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Rigidity of Riemann surface laminations associated with infinitely renormalizable quadratic maps *

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Abstract

In this note we describe the well studied process of renormalization of quadratic polynomials from the viewpoint of their associated Riemann surface laminations. The main result is that, when an infinitely renormalizable quadratic map has a-priori bounds, the topology of the lamination is rigid modulo its combinatorial equivalence. This is a joint work with C. Cabrera (University of Warwick).

1 Renormalization and its combinatorics

Quadratic-like maps. Let U and V be topological disks in $\mathbb{C}$ with U compactly contained in V. A *quadratic-like map* $g : U \to V$ is a proper holomorphic map of degree two. The *filled Julia set* is defined by $K(g) := \bigcap_{n\geq 1} g^{-n}(V)$.

In this note we assume that any quadratic-like map $g : U \to V$ has a connected $K(g)$; equivalently, the forward orbit of the critical point is contained in $K(g)$. [1] We define the *postcritical set* $P(g)$ by the closure of the forward orbit of the critical point.

By the Douady-Hubbard straightening theorem, there exists a unique $c = c(g) \in \mathbb{C}$ and a quasiconformal map $h : V \to V'$ such that h conjugates $g : U \to V$ to $f_c : U' \to V'$ where $U' = h(U) = f_c^{-1}(V')$ and $\bar{\partial} h = 0$ a.e. on $K(g)$. The quadratic map f_c is called the *straightening* of g and h is called a *straightening map.* Though such an h is not uniquely determined, we always assume that any quadratic-like map g is accompanied by one fixed straightening map $h = h_g$.

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[1] Note that any quadratic-like map g has only one simple (= local degree two) critical point in U.

Renormalization of quadratic maps. A quadratic-like map $g : U \to V$ is said to be *renormalizable*, if there exist a number $m > 1$, called the *order of renormalization*, and two open sets $U_1 \subset U$ and $V_1 \subset V$ containing the critical point of g, such that $g_1 = g^m | U_1 \to V_1$ is again a quadratic-like map with connected filled Julia set $K_1 := K(g_1)$.

We also assume that m is the minimal order with this property and that K_1 has the following property: *For any* $1 \leq i < j \leq m$, $g^i(K_1) \cap g^j(K_1)$ *is empty or just one point that separates neither* $g^i(K_1)$ *nor* $g^j(K_1)$. (Such a renormalization is called *simple* or *non-crossing.*)

Superattracting parameters associated with renormalizations. For any (simple) renormalization $g_1 = g^m : U_1 \to V_1$ of $g : U \to V$, the combinatorial property of g_1 within the dynamics of g is represented by a uniquely determined superattracting quadratic map $f_s(z) = z^2 + s$ with $s = s(g, g_1)$ and $f_s^m(0) = 0$. (Roughly put, the dynamics of g is given by the dynamics of f_s with its m periodic Fatou components replaced by m small copies of $K_1 = K(g_1)$.)

More precisely, we can determine $s(g, g_1)$ as follows: We can define the β-fixed point β_1 of quadratic-like map g_1 (not g) by pulling back the β-fixed point (the landing point of the external ray of angle 0) of quadratic map f_{c_1} with $c_1 = c(g_1)$ via straightening map $h_1 = h_{g_1}$. Then the forward orbit of β_1 by the dynamics of g gives a repelling or parabolic cycle O. Next by the straightening map $h = h_g$, we can send the cycle O to the cycle $h(O)$ of f_c with $c = c(g)$. The set of angles of external rays that land on $h(O)$ is called the *ray portrait* of $h(O)$. There is a fact that the ray portrait determines a unique superattracting parameter s such that the boundaries of the periodic Fatou components of f_s contain a repelling cycle O_s with the same orbit portrait as $h(O)$. Now we define $s(g, g_1)$ by this s. (Conversely, superattracting parameter s uniquely determines such an orbit portrait. See Milnor's [6])

Example. The diagram in the left of Figure 1 shows a quadratic-like map g_1 as a renormalization of $g = f_c$ with $c \approx -0.1539 + 1.0377i$. (In this case we regard g as a restriction of the quadratic map f_c on a large disk.)

The β-fixed point of g_1 is the landing point of the external rays of angles 2/15 and 9/15 for $g = f_c$. In this case the orbit portrait is

$$\{\{9/15, 2/15\},\ \{3/15, 4/15\},\ \{6/15, 8/15\},\ \{12/15, 1/15\}\}.$$

The diagram in the right shows the corresponding superattracting dynamics f_s with $s = s(g, g_1) \approx -0.15652 + 1.03225i$, which satisfies $f_s^4(0) = 0$.

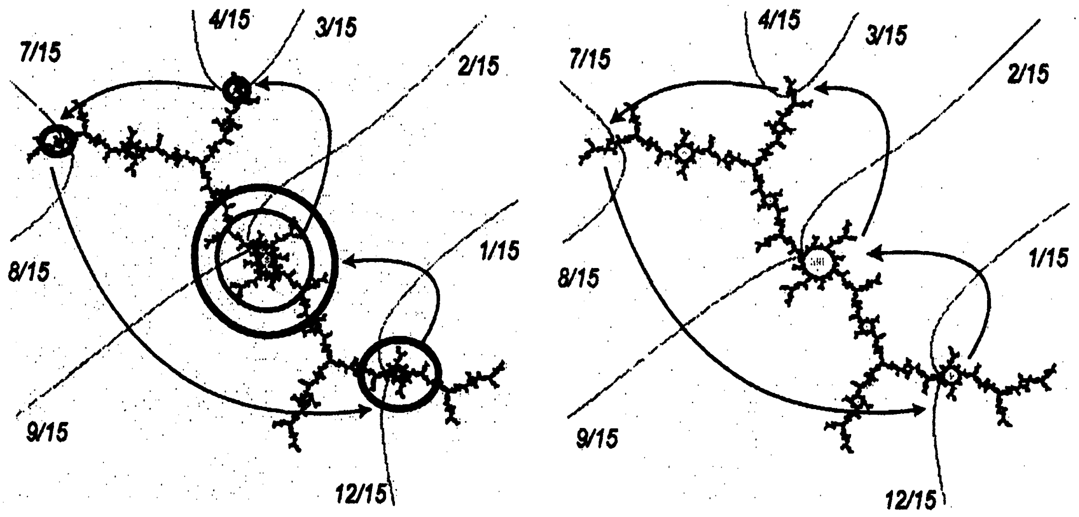

Figure 1: Any simple renormalization determines a unique superattracting parameter.

Infinitely renormalizable maps and its combinatorics. We say f_c is *infinitely renormalizable* if there is an infinite sequence of numbers $p_0 = 1 < p_1 < p_2 < \cdots$ and two sequences of open sets $\{0 \in U_n\}$ and $\{V_n\}$ such that each $g_n = f_c^{p_n} : U_n \to V_n$ is a quadratic-like map, with the property that $g_n'(0) = 0$ and g_{n+1} is a simple renormalization of g_n of order $m_n := p_{n+1}/p_n > 1$. The index n of g_n is called the *level* of renormalization.

For such an f_c, the sequence $\{g_n : U_n \to V_n\}_{n \geq 0}$ uniquely determines the infinite sequence of superattracting parameters $\{s_0, s_1, s_2, \ldots\}$ given by $s_n = s(g_n, g_{n+1})$. We denote the sequence $\sigma(c)$ and call it the *combinatorics* of f_c.

For example, the Feigenbaum parameter $c = -1.4011552..$ has combinatorics $\sigma(c) = \{-1, -1, -1, \ldots\}$, since every level of renormalization determines the superattracting dynamics by $f_{-1}(z) = z^2 - 1$.

2 Inverse limits, natural extensions, and regular parts

Inverse Limits. Consider $\{f_{-n} : X_{-n} \to X_{-n+1}\}_{n=1}^{\infty}$, a sequence of d-to-1 branched covering maps on the manifolds X_{-n} with the same dimension. The *inverse limit* of this sequence is defined as

$$\varprojlim(f_{-n}, X_{-n}) := \{\hat{x} = (x_0, x_{-1}, x_{-2} \ldots) \in \prod_{n\geq 0} X_{-n} : f_{-n}(x_{-n}) = x_{-n+1}\}.$$

The space $\varprojlim(f_{-n}, X_{-n})$ has a *natural topology* which is induced from the product topology in $\prod X_{-n}$. The projection $\pi : \varprojlim(f_{-n}, X_{-n}) \to X_0$ is defined by $\pi(\hat{x}) := x_0$.

Example 1: Natural extensions of quadratic maps. When all the pairs (f_{-n}, X_{-n}) coincide with the quadratic $(f_c, \overline{\mathbb{C}})$, following Lyubich and Minsky [5], we will denote $\varprojlim(f_c, \bar{\mathbb{C}})$ by $\mathcal{N}_c$. The set $\mathcal{N}_c$ is called the *natural extension* of f_c. In this case we denote the projection by $\pi_c : \mathcal{N}_c \to \overline{\mathbb{C}}$. There is a natural homeomorphic action $\hat{f}_c : \mathcal{N}_c \to \mathcal{N}_c$ given by $\hat{f}_c(z_0, z_{-1}, \ldots) := (f_c(z_0), z_0, z_{-1}, \ldots)$. Then π_c semiconjugates the action of $\hat{f}_c : \mathcal{N}_c \to \mathcal{N}_c$ to $f_c : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

Example 2: Dyadic solenoid and solenoidal cones. A well-known example of an inverse limit is the *dyadic solenoid* $\mathcal{S}^1 := \varprojlim(f_0, \mathbb{S}^1)$, where $f_0(z) = z^2$ and $\mathbb{S}^1$ is the unit circle in $\mathbb{C}$. The dyadic solenoid is a connected set but is not path-connected. Any space homeomorphic to $\varprojlim(f_0, \overline{\mathbb{C}} - \overline{\mathbb{D}})$ will be called a *solenoidal cone.* For f_c with connected filled Julia set $K(f_c)$, we have an important example of a solenoidal cone $\varprojlim(f_c, \overline{\mathbb{C}} - K(f_c))$ in $\mathcal{N}_c$ by looking at $\varprojlim(f_0, \overline{\mathbb{C}} - \overline{\mathbb{D}})$ through the inverse Böttcher coordinate $\psi_c^{-1} : \overline{\mathbb{C}} - \overline{\mathbb{D}} \to \overline{\mathbb{C}} - K(f_c)$. One can also find a solenoidal cone in any neighborhood of $\hat{\infty} = \{\infty, \infty, \infty\}$ in $\mathcal{N}_c$.

Example 3: Quadratic-like inverse limits. Let $g : U \to V$ be a quadratic-like map. By $\varprojlim(g, V)$ we denote the inverse limit for the sequence

$$\cdots \to g^{-2}(V) \to g^{-1}(V) = U \to V.$$

By using the Douady-Hubbard straightening, it is not difficult to prove this

Proposition 1. *Let $g : U \to V$ be a quadratic-like map with straightening $f_c(z) = z^2 + c$. Then the inverse limit $\varprojlim(g, V)$ is homeomorphic to $\mathcal{N}_c$ with a compact solenoidal cone at infinity removed.*

Regular parts of quadratic natural extensions. Let f_c be a quadratic map. A point $\hat{z} = (z_0, z_{-1}, \ldots)$ in the natural extension $\mathcal{N}_c = \varprojlim(f_c, \overline{\mathbb{C}})$ is *regular* if there is a neighborhood U_0 of z_0 such that the pull-back of U_0 along $\hat{z}$ is eventually univalent. The *regular part*(or *regular leaf space*) $\mathcal{R}_{f_c} = \mathcal{R}_c$ is the set of regular points in $\mathcal{N}_c$. Let $\mathcal{I}_{f_c} = \mathcal{I}_c$ denote the set of irregular points.

The regular part is analytically well-behaved part of the natural extensions. For example, it is known that all path-connected components ("*leaves*") of $\mathcal{R}_c$ are isomorphic to $\mathbb{C}$ or $\mathbb{D}$. Moreover, $\hat{f}_c$ sends leaves to leaves isomorphically. However, most of such leaves are wildly foliated in the natural extension, indeed dense in $\mathcal{N}_c$. See [5, §3] for more details.

Example: Regular part of superattracting maps. Let f_s be a superattracting quadratic map with superattracting cycle

$$\{\alpha_s(1), \ldots, \alpha_s(m) = 0\}.$$

Under the homeomorphic action $\hat{f}_s : \mathcal{N}_s \to \mathcal{N}_s$, the points

$$\hat{\alpha}_s(i) \ := \ (\alpha_s(i), \alpha_s(i-1), \alpha_s(i-2), \ldots)$$

form a cycle of period m. In this case, the set $\mathcal{I}_s$ of irregular points consists of $\{\hat{\infty}, \hat{\alpha}_s(1), \ldots, \hat{\alpha}_s(m)\}$. Thus the regular part $\mathcal{R}_s$ is $\mathcal{N}_s$ minus these $m+1$ irregular points. Moreover, it is known that $\mathcal{R}_s$ is a Riemann surface lamination with all leaves isomorphic to $\mathbb{C}$.

3 Main Results

Regular part of infinitely renormalizable maps. An infinitely renormalizable f_c is said to have *a-priori bounds* if there exist $\eta > 0$, independent of n, such that $mod(V_n \setminus U_n) > \eta$. In this case the nested domains of infinite renormalizations nicely shrink and the "remained" postcritical set $P(f_c)$ is a Cantor set.

The following is due to Kaimanovich and Lyubich [3] [2]:

Theorem 2 (Riemann surface lamination). *If f_c has a-priori bounds, then $\mathcal{R}_c$ is a locally compact Riemann surface lamination, whose leaves are conformally isomorphic to planes.*

The local compactness is important when we consider its *end compactification* in the proof of Theorem 4. It is also known that there exist quadratic maps with locally non-compact regular parts.

In addition to the theorem above, we can show that such an $\mathcal{R}_c$ can be decomposed into "blocks" which are given by combinatorics determined by the sequence of renormalization:

Theorem 3 (Structure Theorem, [2]). *Let f_c be infinitely renormalizable with a priori bounds and $\{g_n = f_c^{p_n}|\ U_n \to V_n\}_{n\geq 0}$ be the associated sequence of renormalizations with combinatorics $\sigma(c) = \{s_0, s_1, \ldots\}$. Set $m_n := p_{n+1}/p_n$. Then there exist disjoint open subsets $\mathcal{B}_0, \mathcal{B}_1, \ldots$ of $\mathcal{N}_c$ such that:*

1. *For $n = 0$, the set $\mathcal{B}_0$ is homeomorphic to $\mathcal{R}_{s_0}$ with the closure of small solenoidal cones near $\mathcal{I}_{s_0} - \{\hat{\infty}\}$ removed.*
2. *For each $n \geq 1$, the set $\mathcal{B}_n$ is homeomorphic to $\mathcal{R}_{s_n}$ with the closure of small solenoidal cones near $\mathcal{I}_{s_n}$ removed.*
3. *For any $n \geq 1$ and $1 \leq i < j \leq p_n$, the sets $\hat{f}_c^i(\mathcal{B}_n)$ and $\hat{f}_c^j(\mathcal{B}_n)$ are disjoint.*
4. *For $0 \leq n < n'$, the closures $\overline{\mathcal{B}_n}$ and $\overline{\mathcal{B}_{n'}}$ intersects iff $n' = n+1$. In this case, for all $0 \leq i \leq m_n - 1$ the closures $\overline{\hat{f}_c^{p_n i}(\mathcal{B}_{n+1})}$ and $\overline{\mathcal{B}_n}$ share just one of their solenoidal boundary components.*
5. *The set $\mathcal{B}_0 \cup \bigcup_{n=1}^{\infty} \bigcup_{i=0}^{p_n - 1} \overline{\hat{f}_c^i(\mathcal{B}_n)}$ is equal to the regular part $\mathcal{R}_c$.*
6. *The original natural extension is given by $\mathcal{N}_c = \mathcal{R}_c \sqcup \widehat{P(f_c)} \sqcup \{\hat{\infty}\}$, where $\widehat{P(f_c)}$ is the set of the backward orbits remain in the postcritical set $P(f_c)$.*

[2] This theorem and Theorem 3 are originally proved under *persistent recurrence*; that is, for any neighborhood U_0 of $z_0 \in$ (the postcritical set) and any backward orbit $\hat{z} = (z_0, z_{-1}, \ldots)$, the pull-backs of U_0 along z_0 contains the critical point $z = 0$. This condition is much weaker.

See Figure 2. The open sets $\mathcal{B}_n$, $\hat{f}_c(\mathcal{B}_n), \cdots, \hat{f}_c^{p_n-1}(\mathcal{B}_n)$ form the "block" of level n. The theorem says that the regular part $\mathcal{R}_c$ is a tree-like structure which consists of the blocks $\left\{\hat{f}_c^i(\mathcal{B}_n) : n \geq 0, 0 \leq i < p_n\right\}$. One may compare this tree-like object with the Riemann surface $\mathbb{C} - P(f_c)$, where the postcritical set $P(f_c)$ is a Cantor set. They have the same configuration.

A remarkable fact is that, for a superattracting parameter s, the object "$\mathcal{R}_s$ with the closure of small solenoidal cones near $\mathcal{I}_s - \{\hat{\infty}\}$ (or $\mathcal{I}_s$) removed" is rigid; i.e., if such objects for s and s' are homeomorphic, then $s = s'$. (See [1].) Thus we may say that $\mathcal{B}_n$ are the "rigid blocks".

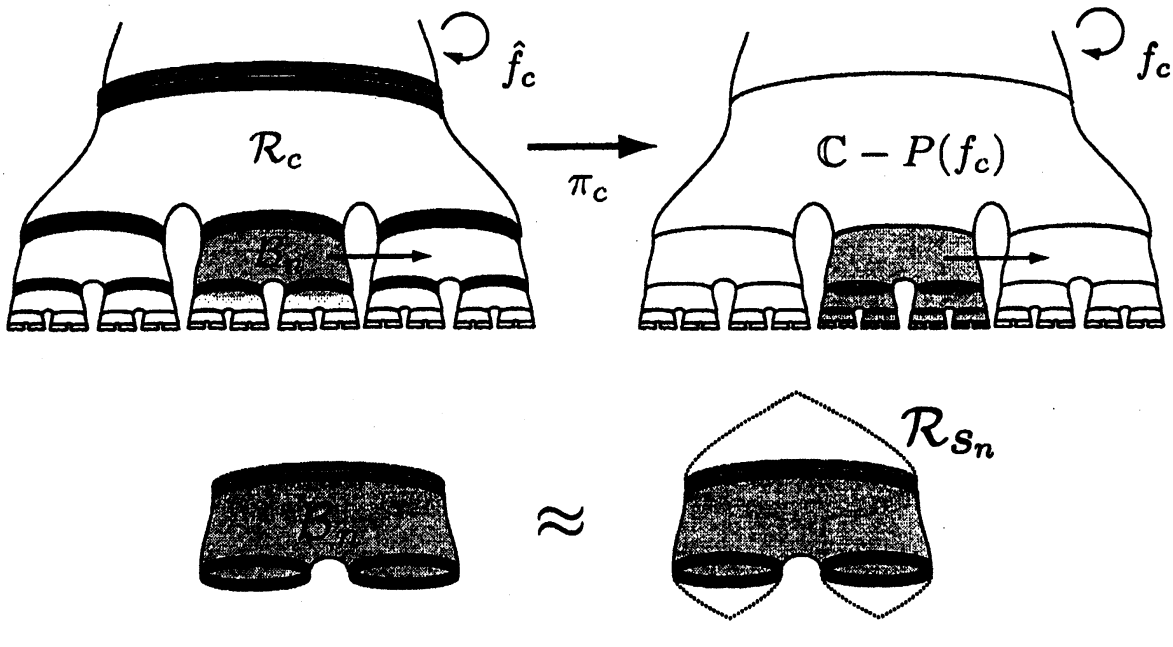

Figure 2: A caricature of blocks in $\mathcal{R}_c$.

Remark also that the statement of Theorem 3 is quite topological. For instance, the block $\mathcal{B}_n$ which we will construct may not be an invariant set of $\hat{f}_c^{p_n}$. Nevertheless, we can prove that the topology of $\mathcal{R}_c$ given by such blocks determines the original dynamics modulo combinatorial equivalence:

Theorem 4 (Rigidity up to combinatorial equivalence, [2]). *Let c be a non-real complex number, such that the map f_c is infinitely renormalizable with a-priori bounds. If there exists an orientation preserving homeomorphism $h : \mathcal{R}_c \to \mathcal{R}_{c'}$, then c and c' belong to the same combi-*

natorial class; i.e., $\sigma(c) = \sigma(c')$.

This implies that the topology of the regular part determines the combinatorics of the map. It is conjectured that for any infinitely renormalizable c and c', if $\sigma(c) = \sigma(c')$ then $c = c'$. (=Rigidity Conjecture, see Lecture 4 of [4].). So the topology of the regular part may determine the map itself. Note that the topology of the Riemann surface $\mathbb{C} - P(f_c)$ does not determine the combinatorics of the map. In fact, $\mathbb{C}$ minus a Cantor set always has the same topology.

From the viewpoint of the parameter plane, it is known that c is combinatorially rigid if and only if the Mandelbrot set is locally connected at c. So we have the following

Corollary 5. *Assume that c is as in the Main Theorem and that the Mandelbrot set is locally connected (MLC) at c, then $c = c'$.*

Lyubich proved MLC for f_c with a-priori bounds with some extra condition on combinatorics, called secondary limb condition. In this direction, there is recent work by Kahn and Lyubich where they prove a-priori bounds and MLC for infinite renormalizable parameters with special combinatorics.

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