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# On biaccessible points in the Julia set of the family $z(a+z^d)$

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**Abstract**

We are interested in biaccessibility in the Julia sets of polynomials with Cremer fixed points. In this paper, we consider $f_a(z)=z(a+z^d)$ where the origin is a Cremer fixed point.

D. Schleicher and S. Zakeri studied which points are biaccessible when $d=1$ [**SZ**]. We consider when $d\geq 1$.

## 1 Preliminaries

In this paper, we set $f_a(z)=z(a+z^d)$ for some $d$ greater than or equal to one. For each $0\leq j\leq d-1$, let $\tau_j(z)=e^{2\pi i\frac{j}{d}}z$ be a $\frac{j}{d}$-rotation. Now $f_a$ has $\tau_j$-symmetric critical points $c_j=\tau_j(c)$, where $c$ is one of the solutions of $a+(d+1)z^d=0$.

Recall that the *filled Julia set* of $f_a$ is

$$K_a=\{z\in\mathbb{C}:\{f_a^{\circ n}(z)\}_{n\geq 0}\text{ is bounded}\}$$

and the *Julia set* of $f_a$ is $J_a=\partial K_a$. Then $f_a\circ\tau_j=\tau_j\circ f_a$ implies $\tau_j(K_a)=K_a$ and thus $\tau_j(J_a)=J_a$.

Now assume that the filled Julia set $K_a$ is connected. Then there exists a unique conformal isomorphism:

$$\psi:\mathbb{C}-\overline{\mathbb{D}}\to\mathbb{C}-K_a$$

such that $\frac{\psi(z)}{z}\to 1$ as $z\to\infty$.

Here it is important that the following holds [**Mi**, Theorem 9.5]:

$$f_a(\psi(z)) = \psi(z^{d+1}). \qquad (*)$$

We say $R_t = \{\psi(re^{2\pi it}) : 1 < r\}$ is the *external ray* with *angle* $t \in \frac{\mathbb{R}}{\mathbb{Z}}$. Then $(*)$ implies $f_a(R_t) = R_{(d+1)t}$. In addition, $\tau_j(K_a) = K_a$ implies $\tau_j \circ \psi = \psi \circ \tau_j$ and thus $\tau_j(R_t) = R_{t+\frac{j}{d}}$.

If $\lim_{r\searrow 1} \psi(re^{2\pi it}) = z \in J_a$, then we say that the external ray $R_t$ *lands* at $z$. If there exist two distinct rays landing at $z \in J_a$, then we say that $z$ is a *biaccessible point*. By a thorem of F. and M. Riesz [**Mi**], the point $z$ is a *cut point* of the Julia set $J_a$, namely $J_a - \{z\}$ is disconnected.

# 2 Some known results

Very little is known about the topology of the Julia set and the dynamics of polynomials with Cremer fixed points. We have the following results:

- If the origin is a Cremer fixed point, then the Julia set $J_a$ cannot be locally connected [**Mi**, Corollary 18.6].
- For a generic choice of $|a| = 1$, the origin has the small cycles property, and therefore is a Cremer fixed point [**Mi**, Theorem 11.13].
- If the origin has the small cycles property, then all critical points $c_j$ cannot be accessible from outside of the Julia set $J_a$ [**Ki**, Theorem 1.1].

Other results about the semi-local dynamics around Cremer fixed points are refered to [**PM**]. The following theorem was proved by Pérez-Marco [**PM**, Theorem 1]:

**Theorem 2.1.** *Let $f(z) = az + \mathcal{O}(z^2)$ be a local holomorphic diffeomorphism. Assume that the origin is a Cremer fixed point. Let $U$ be a Jordan neighborhood of the origin. Assume that $f$ is defined and univalent on a neighborhood of $\overline{U}$. Then there exists a set $H$ such that:*

- *$H$ is compact, connected and full;*
- *$0 \in H \subset \overline{U}$;*
- *$H \cap \partial U \neq \phi$;*
- *$f(H) = H$.*

In addition, the following holds [**SZ**, Proposition 2]:

**Proposition 2.1.** *Assuming the hypothesis in the above theorem, let $H$ be a set given by that theorem. The only point in $H$ which can be a cut point of $H$ is the Cremer fixed point* $0$.

# 3 Main result

Using the preceding results and the following lemma, we can show Theorem 3.1. The method of proof is similar to that of Theorem 3.2.

**Lemma 3.1.** *Assume that the origin is a Cremer fixed point. Assume that* $z$ *is a biaccessible point such that* $0 \notin \{f_a^{\circ n}(z)\}_{n\geq 0}$ *and* $c_j \notin \{f_a^{\circ n}(z)\}_{n\geq 0}$ *for all* $j$. *Then for each* $j$ *there exist two distinct rays* $R_{s_j}$ *and* $R_{t_j}$ *with a common landing point* $w_j$, *such that* $R_{s_j} \cup \{w_j\} \cup R_{t_j}$ *separates* $c_j$ *from the origin.*

**Theorem 3.1.** *Assume that the origin is a Cremer fixed point. Assume that* $z$ *is a biaccessible point. Then* $0 \in \{f_a^{\circ n}(z)\}_{n\geq 0}$ *or there exists* $j_0$ *such that* $c_{j_0} \in \{f_a^{\circ n}(z)\}_{n\geq 0}$.

**Remark 3.1.** In the above theorem, if the origin has the small cycles property, then $c_j \notin \{f_a^{\circ n}(z)\}_{n\geq 0}$ for all $j$ [**Ki**, Theorem 1.1]. Therefore, the conclusion is just $0 \in \{f_a^{\circ n}(z)\}_{n\geq 0}$.

Finaly, we make mention of the theorem in [**SZ**].

**Theorem 3.2.** *Let* $f_a(z) = z(a+z)$ *be a quadratic polynomial. Assume that the origin is a Cremer fixed point. Assume that* $z$ *is a biaccessible point. Then* $0 \in \{f_a^{\circ n}(z)\}_{n\geq 0}$.

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