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On Plane Curve Which Has Similar Caustic

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1. What is a caustic?

A **caustic** is the envelope of rays reflected by a curve. For example, if we put a coffee cup on the table and we make parallel light rays on the coffee cup, then we will see a caustic on the surface of coffee. See Figure 1.

Figure 1

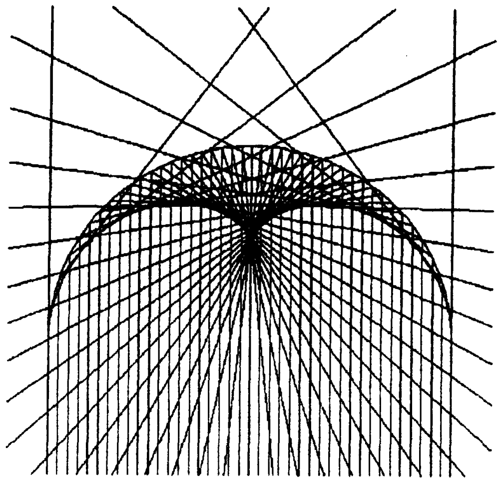

Figure 2

The contents of this paper are as follows: In Section 2, we study how we calculate the caustic from a given curve. As examples, we show that the caustic of a half circle is an epicycloid and that the caustic of a cycloid is also a cycloid whose size is a half of the original cycloid. In Section 3, we study how we calculate the original curve from a given caustic. As an example, we show that, if the caustic is a cycloid, the original curve is also a cycloid. In Section 4, we prove that the cycloid is the unique curve whose caustic is similar to the original curve.

2. Parametrization by angle

Consider a smooth curve on xy-plane. Assume that light rays are parallel to the y-axis. Let θ be the angle between the y-axis and the tangent line of the curve at a point P. Assume that θ is increasing from 0 to π as P varies from end to end of the curve. So we can express the point P by θ. Let $\alpha(\theta) = (x(\theta), y(\theta))$ be a parametrization of a given curve.

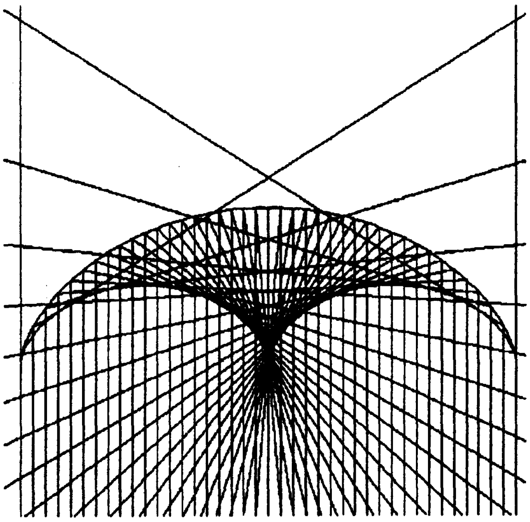

Figure 3

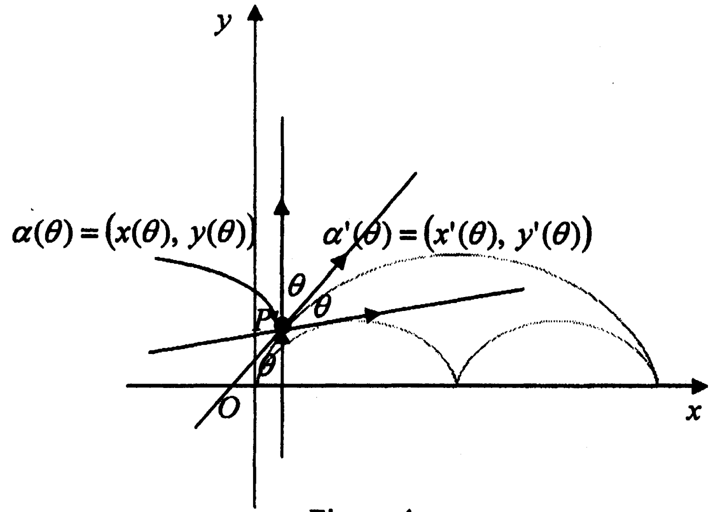

Figure 4

How can we find the caustic from a given curve? By the definition of θ, we have

$$\frac{y'(\theta)}{x'(\theta)}=\cot\theta. \qquad (1)$$

Therefore the equation of reflected ray from $P(x(\theta),\ y(\theta))$ is given by

$$y=\cot 2\theta(x-x(\theta))+y(\theta). \qquad (2)$$

By differentiating both sides with respect to θ and using (1), we have

$$\begin{aligned} y_\theta &= \frac{-2}{\sin^2 2\theta}(x-x(\theta))-\cot 2\theta\, x'(\theta)+y'(\theta) \\ &= \frac{-2}{\sin^2 2\theta}(x-x(\theta))-\frac{\cos 2\theta}{\sin 2\theta}x'(\theta)+\frac{\cos\theta}{\sin\theta}x'(\theta) \\ &= \frac{-2}{\sin^2 2\theta}(x-x(\theta))-\frac{\cos^2\theta-\sin^2\theta}{\sin 2\theta}x'(\theta)+\frac{2\cos^2\theta}{\sin 2\theta}x'(\theta) \\ &= \frac{-2}{\sin^2 2\theta}(x-x(\theta))+\frac{1}{\sin 2\theta}x'(\theta). \end{aligned}$$

Setting $y_\theta=0$ gives the envelope. By setting $y_\theta=0$, we have

$$x=x(\theta)+\frac{1}{2}\sin 2\theta\, x'(\theta)=x(\theta)+\sin\theta\cos\theta\, x'(\theta).$$

By putting it to (2), we have

$$y=y(\theta)+\frac{1}{2}\cos 2\theta\, x'(\theta)=y(\theta)+\frac{1}{2}\frac{\sin\theta(\cos^2\theta-\sin^2\theta)}{\cos\theta}y'(\theta).$$

Therefore, if we put

$$\begin{cases} u(\theta) = x(\theta) + \sin\theta\cos\theta\, x'(\theta) & (3) \\ v(\theta) = y(\theta) + \dfrac{1}{2}\dfrac{\sin\theta\left(\cos^2\theta - \sin^2\theta\right)}{\cos\theta} y'(\theta). & (4) \end{cases}$$

Then $\beta(\theta) = (u(\theta), v(\theta))$ is the caustic of $\alpha(\theta)$. By the definition of θ, we have

$$\frac{v'(\theta)}{u'(\theta)} = \cot 2\theta. \qquad (5)$$

Example 1. When $\alpha(\theta) = (-\cos\theta, \sin\theta)$, find its caustic $\beta(\theta)$.

Solution. Since $\alpha(\theta)$ satisfies (1), we can apply our formulas to this example. By using (3) and (4), we have

$$u(\theta) = -\cos\theta + \frac{1}{2}\sin 2\theta \sin\theta = -\frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta.$$

$$v(\theta) = \sin\theta + \frac{1}{2}\cos 2\theta \sin\theta = \frac{3}{4}\sin\theta + \frac{1}{4}\sin 3\theta.$$

Thus we have $\beta(\theta) = \left(-\frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta, \frac{3}{4}\sin\theta + \frac{1}{4}\sin 3\theta\right)$. Therefore the caustic of a half circle is an epicycloid.

Example 2. When $\alpha(\theta) = (2\theta - \sin 2\theta, 1 - \cos 2\theta)$, find its caustic $\beta(\theta)$.

Solution. Since $\alpha(\theta)$ satisfies (1), we can apply our formulas to this example. By using (3) and (4), we have

$$u(\theta) = 2\theta - \sin 2\theta + \frac{1}{2}\sin 2\theta(2 - 2\cos 2\theta) = (2\theta - \sin 2\theta \cos 2\theta) = \frac{1}{2}(4\theta - \sin 4\theta).$$

$$v(\theta) = 1 - \cos 2\theta + \frac{1}{2}\cos 2\theta(2 - 2\cos 2\theta) = \left(1 - \cos^2 2\theta\right) = \frac{1}{2}(1 - \cos 4\theta).$$

Thus we have $\beta(\theta) = \left(\frac{1}{2}(4\theta - \sin 4\theta), \frac{1}{2}(1 - \cos 4\theta)\right)$. Therefore the caustic of a cycloid is also a cycloid.

3. Inverse problem

From (3), we have

$$x'(\theta) + \frac{1}{\sin\theta\cos\theta}x(\theta) = \frac{u(\theta)}{\sin\theta\cos\theta}.$$

The above equality is equivalent to

$$\{x(\theta)\tan\theta\}' = \frac{u(\theta)}{\cos^2\theta}. \tag{6}$$

When $0 < \theta < \frac{\pi}{2}$, by integrating (6), we have

$$x(\theta)\tan\theta = \int_0^\theta \frac{u(\phi)}{\cos^2\phi}d\phi.$$

When $\frac{\pi}{2} < \theta < \pi$, by integrating (6), we have

$$-x(\theta)\tan\theta = \int_\theta^\pi \frac{u(\phi)}{\cos^2\phi}d\phi.$$

Therefore we obtain

$$x(\theta) = \begin{cases} u(0) & (\theta = 0) \\ \cot\theta\int_0^\theta \frac{u(\phi)}{\cos^2\phi}d\phi & (0 < \theta < \frac{\pi}{2}) \\ u\left(\frac{\pi}{2}\right) & (\theta = \frac{\pi}{2}) \\ -\cot\theta\int_\theta^\pi \frac{u(\phi)}{\cos^2\phi}d\phi & (\frac{\pi}{2} < \theta < \pi) \\ u(\pi) & (\theta = \pi). \end{cases} \tag{7}$$

Example 3. When $\beta(\theta) = \left(\frac{1}{2}(4\theta - \sin 4\theta), \frac{1}{2}(1 - \cos 4\theta)\right)$, find the original curve $\alpha(\theta)$.

Solution. Since $\beta(\theta)$ satisfies (5), we can apply our formula to this example. When $0 < \theta < \frac{\pi}{2}$, by using (7), we have

$$\begin{aligned} x(\theta) &= \cot\theta\int_0^\theta \frac{(4\phi - \sin 4\phi)}{2\cos^2\phi}d\phi \\ &= \frac{1}{2}\cot\theta\left(\int_0^\theta \frac{4\phi}{\cos^2\phi}d\phi - \int_0^\theta \frac{\sin 4\phi}{\cos^2\phi}d\phi\right) \\ &= \frac{1}{2}\cot\theta\left(\int_0^\theta \frac{4\phi}{\cos^2\phi}d\phi - \int_0^\theta \frac{4\sin\phi\cos\phi(\cos^2\phi - \sin^2\phi)}{\cos^2\phi}d\phi\right) \\ &= \frac{1}{2}\cot\theta\left(4\theta\tan\theta - 4\int_0^\theta \tan\phi\, d\phi - 8\int_0^\theta \sin\phi\cos\phi\, d\phi + 4\int_0^\theta \tan\phi\, d\phi\right) \\ &= \frac{1}{2}\cot\theta\left(4\theta\tan\theta - 4\sin^2\theta\right) = 2\theta - \sin 2\theta. \end{aligned}$$

When $\frac{\pi}{2} < \theta < \pi$, by using (7), we have

$$x(\theta) = -\cot\theta \int_\theta^\pi \frac{(4\phi - \sin 4\phi)}{2\cos^2\phi} d\phi$$

$$= -\frac{1}{2}\cot\theta\left(\int_\theta^\pi \frac{4\phi}{\cos^2\phi} d\phi - \int_\theta^\pi \frac{\sin 4\phi}{\cos^2\phi} d\phi\right)$$

$$= -\frac{1}{2}\cot\theta\left(\int_\theta^\pi \frac{4\phi}{\cos^2\phi} d\phi - \int_\theta^\pi \frac{4\sin\phi\cos\phi(\cos^2\phi - \sin^2\phi)}{\cos^2\phi} d\phi\right)$$

$$= -\frac{1}{2}\cot\theta\left(-4\theta\tan\theta - 4\int_\theta^\pi \tan\phi\, d\phi - 8\int_\theta^\pi \sin\phi\cos\phi\, d\phi + 4\int_\theta^\pi \tan\phi\, d\phi\right)$$

$$= -\frac{1}{2}\cot\theta\left(-4\theta\tan\theta + 4\sin^2\theta\right) = 2\theta - \sin 2\theta.$$

Therefore we have $x(\theta) = 2\theta - \sin 2\theta$. By using (1), we have

$$y'(\theta) = \cot\theta\, x'(\theta) = 2\cot\theta \cdot (1 - \cos 2\theta) = 2\sin 2\theta.$$

Therefore we have

$$y(\theta) = 2\int_0^\theta \sin 2\phi\, d\phi = 1 - \cos 2\theta.$$

Thus we obtain $\alpha(\theta) = (2\theta - \sin 2\theta,\ 1 - \cos 2\theta)$.

4. On plane curve which has similar caustic

Example 2 says that the caustic of cycloid is also a cycloid. So a question arises: "Is there another curve which is similar to its caustic?" The following theorem is an answer of this problem.

Theorem. *Suppose that a curve $\alpha(\theta)$ $(0 \le \theta \le \pi)$ with $\alpha(0) = (0,0)$, $\alpha(\pi) = (2\pi, 0)$ has a caustic $\beta(\theta)$ which consists of two curves both similar to $\alpha(\theta)$ in ratio $\frac{1}{2}$, that is,*

$$\beta(\theta) = \begin{cases} \frac{1}{2}\alpha(2\theta) & (0 \le \theta \le \frac{\pi}{2}) \\ (\pi, 0) + \frac{1}{2}\alpha(2\theta - \pi) & (\frac{\pi}{2} \le \theta \le \pi), \end{cases}$$

then $\alpha(\theta) = (2\theta - \sin 2\theta, 1 - \cos 2\theta)$.

Proof. Put $\alpha_0(\theta) = (x_0(\theta), y_0(\theta)) = (2\theta - \sin 2\theta,\ 1 - \cos 2\theta)$. In Example 2, we already proved that $\alpha_0(\theta)$ satisfies the assumption of the theorem. We assume that there is a curve

$\alpha_1(\theta) = (x_1(\theta),\ y_1(\theta))$ which also satisfies the assumption. Then by (7), both $x_0(\theta)$ and $x_1(\theta)$ satisfy

$$x_i(\theta) = \begin{cases} 0 & (\theta = 0) \\ \cot\theta \displaystyle\int_0^\theta \frac{x_i(2\phi)}{2\cos^2\phi}\,d\phi & (0 < \theta < \frac{\pi}{2}) \\ \pi & (\theta = \frac{\pi}{2}) \\ \pi - \cot\theta \displaystyle\int_\theta^\pi \frac{x_i(2\phi - \pi)}{2\cos^2\phi}\,d\phi & (\frac{\pi}{2} < \theta < \pi) \\ 2\pi & (\theta = \pi). \end{cases}$$

Put $M = \max\limits_{0\le\theta\le\pi} |x_1(\theta) - x_0(\theta)|$. Then we can calculate as follows:

$$\begin{aligned} \sup_{0<\theta<\frac{\pi}{2}} |x_1(\theta) - x_0(\theta)| &= \sup_{0<\theta<\frac{\pi}{2}} \left| \cot\theta \int_0^\theta \frac{x_1(2\phi)}{2\cos^2\phi}\,d\phi - \cot\theta \int_0^\theta \frac{x_0(2\phi)}{2\cos^2\phi}\,d\phi \right| \\ &\le \sup_{0<\theta<\frac{\pi}{2}} \left\{ \cot\theta \int_0^\theta \frac{1}{2\cos^2\phi} |x_1(2\phi) - x_0(2\phi)|\,d\phi \right\} \\ &\le \sup_{0<\theta<\frac{\pi}{2}} \left\{ \cot\theta \int_0^\theta \frac{M}{2\cos^2\phi}\,d\phi \right\} = \frac{M}{2}, \end{aligned}$$

$$\begin{aligned} \sup_{\frac{\pi}{2}<\theta<\pi} |x_1(\theta) - x_0(\theta)| &= \sup_{\frac{\pi}{2}<\theta<\pi} \left| \pi - \cot\theta \int_\theta^\pi \frac{x_1(2\phi - \pi)}{2\cos^2\phi}\,d\phi - \pi + \cot\theta \int_\theta^\pi \frac{x_0(2\phi - \pi)}{2\cos^2\phi}\,d\phi \right| \\ &\le \sup_{\frac{\pi}{2}<\theta<\pi} \left\{ \cot\theta \int_\theta^\pi \frac{1}{2\cos^2\phi} |x_1(2\phi - \pi) - x_0(2\phi - \pi)|\,d\phi \right\} \\ &\le \sup_{\frac{\pi}{2}<\theta<\pi} \left\{ \cot\theta \int_\theta^\pi \frac{M}{2\cos^2\phi}\,d\phi \right\} = \frac{M}{2}. \end{aligned}$$

Therefore we have $M \le \max\left\{0, \frac{M}{2}, 0, \frac{M}{2}, 0\right\} = \frac{M}{2}$. Thus we have $M = 0$, that is, $x_1(\theta) = x_0(\theta)$ for every θ. Since $\dfrac{y_1'(\theta)}{x_1'(\theta)} = \dfrac{y_0'(\theta)}{x_0'(\theta)} = \cot\theta$, we have $y_1'(\theta) = y_0'(\theta)$. Since we have $y_1(0) = y_0(0)$, we obtain $y_1(\theta) = y_0(\theta)$ for every θ. Thus $\alpha_0(\theta)$ is the only curve satisfying the assumption.

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References

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