau(f)_{s+1} = \delta^{-1}(\nabla r_s + \tfrac{1}{\nu}\textstyle\sum_{t=1}^{s-1} ad(r_{t+2})\tau(f)_{s-t}). \end{aligned} \tag{5}$$

According to the linear isomorphism above, a product $f * g = \sigma(\tau(f) * \tau(g))$ gives an example of $*$-product.

Proof *A proof is given by the following steps.*

1. *Using the property $D^2 = 0$, we show that there exists a linear isomorphism*

$$\sigma : \Gamma^F(W_M) = \{s \in \Gamma(W_M); Ds = 0\} \overset{1-1}{\leftrightarrow} C^\infty(M)[[\nu]]. \tag{6}$$

2. *We show that $\Gamma^F(W_M)$ is closed under $\hat{*}$.*

3. *Define a product $*_F$ on the space $C^\infty(M)[[\nu]]$ by*

$$a *_F b = \sigma(\sigma^{-1}(a) * \sigma^{-1}(b)). \tag{7}$$

Then we can show $*_F$ satisfies the properties of $*$-product on $C^\infty(M)[[\nu]]$.

Summarizing what mentioned above, we obtain the proposition.

As for $*$-product on a symplectic manifold, we have the following.

Theorem 2.6 **([5], [11], [21])** *Let M be a symplectic manifold. Then*

$$\{\textit{Poincaré-Cartan class on } M\} \cong \check{H}(M)[[\nu^2]] \cong \{*\textit{-product}\}/\sim .$$

where the equivalent relation $\sim$ is defined as follows.

Definition 2.7 *Let $*_0$, $*_1$ be $*$-products of a symplectic manifold. Then a map $T : (C^\infty(M)[[\nu]], *_0) \to (C^\infty(M)[[\nu]], *_1)$ is called an equivalence isomorphism if it satisfies the following conditions:*

1. *$T : \mathbb{R}[[\nu]]$-linear isomorphism,*

2. *$T(f *_0 g) = T(f) *_1 T(g)$,*

3. *Tf has an expansion $Tf = f + T_1 f + \cdots + T_k f + \cdots$, and each T_k is a differential operator.*

We denote $*_0 \sim *_1$ if there exists an equivalence isomorphism $T : (C^\infty(M)[[\nu]], *_0) \to (C^\infty(M)[[\nu]], *_1)$.

Remark As seen as above, the Poincaré-Cartan class is a complete invariant of $*$-product. It was introduced in [21] independent of Deligne's characteristic class of $*$-product introduced in [5].

3 Symplectic diffeomorphisms and automorphisms of $*$-product

3.1 Fundamental properties

Under the notations and facts in the previous section, the automorphism group of $*$-product is defined in the following way.

Definition 3.1

$$\mathrm{Aut}(M,*) = \{\Psi : \textit{automorphism of } * \textit{-product}\}. \tag{8}$$

As for this group, we have

Theorem 3.2 ([17]) *1. The groups* $\mathrm{Diff}(M,\omega)$ *and* $\mathrm{Aut}(M,*)$ *are infinite-dimensional groups which are modeled on a Mackey complete space.*

2. $\underline{\mathrm{Aut}}(M,*)$ *is a closed normal subgroup of* $\mathrm{Aut}(M,*)$*, where* $\underline{\mathrm{Aut}}(M,*) := \{\Psi \in \mathrm{Aut}(M,*) | \Psi$ *induces the identity map on the base manifold* $\}$.

3. The following diagram

$$1 \to \underline{\mathrm{Aut}}(M,*) \to \mathrm{Aut}(M,*) \xrightarrow{p} \mathrm{Diff}(M,\omega) \to 1 \tag{9}$$

gives a short exact sequence of infinite-dimensional groups.

4. The group $\underline{\mathrm{Aut}}(M,*)$ *is regular in the sense of [15, 20, 23].*

5. The group $\mathrm{Aut}(M,*)$ *is also regular.*

3.2 Secondary characteristic forms

In this section, we remark about characteristic forms associated with the infinite-dimensional group of automorphisms of a $*$-product on a symplectic manifold with a real polarization introduced in [18]. See also [2].

For the purpose, we recall *Lagrangian orthonormal frame bundle* associated with a Lagrangian subbundle. Let (E,ω) be a symplectic vector bundle over X of rank $2m$, and $\mathcal{L}$ be a Lagrangian subbundle of E. Choose a compatible complex structure J and set $g(\ ,\ ) = \omega(J\ ,\ )$. Fix an orthonormal frame $\{e_1, \dots, e_m\}$ of $\mathcal{L}$ with respect to g, which is called an $\mathcal{L}$-orthonormal frame, and set

$$\varepsilon_1 = \frac{1}{\sqrt{2}}\left(e_1 - \sqrt{-1}Je_1\right), \dots, \varepsilon_m = \frac{1}{\sqrt{2}}\left(e_m - \sqrt{-1}Je_m\right). \tag{10}$$

Then we get a unitary frame $(\varepsilon_1, \dots, \varepsilon_m)$ of E called an *$\mathcal{L}$-orthogonal unitary frame*, and then get an orthogonal frame bundle $\mathcal{O}(E, J, \mathcal{L})$. Hence, summing up what mentioned as above we see

Lemma 3.3 *Under the above notation, the unitary frame bundle $U(m) \to \mathcal{U}(E, J) \to X$ determined by the complex structure J is reduced to $O(m) \to \mathcal{O}(E, J, \mathcal{L}) \to X$.*

We define a smooth map $\mathfrak{F}$ by

$$\begin{array}{ccc} \mathrm{Diff}(M, \omega) \times M & \xrightarrow{\mathfrak{F}} & (M \times M^-, J \oplus J^-, \omega \oplus \omega^-) \\ (\psi, p) & \mapsto & (p, \psi(p)). \end{array} \tag{11}$$

Then we have a symplectic vector bundle over $\mathrm{Diff}(M, \omega) \times M$.

$$\mathfrak{F}^*T(M \times M^-) \to \mathrm{Diff}(M, \omega) \times M. \tag{12}$$

Assume that $\tilde{\mathcal{L}}$ is a Lagrangian subbundle of $T(M \times M^-)$, then $\mathcal{L}_0 = \mathfrak{F}^*\tilde{\mathcal{L}}$ is a Lagrangian subbundle of $\mathfrak{F}^*T(M \times M^-)$. On the other hand, because the graph of ψ is a Lagrangian submanifold of $M \times M^-$, we can define another Lagrangian subbundle in the following way:

$$\mathcal{L}_{1,(\psi,p)} := \mathfrak{F}^*T_{(p,\psi(p))}\mathrm{Graph}(\psi) \subset \mathfrak{F}^*T(M \times M^-). \tag{13}$$

Applying Lemma 3.3 to the above cases $\mathcal{L} = \mathcal{L}_0$ and $\mathcal{L} = \mathcal{L}_1$, we have

Lemma 3.4 ([18]) *Under the above notation, we have two reductions: For $i = 0, 1$*

$$\begin{array}{cccc} & U(2m) & \to & \mathcal{U}(\mathfrak{F}^*T(M \times M^-), J \oplus J^-) \\ & & & \downarrow \\ & & & \mathrm{Diff}(M, \omega) \times M \\ \overset{\text{reduction}}{\rightsquigarrow} & O(2m) & \to & \mathcal{O}(\mathfrak{F}^*T(M \times M^-), J \oplus J^-, \mathcal{L}_i) \\ & & & \downarrow \\ & & & \mathrm{Diff}(M, \omega) \times M. \end{array} \tag{14}$$

Using these reductions with $\mathfrak{o}(2m)$-valued connections θ_0, θ_1, we would like to define closed forms on $\mathrm{Diff}(M,\omega) \times M$.

Lemma 3.5 ([18]) *Let θ_i be $\mathfrak{o}(2m)$-valued connection of $\mathcal{L}_i$-orthogonal unitary frame bundle $(i = 0,1)$. Then*

$$(15) \qquad \mu_k(\mathrm{Diff}(M,\omega),\mathcal{L}_0) = -\int_0^1 c_{2k-1}(\tilde{\Omega})$$

is a closed $(4k-3)$-form on $\mathrm{Diff}(M,\omega) \times M$, where

$$\tilde{\Omega} = \text{curvature of } \tilde{\theta} := t\theta_0 + (1-t)\theta_1,$$

and c_h is a Chern polynomial with degree h.

Proof *Combining the Gauss-Stokes theorem, Bianchi's identity and skew-symmetricity of elements of Lie algebra $\mathfrak{o}(2m)$ completes the proof of lemma.*

Owing to this lemma, we have the following (See Theorems 1.1, 1.2):

Theorem 3.6 *Homotopy types of the symplectic diffeomorphism group* $\mathrm{Diff}(M,\omega)$ *and the automorphism group* $\mathrm{Aut}(M,*)$ *of $*$-product are not trivial in general.*

Proof *For an appropriate manifold M, we can construct cycles in* $\mathrm{Diff}(M,\omega)$ *whose parings with $\mu_k(\mathrm{Diff}(M,\omega),\mathcal{L}_0)$ do not vanish. Moreover making non-trivial lifts of the cycles with respect to $\mathfrak{p}$ in (9) shows non-triviality of homotopy type of the automorphism group of $*$-product. In order to construct suitable lifts, we need the following.*

Theorem 3.7 ([18]) *1. Assume that F satisfies*

$$(16) \qquad \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{i+j=k-1}\{i;\tilde{D};j\}(\frac{F}{\nu}) = \exp[ad(\frac{F}{\nu})](\frac{1}{\nu}(G-\phi^{-1*}(G)))$$

where $\tilde{D}$, G, $\{i;\tilde{D};j\}$ and ϕ are given by

$$(17) \qquad \tilde{D} := \nabla + ad(\phi^{-1*}G), \quad G := \omega_{ij}dz^iZ^j + \gamma$$

$$(18) \qquad \{i;\tilde{D};j\} := (ad(\frac{F}{\nu}))^i \circ ad(\tilde{D}(\frac{F}{\nu})) \circ (ad(\frac{F}{\nu}))^j,$$

$$(19) \qquad \phi : M \to M : \text{ a symplectic diffeomorphism.}$$

Then we have

$$(20) \qquad D \circ \phi^* \circ \exp[ad(\frac{1}{\nu}F)] = \phi^* \circ \exp[ad(\frac{1}{\nu}F)] \circ D.$$

2. *For any symplectic diffeomorphism ϕ on a symplectic manifold (M,ω), there exists an element $\hat{\phi} \in \mathrm{Aut}(M,*)$ which induces $\phi^* \circ \exp[ad(\frac{1}{\nu}F)]$ on $\Gamma(W_M)$ and the base map ϕ on M.*

It is known that homotopy types of GFIO(N) and GΨDO(N) are not trivial in general. Furthermore, for a generic manifold N, GFIO(N) is not homotopically equivalent to GΨDO(N). However, on the contrast,

Conjecture Diff(M,ω) is homotopically equivalent to Aut$(M,*)$.

4 Appendix

For reader's convenience, this appendix is devoted to give a brief survey of regularity of infinite-dimensional groups. For the purpose, we first recall the definition of Mackey completeness, see the monographs [13] for details.

Definition 4.1 *A locally convex space E is called a* Mackey complete (MC *for short) if one of the following equivalent conditions is satisfied:*

1. *For any smooth curve c in E there is a smooth curve C in E with $C' = c$.*

2. *If $c : \mathbb{R} \to E$ is a curve such that $l \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $\ell \in E^*$, then c is smooth.*

3. *Locally completeness: For every absolutely convex closed bounded[2] subset B, E_B is complete, where E_B is a normed space linearly generated by B with a norm $p_B(v) = \inf\{\lambda > 0 | v \in \lambda B\}$.*

4. *Mackey completeness: a Mackey-Cauchy net converges in E.*

5. *Sequential Mackey completeness: a Mackey-Cauchy sequence converges in E.*

where a net $\{x_\gamma\}_{\gamma\in\Gamma}$ is called Mackey-Cauchy *if there exists a bounded set B and a net $\{\mu_{\gamma,\gamma'}\}_{(\gamma,\gamma')\in\Gamma\times\Gamma}$ in $\mathbb{R}$ converging to 0, such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'}B = \{\mu_{\gamma,\gamma'} \cdot x | x \in B\}$.*

[2]A subset B is called *bounded* if it is absorbed by every 0-neighborhood in E, i.e. for every 0-neighborhood $\mathcal{U}$, there exists a positive number p such that $[0,p] \cdot B \subset \mathcal{U}$.

Next we recall the fundamentals relating to infinite-dimensional differential geometry.

1. Infinite-dimensional manifolds are defined on Mackey complete locally convex spaces in much the same way as ordinary manifolds are defined on finite-dimensional spaces. In this article, a manifold equipped with a smooth group operation is referred to as a Lie group. Remark that in the category of infinite-dimensional groups, the existence of exponential maps is **not** ensured in general, and even if an exponential map exists, the local surjectivity of it does not hold (cf. Definition 4.2).

2. A *kinematic tangent vector* (a tangent vector for short) with a foot point x of an infinite-dimensional manifold X modeled on a Mackey complete locally convex space F is a pair (x, X) with $X \in F$, and let $T_x F = F$ be the space of all tangent vectors with foot point x. It consists of all derivatives $c'(0)$ at 0 of smooth curve $c : \mathbb{R} \to F$ with $c(0) = x$. Remark that operational tangent vectors viewed as derivations and kinematic tangent vectors via curves differ in general. A kinematic vector field is a smooth section of kinematic vector bundle $TM \to M$.

3. We set $\Omega^k(M) = C^\infty(L_{skew}(TM \times \cdots \times TM, M \times \mathbb{R}))$ and call it the space of *kinematic differential forms*, where "skew" denotes "skew-symmetric." Remark that the space of kinematic differential forms turns out to be **the right ones** for calculus on manifolds; especially for them the theorem of de Rham is proved.

Next we recall the precise definition of regularity (cf. [15], [20], [23] and [24]):

Definition 4.2 *An infinite-dimensional group G modeled on a Mackey complete locally convex space $\mathfrak{G}$ is said to be* regular *if one of the following equivalent conditions is satisfied*

1. *For each $X \in C^\infty(\mathbb{R}, \mathfrak{G})$, there exists $g \in C^\infty(\mathbb{R}, G)$ satisfying*

$$g(0) = e, \quad \frac{\partial}{\partial t} g(t) = R_{g(t)}(X(t)), \tag{21}$$

2. *For each $X \in C^\infty(\mathbb{R}, \mathfrak{G})$, there exists $g \in C^\infty(\mathbb{R}, G)$ satisfying*

$$g(0) = e, \quad \frac{\partial}{\partial t} g(t) = L_{g(t)}(X(t)), \tag{22}$$

where $R(X)$ (resp. $L(X)$) is the right (resp. left) invariant vector field defined by the right(resp. left)-translation of a tangent vector X at e.

The following lemma is useful (cf. [13], [15], [23] and [24]):

Lemma 4.3 *Assume that N is a closed normal subgroup of infinite-dimensional group G and*

$$1 \to N \to G \to H \to 1 \tag{23}$$

is a short exact sequence of infinite-dimensional groups with a local smooth section[3] *j from a neighborhood $U \subset H$ of 1_H into G. Suppose that N and H are regular. Then G is also regular.*

Actually, Theorem 3.2 is proved by using the lemma above.

Acknowledgements
The author thanks Professors Kazuyuki Fujii and Hideki Omori for their helpful advice. Especially, he would like to express his thanks to Professors Hitoshi Moriyoshi and Akira Yoshioka for informing him relationships between Maslov index and general secondary characteristic classes, and pointing out mistakes in preliminary version of [17].

References

[1] Adams, M., Ratiu, T. and Schmid, R. *A Lie group structure for Fourier integral Operators*, Math. Ann., 276 (1986), 19-41.

[2] Arnol'd, V. I. *On a characteristic class entering in quantization conditions*, Func. Anal. Appl. 1 (1967), 1-13.

[3] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D. *Deformation theory and quantization I*, Ann. of Phys. 111 (1978), 61-110.

[4] Belov-Kanel, A. and Kontsevich, M. *Automorphisms of the Weyl algebra*, Lett. Math. Phys., 74 (2005), 181-199.

[5] Deligne, P. *Déformations de l'Algébre des Fonctions d'une variété Symplectique: Comparison enter Fedosov et De Wilde, Lecomte*, Selecta Math. N.S.1 (1995), 667-697.

[3] Remark that this does not give global splitting of the short exact sequence.

[6] De Wilde, M. and Lecomte, P. B. *Existence of star-products and formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds,* Lett. Math. Phys. 7 (1983), 487-496.

[7] Dito,G and Sternheimer,D. *Deformation Quantization: Genesis, Developments and Metamorphoses* math.QA/0201168.

[8] Eighhorn, J. and Schmid, R. *Lie gourps of Fourier integral operaotrs on open manifolds,* Commun. Analysis. Geom. 9 (2001), no.5, 983-1040.

[9] Fedosov, B. V. *A simple geometrical construction of deformation quantization,* Jour. Diff. Geom. 40 (1994), 213-238.

[10] Fedosov, B. V. *Deformation quantization and Index theory*, Akademie Verlag (1996)

[11] Gutt, S. and Rawnsley, J. *Equivalence of star products on a symplectic manifold; an introduction of Deligne's Cech cohomology classes,* Jour. Geom. Phys. 29 (1999), 347-392.

[12] Kontsevich, M. *Deformation quantization of Poisson manifolds,* Lett. Math. Phys. 66, (2003), 157-216.

[13] Kriegl, A. and Michor, P. *The convenient setting of Global Analysis,* SURV. 53, (1997), Amer. Math. Soc.

[14] Kumano-go, H.*Pseudodifferential Operators*, MIT, (1982).

[15] Milnor, J. *Remarks on infinite dimensional Lie groups,* Proc. Summer School on Quantum Gravity, (1983), Les Houches.

[16] Miyazaki, N. *Remarks on the characteristic classes associated with the group of Fourier integral operators,* Math. Phys. Stud. 23 (2001), 145-154.

[17] Miyazaki, N. *A Lie Group Structure for Automorphisms of a Contact Weyl Manifold*, Progr. Math. 252 (2007), 25-44, Birkhäuser.

[18] Miyazaki, N. *Lifts of symplectic diffeomorphisms as automorphisms of a Weyl algebra bundle with Fedosov connection,* IJGMMP v.4, N5 (2007) 533-546.

[19] Moyal, J.E. *Quantum mechanics as Statistical Theory,* Proc. Cambridge Phil. Soc., 45 (1949), 99-124.

[20] Omori, H. *Infinite-dimensional Lie groups*, MMONO 158 (1995), Amer. Math. Soc.

[21] Omori, H., Maeda, Y., Miyazaki, N. and Yoshioka, A. *Poincaré-Cartan class and deformation quantization of Kähler manifolds,* Commun. Math. Phys. 194 (1998), 207-230.

[22] Omori, H., Maeda, Y. and Yoshioka, A. *Deformation quantization and Weyl manifolds,* Adv. Math. 85 (1991), 224-255.

[23] Omori, H., Maeda, Y., Yoshioka, A. and Kobayashi, O. *On regular Fréchet-Lie groups IV,* Tokyo Jour. Math. 5 (1982), 365-398.

[24] Omori, H., Maeda, Y., Yoshioka, A. and Kobayashi, O. *The theory of infinite dimensional Lie groups and its applications,* Acta Appl. Math. 3 (1985), 71-105.

[25] Pressley, A. and Segal, G. *Loop groups,* (1988), Clarendon Press, Oxford.

[26] Sternheimer, D. *Deformation quantization twenty years after,* AIP Conf. Proc. 453 (1998), 107-145, (q-alg/9809056).

[27] Yoshioka, A. *Contact Weyl manifold over a symplectic manifold,* in "Lie groups, Geometric structures and Differential equations", Adv. Stud. Pure Math. 37(2002), 459-493.