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Asymptotic forms of slowly decaying positive solutions of second-order quasilinear ordinary differential equations
準線型2階常微分方程式の緩減衰正値解の漸近形について

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1 Introduction

Let us consider the quasilinear ODE

$$(a(t)|u'|^{\alpha-1}u')' + b(t)|u|^{\lambda-1}u = 0, \quad \text{near } +\infty \tag{A}$$

where we assume that $\alpha > 0$ and $\lambda > 0$ are constants, $a(t)$ and $b(t)$ are positive continuous functions satisfying $\int^{\infty} a(t)^{-1/\alpha}dt < \infty$. Every positive solution u of (A) satisfies one of the following three asymptotic properties as $t \to \infty$:

$$u(t) \sim c_1 \quad \text{for some constant } c_1 > 0; \tag{1.1}$$

$$u(t) \sim c_2 \int_t^{\infty} a(s)^{-1/\alpha}ds \quad \text{for some constant } c_2 > 0; \tag{1.2}$$

and

$$u(t) \to 0 \quad \text{and} \quad \frac{u(t)}{\int_t^{\infty} a(s)^{-1/\alpha}ds} \to \infty. \tag{1.3}$$

Asymptotic properties of solutions u satisfying either (1.1) or (1.2) were widely investigated. For example, necessary and sufficient conditions of existence of such solutions were established in [4, 7]. On the other hand there seems to be less information about qualitative properties of solutions u satisfying (1.3). Motivated by this fact, in the article we will discuss about asymptotic behavior of solutions u satisfying (1.3); in particular, we try to find exact asymptotic forms of such solutions near $+\infty$. In what follows we refer solutions u satisfying (1.3) as *slowly decaying solutions.*

Remark 1. When $\int^{\infty} a(t)^{-1/\alpha}dt = \infty$, Eq (A) reduces to the simpler one of the form

$$(|u'|^{\alpha-1}u')' + \tilde{b}(t)|u|^{\lambda-1}u = 0 \quad \text{near } +\infty,$$

where $\tilde{b}(t)$ is a positive continuous function. Studies of this equation were, for example, the main objective of [6]; and asymptotic properties of solutions have been fully established

2 Preparatory observations and results

Asymptotic forms of slowly decaying solutions may be strongly affected by those of coefficient functions $a(t), b(t)$ and the exponents α and λ. Therefore let us consider the following ODE, which has more restrictive appearance than Eq (A):

$$(t^\beta |u'|^{\alpha-1}u')' + t^\sigma(1+\varepsilon(t))|u|^{\lambda-1}u = 0 \quad \text{near} + \infty. \tag{E}$$

In the sequel we assume the next conditions:

(A_1) α, β, λ and σ are constants satisfying $\lambda > \alpha > 0$ and $\beta > \alpha$;
(A_2) $\varepsilon(t)$ is a continuous (or C^1-)function defined near $+\infty$ satisfying $\lim_{t\to\infty} \varepsilon(t) = 0$.

Additional conditions will be given later.

Since we can regard Eq (E) as a "perturbed equation" of the ODE

$$(t^\beta |u'|^{\alpha-1}u')' + t^\sigma |u|^{\lambda-1}u = 0 \quad \text{near} + \infty, \tag{E_0}$$

we conjecture that slowly decaying solutions of Eq (E) and those of Eq (E_0) may have the same asymptotic behavior near $+\infty$ in some sense, if $\varepsilon(t)$ is sufficiently small. It is easily seen that Eq (E_0) has an exact slowly decaying solution u_0 given by

$$u_0(t) = \hat{C}t^{-k}, \tag{2.1}$$

where

$$k = \frac{1+\sigma-(\beta-\alpha)}{\lambda-\alpha}, \quad \text{and} \quad \hat{C}^{\lambda-\alpha} = k^\alpha\{\beta-\alpha(k+1)\}$$

if

$$(\beta-\alpha)-1 < \sigma < \frac{\lambda}{\alpha}(\beta-\alpha)-1. \tag{2.2}$$

Below we always assume (2.2). We can show that our conjecture is true in various cases:

Theorem 1. *Let $\alpha \le 1$ and $\beta-\alpha(k+1)-k \ne 0$. If*

$$\textit{either} \quad \int^\infty \frac{\varepsilon(t)^2}{t}dt < \infty \quad \textit{or} \quad \int^\infty |\varepsilon'(t)|dt < \infty, \tag{2.3}$$

then every slowly decaying positive solution u of Eq (E) *satisfies $u(t) \sim u_0(t)$ as $t \to \infty$, where $u_0(t)$ is given by* (2.1).

Theorem 2. *Let $\alpha \ge 1$ and $\beta-\alpha(k+1)-k \ne 0$. If*

$$\lim_{t\to\infty} t\varepsilon'(t) = 0 \quad \textit{and} \quad \int^\infty |\varepsilon'(t)|dt < \infty, \tag{2.4}$$

then every slowly decaying positive solution u of Eq (E) *satisfies $u(t) \sim u_0(t)$ as $t \to \infty$.*

Theorem 3. *Let $\alpha \geq 1$ and $\alpha(2k+1)-\beta < 0$. If (2.3) holds, then every slowly decaying positive solution u of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to \infty$.*

Example 1. Let $N > m > 1$ and $N \geq 2$. Consider radial solutions $u = u(|x|)$ of the following quasilinear PDE in an exterior domain of $\mathbf{R}^N$:

$$\operatorname{div}(|Du|^{m-2}Du) + |x|^{\ell}(1+|x|^{-\theta})|u|^{\lambda-1}u = 0 \quad \text{near} \quad \infty,$$

where $\lambda > m-1, \ell \in \mathbf{R}, \theta > 0$, and $-m < \ell < \frac{\lambda}{m-1}(N-m) - N$. We know that u solves the ODE

$$(r^{N-1}|u'|^{m-2}u')' + r^{N-1+\ell}(1+r^{-\rho})|u|^{\lambda-1}u = 0 \quad \text{near} \quad +\infty.$$

By Theorems 1 and 2, if $\lambda \neq (mN - N + m\ell)/(N-m)$, then every slowly decaying positive solution u of this equation satisfies

$$u(r) \sim Ar^{-(\ell+m)/(\lambda-m+1)} \quad \text{as} \quad r \to +\infty,$$

where A is a positive constant given by

$$A^{\lambda-m+1} = \left(\frac{\ell+m}{\ell-m+1}\right)^{m-1} \cdot \frac{N\lambda - Nm + N - m\ell - m\lambda + \ell}{\lambda - m + 1}.$$

Remark 1. For the autonomous equation $\operatorname{div}(|Du|^{m-2}Du) + |u|^{\lambda-1}u = 0$, the assertion of Example 1 was established in [1] based on the theory of autonomous dynamical systems. Related results are found in [3, 5].

3 Sketches of the proof of the results

We give the outline of the proof of Theorems 1 and 2. We begin with several auxiliary results.

Lemma 1. *Let $u(t)$ be a slowly decaying positive solution of (E). Then*

$$u(t) = O(u_0(t)) \quad \text{and} \quad u'(t) = O(|u_0'(t)|) \quad \text{as} \quad t \to \infty. \tag{3.1}$$

Proof. An integration of the both sides of Eq (E) on $[t_0, t]$ gives

$$t^{\beta}(-u'(t))^{\alpha} \geq \int_{t_0}^{t} r^{\sigma}(1+\varepsilon(r))u^{\lambda}dr,$$

where t_0 is a sufficiently large number. Since u is a decreasing function, we have

$$t^{\beta}(-u'(t))^{\alpha} \geq u(t)^{\lambda} \int_{t_0}^{t} r^{\sigma}(1+\varepsilon(r))dr;$$

that is,

$$-u'(t)u(t)^{-\lambda/\alpha} \geq \left(t^{-\beta}\int_{t_0}^{t} r^{\sigma}(1+\varepsilon(r))dr\right)^{1/\alpha}.$$

One more integration of the both sides gives the estimates for u in (3.1).

To get the estimates for u', it suffices to notice the inequality

$$t^{\beta}(-u'(t))^{\alpha} \leq C_1 \int_{t_0}^{t} r^{\sigma}u(r)^{\lambda}dr,$$

where $C_1 > 0$ is a constant. Note that , to get this inequality, we must use the property $\lim_{t\to\infty} t^{\beta}(-u'(t))^{\alpha} = \infty$.

Lemma 2. *Let $u(t)$ be a slowly decaying positive solution of* (E). *Put $t = e^s$ and $u/u_0 = v$. Then*

(i) *v, and $\dot{v}$ are bounded, and $\dot{v} - kv < 0$ near $+\infty$, where $\cdot = d/ds$;*

(ii) *v satisfies the ODE*

$$\{(kv-\dot{v})^{\alpha}\}^{\cdot} + \{\beta - \alpha(k+1)\}(kv-\dot{v})^{\alpha} - \hat{C}^{\lambda-\alpha}\{1+\delta(s)\}v^{\lambda} = 0 \quad near \quad +\infty, \tag{3.2}$$

where $\delta(s) = \varepsilon(e^s)$.

The proof of this lemma is based on direct computations; hence we omit it.

Remark 2. Equation (3.2) can be rewritten as

$$\ddot{v} + \left(\frac{\beta}{\alpha} - 2k - 1\right)\dot{v} - k\left(\frac{\beta}{\alpha} - k - 1\right)v + \hat{C}^{\lambda-\alpha}\{1+\delta(s)\}v^{\lambda} = 0. \tag{3.3}$$

Lemma 3. *Let $f(s)$ be a C^1-function near $+\infty$ satisfying $\dot{f}(s) = O(1)$ as $s \to \infty$ and $\int^{\infty} f(s)^2 ds < \infty$. Then $\lim_{s\to\infty} f(s) = 0$.*

The proof of this lemma will be found in [6].

Proof of Theorem 1. By the change of variables $(t,u) \mapsto (s,v)$ introduced in Lemma 2, we obtain Eq (3.2). We note that the integral conditions indicated in (2.3) are equivalent to

$$\int^{\infty} \delta(s)^2 ds < \infty \tag{3.4}$$

and

$$\int^{\infty} |\dot{\delta}(s)| ds < \infty, \tag{3.5}$$

respectively.

Step 1. We show that $\int^{\infty} \dot{v}(s)^2 ds < \infty$. We multiply Eq (3.2) by $\dot{v}$, and integrate the resulting equation on $[s_0, s]$ to obtain

$$\int_{s_0}^{s}\{(kv-\dot{v})^{\alpha}\}^{\cdot}\dot{v}dr + \{\beta-\alpha(k+1)\}\int_{s_0}^{s}(kv-\dot{v})^{\alpha}\dot{v}dr$$

$$-\frac{\hat{C}^{\lambda-\alpha}}{\lambda+1}v^{\lambda+1}-\hat{C}^{\lambda-\alpha}\int_{s_0}^{s}\delta(r)v^{\lambda}\dot{v}dr=\text{const.} \tag{3.6}$$

Since integral by parts implies that

$$\int_{s_0}^{s}\{(kv-\dot{v})^{\alpha}\}^{\cdot}\dot{v}dr=-\int_{s_0}^{s}\{(kv-\dot{v})^{\alpha}\}^{\cdot}(kv-\dot{v})dr+k\int_{s_0}^{s}\{(kv-\dot{v})^{\alpha}\}^{\cdot}vdr$$

$$=-\frac{\alpha}{\alpha+1}(kv-\dot{v})^{\alpha+1}+kv(kv-\dot{v})^{\alpha}-k\int_{s_0}^{s}(kv-\dot{v})^{\alpha}\dot{v}dr+\text{const},$$

we obtain from (3.6)

$$-\frac{\alpha}{\alpha+1}(kv-\dot{v})^{\alpha+1}+kv(kv-\dot{v})^{\alpha}+\{\beta-\alpha(k+1)-k\}\int_{s_0}^{s}(kv-\dot{v})^{\alpha}\dot{v}dr$$

$$-\frac{\hat{C}^{\lambda-\alpha}}{\lambda+1}v^{\lambda+1}-\hat{C}^{\lambda-\alpha}\int_{s_0}^{s}\delta(r)v^{\lambda}\dot{v}dr=\text{const.}$$

The boundedness of v and $\dot{v}$ shown in Lemma 2 imply that

$$\{\beta-\alpha(k+1)-k\}\int_{s_0}^{s}(kv-\dot{v})^{\alpha}\dot{v}dr-\hat{C}^{\lambda-\alpha}\int_{s_0}^{s}\delta(r)v^{\lambda}\dot{v}dr=O(1)\quad\text{as}\quad s\to\infty. \tag{3.7}$$

Now, since $0<\alpha\le 1$, the inequality

$$(X^{\alpha}-Y^{\alpha})(X-Y)\ge c_0(X-Y)^2(X+Y)^{\alpha-1}$$

$$\text{for all}\quad X,Y\ge 0\quad\text{with}\quad X+Y>0 \tag{3.8}$$

holds for some constant $c_0>0$. Therefore we obtain

$$\{(kv)^{\alpha}-(kv-\dot{v})^{\alpha}\}\dot{v}\ge c_0((kv)+(kv-\dot{v}))^{\alpha-1}\dot{v}^2;$$

that is,

$$(kv-\dot{v})^{\alpha}\dot{v}\le -c_1\dot{v}^2+k^{\alpha}v^{\alpha}\dot{v}, \tag{3.9}$$

where $c_1>0$ is a constant. Let $\beta-\alpha(k+1)-k>0$. From (3.7) and (3.9) we find that

$$-c_1\{\beta-\alpha(k+1)-k\}\int_{s_0}^{s}\dot{v}^2dr+\{\beta-\alpha(k+1)-k\}\frac{k^{\alpha}}{\alpha+1}v^{\alpha+1}$$

$$\ge\hat{C}^{\lambda-\alpha}\int_{s_0}^{s}\delta(r)v^{\lambda}\dot{v}dr+O(1)\quad\text{as}\quad s\to\infty. \tag{3.10}$$

Suppose $\int^{\infty}\varepsilon(t)^2dt/t<\infty$, that is, (3.4) holds. Schwarz's inequality and (3.10) imply that

$$c_2\int_{s_0}^{s}\dot{v}^2dr\le c_3-c_4\int_{s_0}^{s}\delta(r)v^{\lambda}\dot{v}dr$$

$$\le c_3+c_5\left(\int_{s_0}^{s}\delta(r)^2dr\right)^{1/2}\left(\int_{s_0}^{s}\dot{v}^2dr\right)^{1/2}$$

with some positive constants c_2, c_3, c_4 and c_5. We therefore obtain $\int^\infty \dot{v}^2 dr < \infty$. Suppose next $\int^\infty |\varepsilon'(t)| dt < \infty$, that is, (3.5) holds. We find from (3.10) that

$$c_2 \int_{s_0}^s \dot{v}^2 dr \leq c_3 - c_4 \int_{s_0}^s \delta(r) \left(\frac{v^{\lambda+1}}{\lambda+1} \right)^{\cdot} dr$$

$$\leq c_6 - \frac{c_4}{\lambda+1} \delta(s) v^{\lambda+1} - c_7 \int_{s_0}^s \dot{\delta}(r) v^{\lambda+1} dr \leq c_8 + c_9 \int_{s_0}^s |\dot{\delta}(r)| dr,$$

where c_6, c_7, c_8 and c_9 are some positive constants. Hence we obtain $\int^\infty \dot{v}^2 dr < \infty$. The case where $\beta - \alpha(k+1) - k < 0$ can be treated similarly.

Since we have shown $\int^\infty \dot{v}^2 dr < \infty$, and $\alpha \leq 1$, Eq (3.3) shows that $\ddot{v} = O(1)$ as $s \to \infty$. Therefore by Lemma 3 we find that $\lim_{s\to\infty} \dot{v}(s) = 0$.

Step 2. We show that $\liminf_{s\to\infty} v(s) > 0$. To see this by contradiction, we will derive a contradiction by assuming $\liminf_{s\to\infty} v(s) = 0$. The argument is divided into the two cases:

Case (a): $v(s)$ monotonically decreases to 0 (and so, $\dot{v}(s) \leq 0$);

Case (b): $\dot{v}(s)$ changes the sign in any neighborhood of $+\infty$.

Let case (a) occur. Put $v = x_1$ and $\dot{v} = x_2$, and $x = {}^t(x_1, x_2)$. Then, x satisfies the binary system

$$\dot{x} = Ax + f(s, x), \tag{3.11}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ k\left(\frac{\beta}{\alpha} - k - 1\right) & -\left(\frac{\beta}{\alpha} - 2k - 1\right) \end{pmatrix},$$

and

$$f(s, x) = \begin{pmatrix} 0 \\ -\frac{C^{\lambda-\alpha}}{\alpha}\{1 + \delta(s)\}(k|x_1| + |x_2|)^{1-\alpha}|x_1|^\lambda \end{pmatrix}.$$

Here we have used the fact that $v(s) > 0$ and $\dot{v}(s) \leq 0$. Since

$$(k|x_1| + |x_2|)^{1-\alpha}|x_1|^\lambda \leq (\max\{1, k\})^{1-\alpha} (|x_1| + |x_2|)^{\lambda-\alpha+1},$$

and $(v(s), \dot{v}(s))$ corresponds to a solution $x(s)$ of system (3.11) satisfying $\lim_{s\to\infty} x(s) = 0$, by [2, Chapter 3, Theorem 5] we have

$$\lim_{s\to\infty} \frac{\log \|x(s)\|}{s} = \Lambda, \tag{3.12}$$

where Λ is the real part of an eigenvalue of A. All the eigenvalues of A are k and $-(\beta/\alpha - k - 1)$; the former is positive and the latter negative. Since $\|x(s)\| \to 0$, we have $\Lambda = -(\beta/\alpha - k - 1)$. By the assumption (2.2) we find a small $\eta > 0$ satisfying $\sigma + \lambda(-\beta/\alpha + 1) + \lambda\eta < -1$. By (3.12) we obtain

$$v(s) \leq e^{\{-(\beta/\alpha - k - 1) + \eta\}s} \quad \text{near } +\infty.$$

This means that $u(t) \leq t^{-\beta/\alpha+1+\eta}$ near $+\infty$. Then

$$t^{\beta}(-u'(t))^{\alpha} \leq c_1 \int_{t_0}^{t} r^{\sigma+\lambda(-\beta/\alpha+1)+\lambda\eta} dr = O(1) \quad \text{as}\, t \to \infty.$$

This contradicts the property of slowly decaying solution $\lim_{t\to\infty} t^{\beta}(-u'(t))^{\alpha} = \infty$. Hence Case (a) never occurs. As in the proof of [6, Theorem 1.3], we can show that Case (b) never occurs. Hence we have $\liminf_{s\to\infty} v(s) > 0$.

The remainder of the proof of the fact $\lim_{s\to\infty} v(s) = 1$ procceeds as in the proof of [6, Theorem 1.3]. We leave them to the reader.

Proof of Theorem 2. As in the proof of Theorem 1, we will show that $\lim_{s\to\infty} v(s) = 1$, where $v(s)$ is introduced in Lemma 2. Define

$$w = (kv - \dot{v})^{\alpha}. \tag{3.13}$$

By Eq (3.2) we know that

$$\dot{w} + \{\beta - \alpha(k+1)\}w - \hat{C}^{\lambda-\alpha}\{1+\delta(s)\}v^{\lambda} = 0.$$

Let us rewrite this equation as

$$\dot{w} + aw - b\{1+\delta(s)\}v^{\lambda} = 0, \tag{3.14}$$

where we have put $\beta - \alpha(k+1) = a$ and $\hat{C}^{\lambda-\alpha} = b$. We therefore find that

$$v = b^{-1/\lambda}(1+\delta(s))^{-1/\lambda}(\dot{w}+aw)^{1/\lambda},$$

and w satisfied the ODE

$$((1+\delta(s))^{-1/\lambda}(\dot{w}+aw)^{1/\lambda})^{\cdot} - k(1+\delta(s))^{-1/\lambda}(\dot{w}+aw)^{1/\lambda} + b^{1/\lambda}w^{1/\alpha} = 0. \tag{3.15}$$

We note, by the definition (3.13), (3.14), and Lemma 2, that $w, \dot{w} = O(1)$ as $s \to \infty$. By putting $(1+\delta(s))^{-1/\lambda} = h(s), 1/\lambda = \rho$, and $1/\alpha = \gamma$, we can rewrite (3.15) simply as

$$(h(s)(\dot{w}+aw)^{\rho})^{\cdot} - kh(s)(\dot{w}+aw)^{\rho} + b^{\rho}w^{1/\alpha} = 0. \tag{3.16}$$

We notc that our assumptions (2.4) are equivalent to

$$\lim_{s\to\infty} \dot{\delta}(s) = 0 \tag{3.17}$$

and

$$\int^{\infty} |\dot{\delta}(s)| ds < \infty. \tag{3.18}$$

It should be emphasized that Eq (3.16) is equivalent to

$$\ddot{w} + \left[a - \frac{k}{\rho} + \frac{\dot{h}(s)}{\rho h(s)}\right]\dot{w} + \frac{a}{\rho}\left[\frac{\dot{h}(s)}{h(s)} - k\right]w + \frac{b^{\rho}}{\rho h(s)}(\dot{w}+aw)^{1-\rho}w^{\gamma} = 0 \tag{3.19}$$

By using (3.18) and computing as in the proof of Theorem 1, we find from Eq (3.16) that

$$(a-k)\int_{s_0}^{s} h(r)(\dot{w}+aw)^{\rho}\dot{w}dr = O(1) \quad as \quad s \to \infty. \tag{3.20}$$

Notice that the assumption $\beta - \alpha(k+1) - k \neq 0$ means that $a - k \neq 0$. Since $\alpha \geq 1$ and $\lambda > \alpha$, we have $\rho < 1$. So inequality (3.8) implies, as before, that

$$\{(\dot{w}+aw)^{\rho} - (aw)^{\rho})\}\dot{w} \geq c_0\dot{w}^2\{|\dot{w}+aw| + |aw|\}^{\rho-1};$$

that is,

$$h(r)(\dot{w}+aw)^{\rho}\dot{w} \geq a^{\rho}h(r)w^{\rho}\dot{w} + c_1 h(r)\dot{w}^2$$

for some constant $c_1 > 0$. Hence by (3.20) and the fact that $h(\infty) = 1$, we find that

$$c_2\int_{s_0}^{s} h(r)w^{\rho}\dot{w}dr + c_3\int_{s_0}^{s}\dot{w}^2 dr = O(1) \quad \text{as} \quad s \to +\infty.$$

By integral by parts and by using this relation, we find that $\int^{\infty}\dot{w}^2 ds < \infty$. Moreover, since $\rho < 1$, we find that $\lim_{s\to\infty}\dot{w}(s) = 0$ as in the proof of Theorem 1.

We want to show that $\liminf_{s\to\infty} w(s) > 0$. The proof is done by a contradiction. Firstly suppose that $w(s)$ decreases to 0 as $s \to \infty$. Then, as in the proof of Theorem 1, we know by [2, Chapter 3, Theorem 5] that for every $\eta > 0$

$$w(s) \leq e^{(-\beta+\alpha(k+1)+\eta)s} \quad \text{as} \quad s \to \infty. \tag{3.21}$$

The definition (3.13) is equivalent to $(e^{-ks}v)^{\cdot} = -e^{-ks}w^{1/\alpha}$; and so

$$v(s) = e^{ks}\int_{s}^{\infty} e^{-kr}w^{1/\alpha}dr. \tag{3.22}$$

Here we have employed the fact that $\lim_{s\to\infty} v(s)/e^{ks} = 0$. Combining (3.21) with (3.22), we get the estimate $t^{\beta}|u'(t)| = O(1)$. Recall that this yields a contradiction.

Next, let $\liminf_{s\to\infty} w(s) = 0$ and $\dot{w}$ change the sign in any neighborhood of $+\infty$. Define the auxiliary function $H(s)$ by

$$H(s) = k^{\alpha}\left[1 - \frac{\dot{h}(s)}{kh(s)}\right]^{\frac{\lambda\alpha}{\lambda-\alpha}}.$$

Then, in the region $0 < w < H(s)$, we have $\ddot{w} > 0$. On the other hand in the region $w > H(s)$, we have $\ddot{w} < 0$. Hence, we can find out two sequences $\{\xi_n\}$ and $\{\eta_n\}$ satisfying

$$\xi_n < \eta_n < \xi_{n+1} < \eta_{n+1} < \cdots; \lim_{n\to\infty}\xi_n = \lim_{n\to\infty}\eta_n = \infty;$$

and

$$w(\eta_n) \to 0, \quad w(\xi_n) = H(\xi_n) \to k^{\alpha} \quad \text{as} \quad n \to \infty \quad \text{and} \quad \dot{w} \leq 0 \quad \text{on } [\xi_n, \eta_n]. \tag{3.23}$$

Multiplying (3.19) by $\dot{w}$ and integrating the resulting equation on $[\xi_n, \eta_n]$, we have

$$\frac{1}{2}(\dot{w}(\eta_n)^2 - \dot{w}(\xi_n)^2) + \left(a - \frac{k}{\rho}\right)\int_{\xi_n}^{\eta_n} \dot{w}^2 dr + \frac{1}{\rho}\int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)}\dot{w}^2 dr$$

$$-\frac{ak}{2\rho}(w(\eta_n)^2 - w(\xi_n)^2) + \frac{a}{\rho}\int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w\dot{w} dr + \frac{b^\rho}{\rho}\int_{\xi_n}^{\eta_n} \frac{1}{h(r)}(\dot{w} + aw)^{1-\rho} w^\gamma \dot{w} dr = 0.$$

Noting the facts $\dot{w}(\infty) = 0$ and $\int^\infty \dot{w}^2 dr < \infty$, we have as $n \to \infty$

$$o(1) + \frac{1}{\rho}\int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)}\dot{w}^2 dr - \frac{ak}{2\rho}(o(1) - k^{2\alpha})$$

$$+\frac{a}{\rho}\int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w\dot{w} dr + \frac{a^{1-\rho}b^\rho}{\rho}\int_{\xi_n}^{\eta_n} \frac{1}{h(r)} w^{1+\gamma-\rho}\dot{w} dr \le 0. \tag{3.24}$$

Now, let us estimate each term of the above. We have firstly

$$\left|\int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)}\dot{w}^2 dr\right| \le C_0 \sup_{[\xi_n,\infty)} |\dot{h}| \int_{\xi_n}^\infty \dot{w}^2 dr = o(1) \quad \text{as} \quad n \to \infty;$$

and

$$\left|\int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w\dot{w} dr\right| = \left|\frac{\dot{h}(c_n)}{h(c_n)}\int_{\xi_n}^{\eta_n} w\dot{w} dr\right| = \left|\frac{\dot{h}(c_n)}{2h(c_n)}(w(\xi_n)^2 - w(\eta_n)^2)\right| = o(1) \quad \text{as} \quad n \to \infty.$$

Here $C_0 > 0$ is a constant, and we have used a variant of the mean value theorem for integrals; that is c_n is a number satisfying $\xi_n < c_n < \eta_n$. Finally, we obtain

$$\int_{\xi_n}^{\eta_n} \frac{1}{h(r)} w^{1+\gamma-\rho}\dot{w} dr = \int_{\xi_n}^{\eta_n} [h(r)^{-1} - 1] w^{1+\gamma-\rho}\dot{w} dr + \frac{1}{2+\gamma-\rho}(w(\eta_n)^{2+\gamma-\rho} - w(\xi_n)^{2+\gamma-\rho})$$

$$= (h(d_n)^{-1} - 1)\int_{\xi_n}^{\eta_n} w^{1+\gamma-\rho}\dot{w} dr + \frac{1}{2+\gamma-\rho}(o(1) - k^{\alpha(2+\gamma-\rho)})$$

$$= o(1) - \frac{k^{\alpha(2+\gamma-\rho)}}{2+\gamma-\rho} \quad \text{as} \quad n \to \infty.$$

Here d_n is a number satisfying $\xi_n < d_n < \eta_n$. Therefore (3.24) can be simplified into

$$\frac{ak^{2\alpha+1}}{2\rho} + o(1) \le \frac{a^{1-\rho}b^\rho k^{\alpha(2+\gamma-\rho)}}{\rho(2+\gamma-\rho)} \quad as \quad n \to \infty.$$

This gives a contradiction. Hence we find that $\liminf_{s\to\infty} v(s) > 0$.

Arguing as in the proof of Theorem 1, we will show that $\lim_{s\to\infty} v(s) = 1$. The details are left to the reader.

To see Theorem 3, we will show that $\lim_{s\to\infty} v(s) = 1$, where $v(s)$ is introduced by Lemma 2, as before. However, we can not help omitting the proof for the lack of space.

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