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Existence of invariant manifolds at an indeterminate point

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Abstract

In this note, consider dynamics of a rational mapping F on 2-dimensional complex projective space $\mathbf{P}^2$ which has a periodic indeterminate point p. By using a symbol sequence $j \in \{1,2\}^{\mathbf{N}}$, we will define some family $\{V_j\}_{j\in J}$ which consists of locally invariant holomorphic curves at p by F, algebraically.

1. Introduction.

In this note, we consider a local dynamical structure of a rational mapping F of $\mathbf{P}^2$ near a periodic indeterminate point p. Using a blow up, we construct a family $\{V_j\}_{j\in J}$ which consists of locally invariant curves at p by F, where J is a subset of the Cantor set $\{1,2\}^{\mathbf{N}}$.

Here, prepare some notation and terminology. Let $f_i(x,y,t)(i=0,1,2)$ be homogeneous polynomials with degree d, $F:[x:y:t]\mapsto[f_0:f_1:f_2]$ a rational mapping on $\mathbf{P}^2$ and $\tilde{F}:(x,y,t)\mapsto(f_0,f_1,f_2)$ a polynomial mapping on $\mathbf{C}^3$. Then, we have $\pi\circ\tilde{F}=F\circ\pi$ on $\mathbf{C}^3$ except some analytic sets, where $\pi:\mathbf{C}^3\setminus\{(0,0,0)\}\to\mathbf{P}^2$ is the canonical projection. A point $p\in\mathbf{P}^2$ is said to be an *indeterminate point* of F if $\tilde{F}(\tilde{p})=(0,0,0)$ for some point $\tilde{p}\in\pi^{-1}(p)$. In general, if p is an indeterminate point, then $\bigcap_{U_p}\overline{F(U_p\setminus\{p\})}$ is not a single point, where the intersection is taken over all open neighborhoods U_p of p. So, no definition of the image $F(p)$ makes the mapping F be continuous. Moreover, if $p\in\bigcap_{U_p}\overline{F(U_p\setminus\{p\})}$, it is called a *periodic indeterminate point*. It can be seen from the definition that a periodic indeterminate point has a certain recurrent property, hence we expect a local dynamical structure at this point.

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In this note, we assume that $F : \mathbf{P}^2 \to \mathbf{P}^2$ is a rational mapping with an indeterminate point $p = [0:0:1]$. We often identified $\mathbf{C}^2$ with the affine chart of $\mathbf{P}^2$ which is defined by $\{[x:y:t] \in \mathbf{P}^2 \mid t \neq 0\}$, and put $p = (0,0)$. Let

$$X := \left\{(x,y) \times [u:v] \in \mathbf{C}^2 \times \mathbf{P}^1 \mid xv - yu = 0\right\}$$

be a subset of $\mathbf{C}^2 \times \mathbf{P}^1$. Then, X is a subvariety of $\mathbf{C}^2 \times \mathbf{P}^1$ and covered by the following two coordinate neighborhoods $\{(U^j, \mu^j)\}_{j=0,1}$,

$$U^0 := \left\{(x,y) \times [u:v] \in X \mid y = \frac{v}{u}x\right\}, \ \mu^0 : U^0 \ni (x,y) \times [u:v] \mapsto \left(x, \frac{v}{u}\right) \in \mathbf{C}^2,$$

$$U^1 := \left\{(x,y) \times [u:v] \in X \mid x = \frac{u}{v}y\right\}, \mu^1 : U^1 \ni (x,y) \times [u:v] \mapsto \left(\frac{u}{v}, y\right) \in \mathbf{C}^2.$$

Definition 1 (see [4]). *The mapping $\pi : X \to \mathbf{C}^2$ defined by restricting the first projection $\mathbf{C}^2 \times \mathbf{P}^1 \to \mathbf{C}^2$ is called the blow up of $\mathbf{C}^2$ centered at $p = (0,0)$ and $E := \pi^{-1}(0,0) = (0,0) \times \mathbf{P}^1$ is called the exceptional curve.*

It is remarked here that $\pi : X \setminus E \to \mathbf{C}^2 \setminus \{(0,0)\}$ is a biholomorphic mapping, and by replacing $\mathbf{C}^2$ with the affine chart $\{[x:y:t] \in \mathbf{P}^2 \mid t \neq 0\}$ of $\mathbf{P}^2$, it can be naturally extended as the blow up of $\mathbf{P}^2$ centered at $p = [0:0:1]$. To simplify the notation, we call it blow up of $\mathbf{P}^2$ centered at p, too.

The study of local dynamics of a periodic indeterminate point was started by Y. Yamagishi (see [8] and [9]). Here, introduce his results. Set $\tilde{F} := F \circ \pi : X \to \mathbf{P}^2$. Assume that F satisfies the following assumption.

$$(A.0)\begin{cases} (1)\ \tilde{F} \text{ is a holomorphic mapping on some open neighborhood of } E, \\ (2)\ \tilde{F}^{-1}(p) \cap E = \{p_1, p_2\} \text{ and}, \\ (3)\ \text{there exist open neighborhoods } N_i \text{ of } p_i \text{ such that } \tilde{F}|_{N_i} \text{ is} \\ \quad \text{a biholomorphic mapping for } i = 1, 2. \end{cases}$$

Here, we remark that p is a periodic indeterminate point of F. Moreover, he assumed that $\tilde{F}$ is contracting in the horizontal direction on N_i. Then, it has been proved that there exists a family of local stable manifolds of p which is indexed by the Cantor set

$$\{1,2\}^{\mathbf{N}} := \{j = (j_1, j_2, \ldots) \mid j_n = 1, 2 \ \text{ for } n \in \mathbf{N}\}.$$

It is called the *Cantor bouquet* (for detail, see [8] and [9]).

In this note, we consider the following family of curves which is a generalization of the Cantor bouquet.

Definition 2. *A family $\{W_\lambda\}_{\lambda\in\Lambda}$ of curves is locally invariant at p by F if*
(1) *every W_λ is given by a graph of some continuous function*

$$\phi_\lambda : \Delta_{\rho_\lambda} \ni x \mapsto y = \phi_\lambda(x) \in \mathbf{C}$$

with $\phi_\lambda(0) = 0$, where $\Delta_{\rho_\lambda} := \{x \in \mathbf{C} \mid |x| < \rho_\lambda\}$, and
(2) *for any W_λ there is a $\lambda' \in \Lambda$ and some open neighborhood $N_{\lambda'}$ of p such that $\lim_{x\to 0} F(x, \phi_\lambda(x)) = p$ and*

$$F(x, \phi_\lambda(x)) \cap N_{\lambda'} \subset W_{\lambda'} \quad for \quad x \in \Delta_{\rho_\lambda} \setminus \{0\}.$$

Especially, if every ϕ_λ is a holomorphic function, then $\{W_\lambda\}$ is called a family of holomorphic curves.

Remark. Let be a mapping $\Phi_\lambda : \Delta_{\rho_\lambda} \to \mathbf{C}^2$ by $\Phi_\lambda(x) = (x, \phi_\lambda(x))$. Assume that Φ_λ is a holomorphic mapping. Then, $F \circ \Phi_\lambda$ is well-defined on Δ_{ρ_λ}, even if p is an indeterminate point of F, that is, there is a unique holomorphic mapping $g : \Delta_{\rho_\lambda} \to \mathbf{C}^2$ such that $g(z) = F \circ \Phi_\lambda(z)$ for $z \in \Delta_{\rho_\lambda} \setminus \{0\}$ (for detail, see [1]).

Now, we state our Main theorems. *In the reminder of this note, denote $j_n = 1, 2$ for every $n \in \mathbf{N}$.* Assume that F satisfies the condition (A.0). Then, the following claim $(A.1)$ holds.

$$(A.1)\begin{cases} (1)\ F_0 := \pi^{-1} \circ \tilde{F} \text{ is a meromorphic mapping on } N(E) \text{ and } \{p_1, p_2\} \text{ are} \\ \quad \text{indeterminate points of } F_0, \text{ where } N(E) \text{ is an open neighborhood of } E. \\ \quad \text{Let } \pi_{j_1} : X_{j_1} \to X \text{ be the blow up of } X \text{ centered at } p_{j_1} \text{ and} \\ \quad \tilde{F}_{j_1} := F_0 \circ \pi_{j_1} : X_{j_1} \to X. \text{ Then,} \\ (2)\ \tilde{F}_{j_1}|_{E_{j_1}} : E_{j_1} \to E \text{ is bijective, and one can set } p_{j_1 j_2} := \tilde{F}_{j_1}^{-1}(p_{j_2}) \in E_{j_1}. \\ (3)\ \text{There is an open neighborhood } N_{j_1 j_2} \text{ of } p_{j_1 j_2} \text{ such that} \\ \quad \tilde{F}_{j_1}|_{N_{j_1 j_2}} \text{ is a biholomorphic mapping.} \end{cases}$$

Theorem 1 (see [5]). *We can repeat this process inductively for all $n \in \mathbf{N}$ and symbol sequences $j = (j_1, \ldots) \in \{1, 2\}^{\mathbf{N}}$, and succeed with infinitely many times of* *blow ups* $\pi_{j_1\cdots j_{n+1}} : X_{j_1\cdots j_{n+1}} \to X_{j_1\cdots j_n}$. *In particular, there exist sequences of points* $p_{j_1\cdots j_{n+1}} \in X_{j_1\cdots j_n}$.

In addition to the condition $(A.0)$, we suppose the following condition (B).

$$(B)\quad p_{j_1\cdots j_n} \in U^0_{j_1\cdots j_{n-1}} \text{ for every } n \in \mathbf{N},$$

where $U^0_{j_1\cdots j_{n-1}}$ is the coordinate neighborhood of $X_{j_1\cdots j_{n-1}}$ analogue to that defined for X. Then, we can set $p_{j_1\cdots j_n} := (0, \alpha_{j_1\cdots j_n})$ by using the local coordinates system of $U^0_{j_1\cdots j_{n-1}}$. Finally, for all symbol sequences $j \in \{1,2\}^{\mathbf{N}}$ with $j = (j_1, j_2, \ldots)$, define a formal power series

$$y = \phi_j(x) := \alpha_{j_1}x + \alpha_{j_1j_2}x^2 + \cdots,$$

$$J := \left\{ j \in \{1,2\}^{\mathbf{N}} \mid \phi_j(x) \text{ has a positive convergent radius } \rho_j > 0 \right\},$$

$$V_j := \left\{ (x,y) \in N_j \mid y = \phi_j(x) \text{ on } \Delta_{\rho_j} \right\} \quad \text{for all } j \in J.$$

Then, we have the following Theorem 2.

Theorem 2 (see [5]). *$\{V_j\}_{j\in J}$ is a family of locally invariant holomorphic curves at p by F. In particular, every family $\{W_\lambda\}_{\lambda\in\Lambda}$ of locally invariant holomorphic curves at p by F must be a subfamily of $\{V_j\}_{j\in J}$.*

As applications, consider the following rational mappings of $\mathbf{C}^2$.

$$(*1)\quad F(x,y) = \left(ax, \frac{y(y-x)}{x^2} \right), \quad |a| > 4,$$

$$(*2)\quad F(x,y) = \left(x + ax^2, \frac{y(2y-x)}{x^2} \right), \quad |a| \neq 0.$$

Theorem 3 (see [6]). *Suppose that F is the rational mapping in $(*1)$. For all symbol sequences $j = (j_1, j_2, \ldots) \in \{1,2\}^{\mathbf{N}}$, one of the following claims holds.*
(1) If there exists an integer n_0 such that $j_n = 1$ for any $n \ge n_0$, then $V_j \neq \emptyset$ and $V_j \subset F^{-n_0}(V_{11\cdots}) = F^{-n_0}(\{y=0\})$. Especially, V_j are unstable manifolds of p.
(2) If there exist infinitely many $n_0 \in \mathbf{N}$ with $j_{n_0} = 2$, then $V_j = \emptyset$.

For the rational mapping F in $(*2)$, the following theorems 4 and 5 hold.

Theorem 4. *For every symbol sequence* $j \in \{1,2\}^{\mathbf{N}}$ *there exists a continuous function* $y = \psi_j(x)$ *on* Δ_δ. *Put*

$$W_j := \left\{(x,y) \in \mathbf{C}^2 \mid y = \psi_j(x) \text{ on } \Delta_\delta\right\}.$$

In particular, $\{W_j\}_{j\in\{1,2\}^{\mathbf{N}}}$ *is a family of curves which is locally invariant at* p *by* F.

Theorem 5. *For any symbol sequence* $j \in \{1,2\}^{\mathbf{N}}$, *there exists* $j' = (j'_1, j'_2, \dots) \in \{1,2\}^{\mathbf{N}}$ *such that the formal power series* $\phi_{j'}(x) = \sum \alpha_{j'_1\dots j'_n} x^n$ *is the asymptotic expansion of* $\psi_j(x)$. *That is, for all* $n \in \mathbf{N}$, *there exist positive constants* δ_n *and* M_n *such that*

$$\left|\psi_j(x) - \alpha_{j'_1}x - \cdots - \alpha_{j'_1\dots j'_{n-1}}x^{n-1}\right| \le M_n|x|^n,$$

for any $x \in \Delta_{\delta_n}$.

Remark. Although $\phi_{j'}$ may not be a convergent power series, for any fixed $n \in \mathbf{N}$, $\psi_j(0)$ is approximated by the polynomial $\alpha_{j'_1}x + \cdots + \alpha_{j'_{n-1}}x^{n-1}$ with the order $O(|x|^n)$ by taking the limit as $x \to 0$.

Theorems 1, 2 and 3 have been obtained by [5] and [6]. In this note, we will give an outline of proof of Theorems 4 and 5.

2. Proof of Theorem 4.

Put $q(x) := x + ax^2$. This is the first component of F in (*2). We begin with basic facts on dynamics of the polynomial $q(x)$ at $x = 0$ (for detail, see [7]). For the polynomial $q(x)$, $x = 0$ is a rationally indifferent fixed point and there exist an *attracting petal* P and a *repelling petal* R such that

(1) $q(\overline{P}) \subset P \cup \{0\}$, $\bigcap_{n=1}^{\infty} (\overline{P}) = \{0\}$,

(2) $(q|_R)^{-1}(\overline{R}) \subset R \cup \{0\}$, $\bigcap_{n=1}^{\infty} (q|_R)^{-n}(\overline{R}) = \{0\}$,

(3) $\{0\} \cup P \cup R$ is an open neighborhood of 0.

Now, let us start the proof of Theorem 4. In the following part, we shall give a proof which is based on an argument by Hadamard-Perron Theorem in [3] and the construction of the Cantor bouquet in [8].

For $\alpha_j \in \mathbf{C}$, $p = (0,0) \in \mathbf{C}^2$ and $p_j := (0, \alpha_j) \in \mathbf{C}^2$, define the following sets;

$$\Delta_r(\alpha_j) := \{x \in \mathbf{C} \mid |x - \alpha_j| < r\}, \quad \Delta_r := \Delta_r(0),$$

$$\Delta_r^2(p) := \Delta_r \times \Delta_r, \quad \Delta_r^2(p_j) := \Delta_r \times \Delta_r(\alpha_j).$$

From some easy calculation, one can check that our F satisfies the conditions $(A.0)$ and (B). Hence, Theorems 1 and 2 hold and for any infinite symbol sequence $j \in \{1,2\}^{\mathbf{N}}$, there exists the sequence of points $\{\alpha_{j_1 \ldots j_n}\}_{n \geq 1}$.

In the reminder of this note, denote $k, l = 1, 2$. From $(A.0)$, $\tilde{F}$ is a locally biholomorphic mapping on some neighborhoods of p_l, and there are positive constants r and r' and branches $G_l : \Delta_r^2(p) \to \Delta_{r'}^2(p_l)$ of $\tilde{F}$. Let $\rho : \mathbf{C}^2 \to [0,1]$ be a C^1-function such that

$$\rho(x,y) = \begin{cases} 1 & \text{on } \Delta_r^2(p_k) \\ 0 & \text{on } \Delta_{2r}^2(p_k)^c. \end{cases}$$

By using this C^1-function ρ, define a C^1-mapping $f_{kl} : \mathbf{C}^2 \to \mathbf{C}^2$ such that

$$f_{kl} = \begin{cases} G_l \circ \pi & \text{on } \Delta_r^2(p_k) \\ J(G_l \circ \pi)_{p_k} & \text{on } \Delta_{2r}^2(p_k)^c, \end{cases}$$

where $J(G_l \circ \pi)_{p_k}$ is the Jacobian matrix of $G_l \circ \pi$ at the point p_k. Set

$$C_\gamma^{p_k} := \{\phi : \mathbf{C} \to \mathbf{C},\ \text{Lipshitz ft. with Lipshitz constant } \gamma \text{ and } \phi(0) = \alpha_k\},$$

$$C_\gamma := C_\gamma^{p_1} \cup C_\gamma^{p_2}.$$

Then, C_γ is a complete metric space with respect to the metric d defined as follows;

$$d(\phi, \psi) := \begin{cases} \sup_{x \in \mathbf{C} \setminus \{0\}} \frac{|\phi(x) - \psi(x)|}{|x|} & \text{if } \phi, \psi \in C_\gamma^{p_k} \\ 3 & \text{if } \phi \in C_\gamma^{p_k} \text{ and } \psi \in C_\gamma^{p_l}\ (k \neq l). \end{cases}$$

It can be seen that for any $\phi \in C_\gamma^{p_k}$ there exists $\psi \in C_\gamma^{p_l}$ such that

$$f_{kl}(\text{graph } \phi) = \text{graph } \psi.$$

By using this fact, one can define the action of g_l on C_γ by

$$g_l(\text{graph } \phi) := \text{graph}\left((f_{kl})_* \phi\right), \quad \text{if } \phi \in C_\gamma^{p_k}$$

and know that $g_l : C_\gamma \to C_\gamma^{p_l}$ is a contraction mapping.

Let S be the space of non-empty compact subsets of C_γ. Then, S is a complete metric space with respect to the Hausdorff metric. Setting a mapping

$$G : S \to S, \quad \text{by} \quad A \mapsto G(A) := g_1(A) \cup g_2(A)$$

we can show that G is contraction on S, since g_l is a contraction mapping.

Thus, it follows from Banach's contraction mapping theorem that G has the unique fixed point $E \in S$, and $G^n(A)$ converges to E for any $A \in S$. Here, we choose a subset A of S satisfying $g_l(A) \subset A$ for $l = 1, 2$. Then

$$\bigcap_{n=0}^{\infty} G^n(A) = E.$$

Consequently, since $g_1(A) \cap g_2(A) = \emptyset$, there exists the unique point $\tilde{\phi}_j \in C_\gamma$ such that $g_{j_1} \circ \cdots \circ g_{j_n}(A) \to \tilde{\phi}_j$ $(n \to \infty)$ for every symbol sequence $j \in \{1, 2\}^{\mathbf{N}}$.

By using $\tilde{\phi}_j$, let us set

$$\tilde{W}_j := \left\{(x, y) \in \mathbf{C}^2 \mid y = \tilde{\phi}_j(x)\right\}.$$

Then, it implies that $g_l(\tilde{W}_j) = \tilde{W}_{\sigma(j)}$, where σ is the shift mapping on $\{1, 2\}^{\mathbf{N}}$. Take a small positive constant δ with $0 < \delta < r$, and put

$$\tilde{W}_j^\delta := \tilde{W}_j \cap \Delta_\delta \times \mathbf{C} \quad \text{and} \quad W_j := \pi(\tilde{W}_j^\delta).$$

Finarlly, we can prove that

$$W_j = \left\{(x, y) \in \mathbf{C}^2 \mid y = x\tilde{\phi}_j(x)\right\}$$

and $\{W_j\}_{j \in \{1,2\}^{\mathbf{N}}}$ is a family of curves which is locally invariant at p by F. This is required.

Remark. Unfortunatelly, $\tilde{\phi}_j$ depends on the construction of an extension mapping f_{kl} and does not have uniqueness. However, $\tilde{\phi}_j(x)$ is determined uniquely for any $x \in P$, where P is an attracting petal of $q(x)$ at 0, and

$$F^n(x, y) \to p \text{ as } n \to \infty \quad \text{for any } (x, y) \in W_j \cap \{P \times \mathbf{C}\} \text{ with } x \neq 0.$$

3. Proof of Theorem 5.

To prove Theorem 5, we need the following Lemmas 1 and 2.

Lemma 1. For every symbol sequence $j \in \{1,2\}^{\mathbf{N}}$ the following claims hold;
(1) there exists a point $p_{j_1'} \in \{p_1, p_2\}$ such that $\overline{\pi^{-1}(W_j \setminus \{p\})} \cap E = \{p_{j_1'}\}$, and put

$$(W_j)_{j_1'} := \overline{\pi^{-1}(W_j \setminus \{p\})},$$

(2) there exists a continuous function $\phi_{j_1'}$ on Δ_δ such that

$$(W_j)_{j_1'} = \left\{(x,y) \in \Delta_\delta \times \mathbf{C} \mid y = \phi_{j_1'}(x) \text{ on } \Delta_\delta\right\}.$$

Since $\{W_j\}_{j\in\{1,2\}^{\mathbf{N}}}$ is a family of curves which is locally invariant at p by F, for every W_j there exists a symbol sequence $i = (i_1, i_2, \ldots) \in \{1,2\}^{\mathbf{N}}$ and an open neighborhood N_i of p such that $F(W_j \setminus \{p\}) \cap N_i \subset W_i$. From Lemma 1 (1), there is a point $p_{i_1'} \in \{p_1, p_2\}$ such that $\overline{\pi^{-1}(W_i \setminus \{p\})} \cap E = \{p_{i_1'}\}$. Put

$$(W_i)_{i_1'} := \overline{\pi^{-1}(W_i \setminus \{p\})} \quad \text{and} \quad F_0 := \pi^{-1} \circ \tilde{F}.$$

Then, we have the following lemma.

Lemma 2.
(1) There exists an open neighborhood $(N_i)_{i_1'}$ of $p_{i_1'}$ such that

$$\lim_{x\to 0} F_0(x, \phi_{j_1'}(x)) = p_{i_1'} \quad \text{and} \quad F_0\left((W_j)_{j_1'} \setminus \{p_{j_1'}\}\right) \cap (N_i)_{i_1'} \subset (W_i)_{i_1'}$$

(2) There exist positive constants δ_{j_1} and M_{j_1} such that

$$|\psi_j(x) - \alpha_{j_1'} x| \le M_{j_1} \quad \text{for} \quad x \in \Delta_{\delta_{j_1}}.$$

We can repeat this process inductively for every $n \in \mathbf{N}$ and prove Theorem 5.

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