数理解析研究所講究録
第 1588 巻 2008 年 30-45

On Time Local Solvability for the Motion of an Unbounded Volume of Viscous Incompressible Fluid

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1 Introduction

We consider the system of equations describing the motion of the viscous incompressible fluid which occupies an unbounded domain without taking into account surface tension. For given initial fluid domain $\Omega \equiv \Omega(0) \subset \mathbb{R}^3$ with its boundary $\{F_0(x)=0\}$, the initial velocity field $u_0(x)$ in Ω, and an outer force $f=f(x,t)$ defined in $\mathbb{R}^3 \times [0,\infty]$, we would like to know the domain $\Omega(t)$, $t>0$ occupied by the fluid, with its free boundary $S_F(t)=\{F\equiv F(x,t)=0\}$, $t>0$, the velocity field $u=u(x,t)=(u^1(x,t),u^2(x,t),u^3(x,t))$ and the pressure $q=q(x,t)$ satisfying the following system as

$$
\begin{aligned}
\frac{Du}{Dt} = \nabla\cdot T(u,q)+f, \quad \nabla\cdot u = 0 \qquad & \text{for } (x,t)\in\Omega(t)\times(0,\infty),\\
u|_{t=0}=u_0(x) \qquad & \text{for} \quad x\in\Omega,\\
Tn = 0 \qquad & \text{for } (x,t)\in S_F(t)\times(0,\infty),\\
\frac{DF}{Dt}=0 \qquad & \text{for } (x,t)\in S_F(t)\times(0,\infty),\\
F|_{t=0}=F_0(x) \qquad & \text{for} \quad x\in\Omega.
\end{aligned}
$$

Here, $\dfrac{D}{Dt}=\dfrac{\partial}{\partial t}+u\cdot\nabla$ is the material derivative, $\nabla=\nabla_x=(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3})$, and $\nabla\cdot T=\operatorname{div}T=(\sum_{j=1}^{3}\partial_j T_{ij})_i$ with the stress tensor $T=T(u,q)=-q\mathrm{I}+\mu D$. The tensor $D=D(u)$ is the deformation of the velocity with elements $D_{ij}=\partial u^i/\partial x_j+\partial u^j/\partial x_i$ $(i,j=1,2,3)$. Moreover, I is the unit matrix of degree three, $n=n(x,t)$ is the unit vector of the outward normal to $S_F(t)$ at the point x.

In the case the domain Ω is bounded, hence, so is $\Omega(t)$, local-in-time unique existence [7] and global-in-time unique existence [8] were obtained in Solonnikov. Also, Y. Shibata-S.Shimizu [5] proved global-in-time unique existence in more general setting of the spaces than [8] by the L^p-L^q maximal regularity theorem for the linealized problem. In both [8] and [5] surface tension is not taken into account. On the other hand, in the surface wave problems, that is, if the domain is a semi-infinite domain between the moving upper surface and a fixed bottom, in the Hölder space setting Beale showed local-in-time existence [2] and global-in-time existence with surface tension ([3]), Also, global-in-time existence even the case without surface tension was proved by Sylvester [6]. In the L^2-space framework Tani-Tanaka [10] showed global-in-time unique existence, while in the L^p-space framework by Abels [1].

The aim of this paper is to extend the local-in-time existence result, a part of results in Solonnikov [7], to the case that the domain is unbounded. For this aim we utilise the transformation from $x\in\Omega(t)$, the Euler coordinate, into the Lagrange coordinate, $\xi\in\Omega$, as

$$x=\xi+\int_0^t u(\xi,\tau)\,\mathrm{d}\tau=:X_u(\xi,t), \tag{1.1}$$

which shifts the above system into the following initial boundary value problem in the initial domain Ω and its boundary $S_F(0)$ on $u(\xi,t):=u(X_u(\xi,t),t)$ and $q(\xi,t):=q(X_u(\xi,t),t)$.

$$\frac{\partial u}{\partial t}-\nu\nabla_u^2u+\nabla_u q=f(X_u(\xi,t),t),\quad \nabla_u\cdot u=0\qquad\text{for }(\xi,t)\in\Omega\times(0,\infty), \tag{1.2}$$

$$u(\xi,0)=u_0(\xi)\quad\text{for }\xi\in\Omega,\qquad T_u(u,q)\mathcal{A}n_0|_{S_F(0)}=0. \tag{1.3}$$

Here,

$$\nabla_u\ (:=A^{*-1}\nabla)=\mathcal{A}\nabla=\nabla\cdot\mathcal{A}^*,\qquad T_u(u,q)=((T_u)_{ij})=\sum_{k=1}^{3}\left(A_{jk}\frac{\partial u^i}{\partial\xi_k}+A_{ik}\frac{\partial u^j}{\partial\xi_k}\right),$$

$\nabla=\nabla_\xi$, and $A=(a_{ij})_{ij}$ is the Jacobian matrix of the transform (1.1), $a_{ij}=\delta_{ij}+\int_0^t\frac{\partial u^i}{\partial\xi_j}\,\mathrm{d}\tau$. Also, $\mathcal{A}$ is the matrix whose (i,j)-components is the (i,j)-cofactor A_{ij} of the matrix A, and $n_0=n_0(\xi)$ is the unit outer normal vector to $S_F(0)$.

The proof of local-in-time existence, in [7], relies on the usual successive approximation, that is, solving a system, corresponding to (1.2)-(1.3), for $(u^{(m+1)},q^{(m+1)})$ from a given $(u^{(m)},q^{(m)})$ for $m\in\mathbf{N}$, by defining the transformation

$$x=\xi+\int_0^t u^{(m)}(\xi,\tau)\,\mathrm{d}\tau(=:X_{u^{(m)}}(\xi,t))=:X_m(\xi,t). \tag{1.4}$$

The reason why boundedness of the domain was assumed in [7], is the way of estimating the transformed outer force term $f(X_m(\xi,t),t)$. Since the space variable $X_m(\xi,t)$ is a different variable from the variable ξ, in which iteration scheme is taken place by estimating integral norm. So, in [7], the term is estimated by pulling out the supremum norm of f in the whole space $\mathbb{R}^3$ from the integral in ξ as

$$\left(\int_\Omega |f(X_m(\xi,t),t)|^p \,\mathrm{d}\xi\right)^{1/p} \leq |\Omega|^{1/p} \sup_{\xi\in\mathbb{R}^3} |f(\xi,t)| \leq |\Omega|^{1/p} \sup_{\xi\in\mathbb{R}^3,\ 0\leq t\leq T} |f(\xi,t)|, \tag{1.5}$$

which, of course, causes appearance of $|\Omega|$, the volume of domain Ω, imposing boundedness of the domain.

On the other hand, the main idea of proof of current paper is based on transforming back to the Euler coordinate $x \in \Omega(t)$ where the original equation is described, from the Lagrange coordinate $\xi \in \Omega$ where iteration is taken place. Then we have the following alternative estimate to (1.5).

$$\begin{aligned}
\left(\int_\Omega |f(X_m(\xi,t),t)|^p \,\mathrm{d}\xi\right)^{1/p} &= \left(\int_{\Omega(t)} |f(x,t)|^p \frac{1}{\det A^{(m)}(\xi,t)} \,\mathrm{d}x\right)^{1/p} \\
&\leq \sup_{x\in\Omega(t)} \left|\frac{1}{\det A^{(m)}(\xi,t)}\right|^{1/p} \left(\int_{\Omega(t)} |f(x,t)|^p \,\mathrm{d}x\right)^{1/p} \\
&= \sup_{\xi\in\Omega} \left|\frac{1}{\det A^{(m)}(\xi,t)}\right|^{1/p} \left(\int_{\Omega(t)} |f(x,t)|^p \,\mathrm{d}x\right)^{1/p} \\
&\leq (b_m^0(t))^{1/p} \, ||f(x,t)||_{L^p(\mathbb{R}^3)},
\end{aligned}$$

where, $b_m^0(t)$ is the positive increasing function on small $t > 0$ defined later (in (3.13)). Here, by $A^{(m)} = (a_{ij}^{(m)}) = (\delta_{ij} + \int_0^t \frac{\partial y^{(m)i}}{\partial \xi_j}\,\mathrm{d}\tau)$ we denote the Jacobian matrix of the transform (1.4), and by $W_p^m(\Omega)$ $(m \geq 0)$, the usual Sobolev spaces in space variables. Moreover, we denote by $W_p^{l,m}(Q_T)$, the anisotropic spaces in space-time domain $Q_T = \Omega \times [0,T]$ with the order $l \in \mathbb{N}$ in space and the order $m \in \mathbb{N}$ in time. All the spaces will be defined formally later.

Now the theorem reads as.

Theorem 1.1. *Assume $p > 6$ and that $\Omega(\subset \mathbb{R}^3)$ be a domain (which can be unbounded). Let $S_F(0) \in W_p^{2-1/p}$, and an outer force $f(x,t)$ defined in $\mathbb{R}^3 \times [0,T_0]$ be satisfying the Lipschitz condition with respect to $x \in \mathbb{R}^3$. Then, for any $v_0 \in W_p^{2-2/p}(\Omega)$ satisfying the compatibility condition*

$$\operatorname{div} v_0 = 0 \quad \text{in } \Omega \quad \text{and} \quad \{D(v_0)n_0 - n_0(n_0 \cdot D(v_0)n_0)\}|_{\xi\in S_F(0)} = 0,$$

there exist $T_1 = T_1\left(||v_0||_{W_p^{2-2/p}(\Omega)}, \sup_{0\leq t\leq T_0} ||f||_{L^p(\mathbb{R}^3)}(t)\right)$ $(\leq T_0)$ and a unique time-local solution $u \in W_p^{2,1}(Q_{T_1})$, $q \in W_p^{1,0}(Q_{T_1})$ of (1.2) - (1.3).

This paper is an excerpt from the forthcoming paper [4].

2 Preliminaries

2.1 Function spaces

In this subsection we define function spaces. First, for a function $u(x)$ in space variable $x \in \Omega$ we denote the L^p $(1 < p < \infty)$ and L^∞ norms as

$$||u||_{p,\Omega} := ||u||_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \,\mathrm{d}x\right)^{1/p} \quad \text{and} \quad |u|_\Omega := ||u||_{L^\infty(\Omega)} = \sup_{x\in\Omega} |u(x)|,$$

respectively. Also, as usual, we define for an integer $l > 0$

$$||u||_{W_p^l(\Omega)} := \left(\sum_{|\alpha| \le l} ||D^\alpha u||_{p,\Omega}^p \right)^{1/p},$$

and for a non integer $l = [l] + \lambda > 0$ ($\lambda \in (0,1)$),

$$||u||_{W_p^l(\Omega)} := \left(||u||^p_{W_p^{[l]}(\Omega)} + \sum_{|\alpha|=[l]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{3+p\lambda}} \, dx \, dy \right)^{1/p}.$$

Next, for a function $u(x,t)$ in space-time variable $(x,t) \in Q_T := \Omega \times (0,T)$ we denote by $W_p^{2,1}(Q_T)$ the Banach space with the norm

$$||u||_{W_p^{2,1}(Q_T)} := \left(||\frac{\partial u}{\partial t}||^p_{p,Q_T} + \sum_{|\alpha| \le 2} ||D_x^\alpha u||^p_{p,Q_T} \right)^{1/p},$$

where

$$||u||_{p,Q_T} := ||u||_{L^p(0,T;L^p(\Omega))} = \left(\int_0^T ||u(\cdot,\tau)||^p_{p,\Omega} \, d\tau \right)^{1/p}.$$

The spaces of $W_p^{1,0}(Q_T)$ and $W_p^{2,0}(Q_T)$ are similarly defined.

For a function $u(x,t)$ in space-time variable $(x,t) \in G_T := \Gamma \times (0,T)$, where $\Gamma = \partial\Omega$, we also introduce the anisotropic spaces $W_p^{l,l/2}(G_T)$ for $l \in (0,1)$ with the norm

$$||u||_{W_p^{l,l/2}(G_T)} := \left(||u||^p_{W_p^{l,0}(G_T)} + << u >>^p_{G_T,l/2} \right)^{1/p},$$

where

$$||u||_{W_p^{l,0}(G_T)} = \left(||u||^p_{p,G_T} + \int_0^T dt \int_\Gamma \int_\Gamma \frac{|u(x,t) - u(y,t)|^p}{|x-y|^{2+pl}} \, dS_x \, dS_y \right)^{1/p},$$

$$<< u >>_{G_T,l/2} = \left(\int_\Gamma dS_x \int_0^T dt \int_0^t \frac{|u(x,t) - u(x,t-\tau)|^p}{|\tau|^{1+p(l/2)}} \, d\tau \right)^{1/p}.$$

2.2 Linear problem

In order to solve the equations (1.2) - (1.3) in the Lagrange coordinate, we consider the linear problem with inhomogeneous datum in a fixed domain Ω of the form;

$$\frac{\partial v}{\partial t} - \nu \nabla^2 v + \nabla q = f, \quad \nabla \cdot v = \rho \qquad \text{for } (\xi,t) \in \Omega \times (0,\infty), \tag{2.1}$$

$$v(\xi,0) = v_0(\xi) \quad \text{for } \xi \in \Omega, \qquad T(v,p)n|_{S_F(0)} = d. \tag{2.2}$$

Then we have

Proposition 2.1. [8, Theorem 2] *Assume $p > 3$ and $S_F(0) \in W_p^{2-1/p}$, and*

$$f \in L^p(Q_T), \quad \rho \in W_p^{1,0}(Q_T), \quad \rho = \nabla \cdot R(x,t), \quad R,\ R_t \in L^p(Q_T),$$
$$v_0 \in W_p^{2-2/p}(\Omega), \quad d \in W_p^{1-1/p,(1/2)(1-1/p)}(G_T).$$

Moreover, let $d|_{t=0} = 0$, $\rho(x,0) = 0$ and also suppose that compatibility condition

$$div\ v_0 = 0 \quad in\ \Omega, \qquad \{D(v_0)n_0 - n_0(n_0 \cdot D(v_0)n_0)\}|_{\xi \in S_F(0)} = 0$$

Then, the problem

$$\frac{\partial v}{\partial t} - \nu \nabla^2 v + \nabla q = g, \quad \nabla \cdot v = \rho \qquad for\ (\xi, t) \in \Omega \times (0,\infty),$$
$$v(\xi,0) = v_0(\xi) \quad for\ \xi \in \Omega, \qquad T(v,p)n|_{S_F(0)} = d.$$

has a unique solution $v \in W_p^{2,1}(Q_T)$, $q \in W_p^{1,0}(Q_T)$, and $q \in W_p^{1-1/p,(1/2)(1-1/p)}(G_T)$ which satisfy

$$||v||_{W_p^{2,1}(Q_T)} + \sup_{t \le T} ||v||_{W_p^{2-2/p}(\Omega)} + ||q||_{W_p^{1,0}(Q_T)} + ||q||_{W_p^{1-1/p,(1/2)(1-1/p)}(G_T)}$$
$$\le c(T)\ \{||f||_{p,Q_T} + ||\rho||_{W_p^{1,0}(Q_T)} + ||\frac{\partial R}{\partial t}||_{p,Q_T} + ||v_0||_{W_p^{2-2/p}(\Omega)} + ||d||_{W_p^{1-1/p,(1/2)(1-1/p)}(G_T)}\},$$

where $c(T)$ is a nondecreasing function of $T > 0$. Here, $G_T := S_F(0) \times (0,T)$, $Q_T := \Omega \times (0,T)$.

2.3 Trace and interpolation inequalities

Proposition 2.2. **(trace)** *Assume $1 < p < \infty$. Let $u \in W_p^{2,1}(Q_T)$. Then, $u(\cdot,t) \in W_p^{2-2/p}(\Omega)$ for all $0 \le t \le T$, $u_{x_j}|_{G_T} \in W_p^{1-1/p,(1/2)(1-1/p)}(G_T)$ $(j = 1,2,3)$, and*

$$\sup_{0 \le t \le T} ||u(x,t)||_{W_p^{2-2/p}(\Omega)} \le c_1 ||u||_{W_p^{2,1}(Q_T)},$$
$$||D_x u||_{W_p^{1-1/p,(1/2)(1-1/p)}(G_T)} \le c_2 ||u||_{W_p^{2,1}(Q_T)}.$$

Proposition 2.3. **(interpolation I)** *Assume $1 < p < \infty$. Then,*

$$||Du||_{p,\Omega} \le c_3 ||u||^{1/2}_{W_p^2(\Omega)} ||u||^{1/2}_{p,\Omega}, \tag{2.3}$$
$$|u|_\Omega \le c_4 ||u||^{3/(2p)}_{W_p^2(\Omega)} ||u||^{1-3/(2p)}_{p,\Omega}. \tag{2.4}$$

Proposition 2.4. **(interpolation II)** *Let $p > 3$. Then,*

$$|Du|_\Omega \le c_5 ||u||^{\frac{1}{2}+\frac{3}{2p}}_{W_p^2} ||u||^{\frac{1}{2}-\frac{3}{2p}}_{p,\Omega}. \tag{2.5}$$

Proof. By the Sobolev imbedding theorem

$$|Du|_\Omega \le C||Du||_{W_p^{\frac{3}{p}}} \qquad \text{for } p > 3$$

and the interpolation inequality

$$||f||_{W_p^{2\theta}} \le C||f||^\theta_{W_p^2} ||f||^{1-\theta}_{p,\Omega} \qquad \text{for } 0 < \theta < 1,\ 1 < p < \infty$$

we have for $p > 3$ that

$$|Du|_\Omega \le C||Du||_{W_p^{\frac{3}{p}}} \le C||u||_{W_p^{1+\frac{3}{p}}} \le C||u||^{\frac{1}{2}+\frac{3}{2p}}_{W_p^2} ||u||^{\frac{1}{2}-\frac{3}{2p}}_{p,\Omega}.$$

□

3 Estimate of the Jacobian matrix of the transformation to the Lagrange coordinate

In this section we give estimates for determinant and cofactor of the transformations (1.1) and (1.4).

Firstly, we list estimates for the determinant and the (i,j)-cofactor $A_{ij}^{(m)}$ of the Jacobian matrix $A = (a_{ij})_{ij} = (\delta_{ij} + \int_0^t \frac{\partial u^i}{\partial \xi_j}\,\mathrm{d}\tau)_{ij}$ of the transformation (1.1).

Lemma 3.1. *Let $p > 3$ and $u \in W_p^{2,1}(Q_T)$. Then there hold*

$$\begin{aligned}
&|\mathrm{I} - (A^*)^{-1}| = |(\delta_{ij} - A_{ij})_{ij}| \\
&\leq C\int_0^t |Du(\xi,\tau)|\,\mathrm{d}\tau \left(1 + \int_0^t |Du(\cdot,\tau)|_\Omega\,\mathrm{d}\tau\right) \leq Cb(t), && (3.1)\\
&|(A^*)^{-1}| = |A_{ij}| \leq |A_{ij}|_\Omega \leq C(1 + b(t)), && (3.2)\\
&||\mathrm{I} - (A^*)^{-1}||_{p,\Omega} = ||\delta_{ij} - A_{ij}||_{p,\Omega} \leq Cb'(t), && (3.3)\\
&|D(A^*)^{-1}| = |DA_{ij}| \\
&\leq C\int_0^t |D^2u(\xi,\tau)|\,\mathrm{d}\tau \left(1 + \int_0^t |Du(\xi,\tau)|_\Omega\,\mathrm{d}\tau\right), \\
&||D(A^*)^{-1}||_{p,\Omega} = ||DA_{ij}||_{p,\Omega} \leq Cb(t), && (3.4)
\end{aligned}$$

where

$$\begin{aligned}
b(t) &= \int_0^t ||D^2u(\cdot,\tau)||_{p,\Omega}\,\mathrm{d}\tau \left(1 + \int_0^t |Du(\cdot,\tau)|_\Omega\,\mathrm{d}\tau\right), \\
b'(t) &= \int_0^t ||Du(\cdot,\tau)||_{p,\Omega}\,\mathrm{d}\tau \left(1 + \int_0^t |Du(\cdot,\tau)|_\Omega\,\mathrm{d}\tau\right).
\end{aligned}$$

Proof. All the inequalities are direct consequences from calculation of the cofactor matrices of $A = (a_{ij}) = (\delta_{ij} + b_{ij})$ with $b_{ij} = \int_0^t \frac{\partial u^i}{\partial \xi_j}(\xi,\tau)\,\mathrm{d}\tau$. For (3.1) and (3.2) we used the embedding $|u|_\Omega \leq ||Du||_{p,\Omega}$ for $p > 3$. For (3.3) and (3.4) we note the estimate

$$||\int_0^t f(\cdot,\tau)\,\mathrm{d}\tau||_{p,\Omega} \leq \int_0^t ||f(\cdot,\tau)||_{p,\Omega}\,\mathrm{d}\tau \tag{3.5}$$

for all $f(\cdot,t) \in L^p(\Omega)$. □

In the next lemma we give estimates for the determinant and the (i,j)-cofactor $A_{ij}^{(m)}$ of the Jacobian matrix $A^{(m)} = (a_{ij}^{(m)})_{ij} = (\delta_{ij} + \int_0^t \frac{\partial u^{(m)i}}{\partial \xi_j}\,\mathrm{d}\tau)_{ij}$ of the transformation (1.4). We also define $\underline{A}^{(m)}$ and $\underline{A}_{ij}^{(m)}$ by

$$\underline{A}^{(m)}(:= (\underline{A}_{ij}^{(m)})_{ij}) := (\frac{1}{\det A^{(m)}} A_{ij}^{(m)})_{ij}.$$

Lemma 3.2. *Assume that $p > 3$ and $u^{(m)} \in W_p^{2,1}(Q_T)$ satisfy*

$$t^{1/p'} < x_0/||u^{(m)}||_{W_p^{2,1}(Q_T)}, \tag{3.6}$$

where x_0 *is a positive root of* $1 - 3x - 6x^2 - 6x^3 = 0$. *Then there hold*

$$|\frac{1}{\det A^{(m)}}| \leq b_m^0(t), \qquad |\det A^{(m)}| \leq 16, \qquad |D \det A^{(m)}| \leq 16, \tag{3.7}$$

$$\begin{aligned}
|\mathrm{I} - (A^{(m)*})^{-1}| &= |(\delta_{ij} - \underline{A}_{ij}^{(m)})_{ij}| \leq |(\delta_{ij} - \underline{\delta}_{ij})_{ij}| + |(\underline{\delta}_{ij} - \underline{A}_{ij}^{(m)})_{ij}| \\
&\leq (1 + b_m^0(t)) + Cb_m^0(t) \int_0^t |Du^{(m)}(\xi,\tau)| \, \mathrm{d}\tau \, \left(1 + \int_0^t |Du^{(m)}(\cdot,\tau)|_\Omega \, \mathrm{d}\tau\right) \\
&\leq 1 + b_m^0(t) + Cb_m^0(t) \, b_m(t) \ (=: b_m^1(t)),
\end{aligned} \tag{3.8}$$

$$|(A^{(m)*})^{-1}| = |\underline{A}_{ij}^{(m)}| \leq |\underline{A}_{ij}^{(m)}|_\Omega \leq Cb_m^0(t)(1 + b_m(t)) \ (=: b_m^2(t)), \tag{3.9}$$

$$\begin{aligned}
|D(A^{(m)*})^{-1}| &= |D\underline{A}_{ij}^{(m)}| \leq 16C \frac{1}{(\det A^{(m)}(\xi,t))^2}(|DA_{ij}^{(m)}(\xi,t)| + |A_{ij}^{(m)}(\xi,t)|) \\
&\leq C(b_m^0(t))^2(|DA_{ij}^{(m)}(\xi,t)| + |A_{ij}^{(m)}(\xi,t)|),
\end{aligned} \tag{3.10}$$

$$||DA_{ij}^{(m)}||_{p,\Omega} \leq b_m(t), \tag{3.11}$$

$$|A_{ij}^{(m)}|_\Omega \leq C(1 + b_m(t)), \tag{3.12}$$

where $\underline{\delta}_{ij} := \dfrac{1}{\det A^{(m)}}\delta_{ij}$ *and*

$$b_m(t) = \int_0^t ||D^2u^{(m)}(\cdot,\tau)||_{p,\Omega} \, \mathrm{d}\tau \, \left(1 + \int_0^t |Du^{(m)}(\cdot,\tau)|_\Omega \, \mathrm{d}\tau\right),$$

and $b_m^0(t)$ *is the positive increasing function on* $t > 0$ *under the assumption (3.6), defined by*

$$1/(b_m^0(t)) = 1 - 3(t||u^{(m)}||_{W_p^{2,1}(Q_T)}) - 6(t||u^{(m)}||_{W_p^{2,1}(Q_T)})^2 - 6(t||u^{(m)}||_{W_p^{2,1}(Q_T)})^3 \ (> 0). \tag{3.13}$$

Proof. Most of the estimates can be obtained similarly to Lemma 3.1, however, we note that the assumption (3.6) yields that

$$\begin{aligned}
&|\int_0^t \frac{\partial u_i^{(m)}}{\partial \xi_j} \, \mathrm{d}\tau|_\Omega \leq \int_0^t |\frac{\partial u_i^{(m)}}{\partial \xi_j}|_\Omega \, \mathrm{d}\tau \leq (\int_0^t \, \mathrm{d}\tau)^{1/p'}(\int_0^t |Du^{(m)}|_\Omega \, \mathrm{d}\tau)^{1/p} \\
&\leq t^{1/p'}||Du^{(m)}||_{W_p^{1,0}(Q_t)} \leq t^{1/p'}||u^{(m)}||_{W_p^{2,1}(Q_T)} (\leq t^{1/p'} Z_m),
\end{aligned}$$

which gives

$$|\frac{1}{\det A^{(m)}}| \leq b_m^0(t).$$

For (3.10) we calculate

$$|D\underline{A}_{ij}^{(m)}|_\Omega \leq \frac{1}{(\det A^{(m)}(\xi,t))^2}(|DA_{ij}^{(m)}(\xi,t)||\det A^{(m)}(\xi,t))| + |A_{ij}^{(m)}(\xi,t)||D\det A^{(m)}(\xi,t))|) \tag{3.14}$$

and adapted (3.7). □

4 Key Lemma - Estimate for outer force

In this section we give the key lemma and its proof again, though they are already described in Introduction.

Lemma 4.1. *Let $f = (f^1(x,t), f^2(x,t), f^3(x,t))$ be any vector field defined in $\mathbb{R}^3 \times [0,\infty]$. Assume that t is small so that (3.6) holds. Then its representation in the Lagrange coordinate $f(\xi,t) := f(X_m(\xi,t),t)$ can be estimated as follows.*

$$\left(\int_\Omega |f(X_m(\xi,t),t)|^p \, \mathrm{d}\xi\right)^{1/p} \le (b_m^0(t))^{1/p} \|f(x,t)\|_{L^p(\mathbb{R}^3)}, \tag{4.1}$$

where, $b_m^0(t)$ is defined in (3.13).

Proof. We estimate the norm in the Lagrange coordinate by passing back into the Euler coordinate

$$\begin{aligned}
\int_\Omega |f(X_m(\xi,t),t)|^p \, \mathrm{d}\xi &= \int_{\Omega(t)} |f(x,t)|^p \frac{1}{\det A^{(m)}(\xi,t)} \, \mathrm{d}x \\
&\le \sup_{x\in\Omega(t)} \left|\frac{1}{\det A^{(m)}(\xi,t)}\right| \int_{\Omega(t)} |f(x,t)|^p \, \mathrm{d}x \\
&= \sup_{\xi\in\Omega} \left|\frac{1}{\det A^{(m)}(\xi,t)}\right| \int_{\Omega(t)} |f(x,t)|^p \, \mathrm{d}x \\
&\le b_m^0(t) \|f(x,t)\|_{L^p(\Omega(t))}^p \\
&\le b_m^0(t) \|f(x,t)\|_{L^p(\mathbb{R}^3)}^p.
\end{aligned}$$

□

5 Outline of Proof

In this section we give outline of proof of the theorem, the method of iteration scheme, especially boundedness. Full proof will be given in [4].

Let $u^{(0)} := 0$ and $q^{(0)} := 0$, and let the pair $(u^{(m+1)}, q^{(m+1)})$ be a solution of the following linear problem, regarding the pair $(u^{(m)}, q^{(m)})$ is given.

$$\begin{aligned}
&\frac{\partial u^{(m+1)}}{\partial t} - \nu\nabla^2 u^{(m+1)} + \nabla q^{(m+1)} \\
&\quad = f(X_m(\xi,t),t) + \nu(\nabla^2 - \nabla_m^2)u^{(m)} + (\nabla - \nabla_m)q^{(m)} (=: f^{(m)}(\xi,t)), && (5.1)\\
&\nabla \cdot u^{(m+1)} = (\nabla - \nabla_m)\cdot u^{(m)} (=: \rho^{(m)}(\xi,t)) \quad \text{for } (\xi,t) \in \Omega \times (0,\infty), && (5.2)\\
&u^{(m+1)}(\xi,0) = v_0(\xi) \quad \text{for } \xi \in \Omega, && (5.3)\\
&T(u^{(m+1)}, q^{(m+1)})n|_{S_F(0)} = T(u^{(m)}, q^{(m)})(1 - \underline{A}^{(m)})n_0|_{S_F(0)} \\
&\quad + (T(u^{(m)}, q^{(m)}) - T_m(u^{(m)}, q^{(m)}))\underline{A}^{(m)} n_0|_{S_F(0)} \; (=: d^{(m)}(\xi,t)). && (5.4)
\end{aligned}$$

Here, the variable ξ is determined by the transform

$$x = \xi + \int_0^t u^{(m)}(\xi,\tau) \, \mathrm{d}\tau (=: X_{u^{(m)}}(\xi,t)) =: X_m(\xi,t). \tag{5.5}$$

Also, we denote its Jacobian matrix by $A^{(m)} = (a_{ij}^{(m)})$. Moreover, by $A_{ij}^{(m)}$ we denote the (i,j)-cofactor of the matrix $A^{(m)}$, and define

$\underline{A}^{(m)}(:= (\underline{A}_{ij}^{(m)})_{ij}) := (\frac{1}{\det A^{(m)}} A_{ij}^{(m)})_{ij}, \quad \nabla_m (:= \nabla_{u^{(m)}}) := (A^{(m)*})^{-1}\nabla = \underline{A}^{(m)}\nabla = \nabla\cdot\underline{A}^{(m)*}$, and

$$(T_m)_{ij} = (T_m(u^{(m)}, q^{(m)}))_{ij} = -q^{(m)}\delta_{ij} + \nu\sum_{k=1}^{3}(\underline{A}_{ik}^{(m)}\frac{\partial u_j^{(m)}}{\partial \xi_k} + \underline{A}_{jk}^{(m)}\frac{\partial u_i^{(m)}}{\partial \xi_k}).$$

By virtue of Propositon 2.1

$$\begin{aligned}
&||u^{(m+1)}||_{W_p^{2,1}(Q_T)} + \sup_{t\le T}||u^{(m+1)}||_{W_p^{2-2/p}(\Omega)} + ||q^{(m+1)}||_{W_p^{1,0}(Q_T)} + ||q^{(m+1)}||_{W_p^{1-1/p,(1/2)(1-1/p)}(G_T)}\\
&\le c(T)\ \{||f^{(m)}||_{p,Q_T} + ||\rho^{(m)}||_{W_p^{1,0}(Q_T)} + ||\frac{\partial R^{(m)}}{\partial t}||_{p,Q_T}\\
&\qquad + ||v_0||_{W_p^{2-2/p}(\Omega)} + ||d^{(m)}||_{W_p^{1-1/p,(1/2)(1-1/p)}(G_T)}\},
\end{aligned} \tag{5.6}$$

Then we estimate each term of RHS of the inequality. The most typical difference between our unbounded domain case and bounded domain case [8] appears in the estimate for $f(X_m(\xi,t),t)$ in $f^{(m)}(\xi,t)$, whose $||\cdot||_{p,Q_T}$ norm is estimated by virtue of Lemma 4.1 by

$$(b_m^0(T))T^{1/p}\sup_{t\le T}||f(x,t)||_{L^p(\mathbf{R}^3)}.$$

Although, in this note we do not mention all the estimates, mainly thanks to Lemma 3.2 we estimate other terms.

$$||(\nabla - \nabla_m)\ q^{(m)}||_{p,Q_T} = ||\ |\mathrm{I} - (A^{(m)*})^{-1}|\ \nabla q^{(m)}\ ||_{p,Q_T} \le b_m^1(T)||\nabla q^{(m)}||_{p,Q_T}. \tag{5.7}$$

$$\begin{aligned}
||\rho^{(m)}||_{p,Q_T} &= ||(\nabla - \nabla_m)\cdot u^{(m)}||_{p,Q_T}\\
&= ||\ |\mathrm{I} - (A^{(m)*})^{-1}|\ \nabla\cdot u^{(m)}||_{p,Q_T} \le b_m^1(T)||Du^{(m)}||_{p,Q_T}.
\end{aligned} \tag{5.8}$$

Moreover,

$$||(\nabla^2 - \nabla_m^2)u^{(m)}||_{p,Q_T} \le c_m^1(T)||u^{(m)}||_{W_p^{2,0}(Q_T)}, \tag{5.9}$$

$$||\nabla\rho^{(m)}||_{p,Q_T} = ||\nabla\{(\nabla - \nabla_m)\cdot u^{(m)}\}||_{p,Q_T} \le c_m^2(T)||u^{(m)}||_{W_p^{2,0}(Q_T)}, \tag{5.10}$$

where

$$\begin{aligned}
c_m^1(T) &:= b_m^1(T)(1 + b_m^2(T)) + C(b_m^0(T))^2\ b_m^2(T)\ (1 + 2b_m(T)),\\
c_m^2(T) &:= b_m^1(T) + C(b_m^0(T))^2(1 + 2b_m(T)).
\end{aligned}$$

Hence, the estimates (4.1), (5.9) and (5.7) lead to

$$\begin{aligned}
||f^{(m)}||_{p,Q_T} &\le (b_m^0(T))T^{1/p}\sup_{t\le T}||f(x,t)||_{L^p(\mathbf{R}^3)}\\
&+\nu c_m^1(T)||u^{(m)}||_{W_p^{2,0}(Q_T)} + Cb_m^0(T)b_m(T)||\nabla q^{(m)}||_{p,Q_T}.
\end{aligned} \tag{5.11}$$

Although we do not give detail here, we can also estimate

$$\begin{aligned}
||\frac{\partial R^{(m)}}{\partial t}||_{p,Q_T} &\le Cb_m^1(T)||u_t^{(m)}||_{p,Q_T} + c_m^{11}(T)\left(\int_0^T |u^{(m)}|_\Omega^p||Du^{(m)}||_{p,\Omega}^p\ dt\right)^{1/p}\\
&+c_m^{11}(T)\left(\int_0^T ||Du^{(m)}||_{p,\Omega}\ dt\right)\left(\int_0^T |u^{(m)}|_\Omega^p|Du^{(m)}|_\Omega^p\ dt\right)^{1/p}\\
&+c_m^{11}(T)\left(\int_0^T ||Du^{(m)}||_{p,\Omega}\ dt\right)\left(\int_0^T |Du^{(m)}|_\Omega\ dt\right)\left(\int_0^T |u^{(m)}|_\Omega^p|Du^{(m)}|_\Omega^p\ dt\right)^{1/p}
\end{aligned} \tag{5.12}$$

and

$$||d^{(m)}||_{W_p^{1-1/p,(1/2)(1-1/p)}(G_T)} \le (||d^{(m)}||^p_{W_p^{1,0}(Q_T)} + << d^{(m)} >>^p_{G_T})^{1/p}$$
$$\le \{c_m^5(T) + Cb_m^*(T)(1 + b_m^1(T)b_m^2(T))\}$$
$$\times \left(||u^{(m)}||_{W_p^{2,1}(Q_T)} + ||q^{(m)}||_{W_p^{1,0}(Q_T)} + << q^{(m)} >>_{G_T}\right). \tag{5.13}$$

Here,

$$c_m^5(t) = C(c_m^4(T) + b_m^1(T) + c_m^0(T) + b_m^2(T)b_m^1(T)),$$
$$c_m^4(t) = C(b_m^1(T)b_m^2(T) + c_m^3(T) + b_m^1(T)c_m^0(T)),$$
$$c_m^3(t) = b_m^1(T)b_m^2(T) + b_m^2(T)C(b_m^0(T))^2(1 + 2b_m(T)),$$
$$c_m^{11}(t) := C(b_m^0(t))^2(1 + b_m(t)).$$

Also we denote

$$\widetilde{b_m}(t) := (\int_0^t |Du^{(m)}|^p_\Omega \, d\tau)^{1/p}(1 + 2\int_0^t |Du^{(m)}|_\Omega \, d\tau), \tag{5.14}$$

$$\widehat{b}_m(t) := (\int_0^t |Du^{(m)}|^p_\Omega \, d\tau)^{1/p}\left\{1 + \int_0^t |Du^{(m)}|_\Omega \, d\tau \, \left(1 + \int_0^t |Du^{(m)}|_\Omega \, d\tau\right)\right\}^2, \tag{5.15}$$

$$b_m^*(T) := \frac{4}{(p-1)^{1/p}} T^{1/(2p')}\left\{b_m^0(T)\widetilde{b_m}(T) + (b_m^0(T))^2\widehat{b}_m(T)\right\}. \tag{5.16}$$

Then we conclude that

$$||u^{(m+1)}||_{W_p^{2,1}(Q_T)} + \sup_{t\le T}||u^{(m+1)}||_{W_p^{2-2/p}(\Omega)} + ||q^{(m+1)}||_{W_p^{1,0}(Q_T)} + ||q^{(m+1)}||_{W_p^{1-1/p,(1/2)(1-1/p)}(G_T)}$$
$$\le c(T)\ [\ ||v_0||_{W_p^{2-2/p}(\Omega)}$$
$$+C\{c_m^6(T) + b_m^*(T)(1 + b_m^1(T)b_m^2(T))\}$$
$$\times \left(||u^{(m)}||_{W_p^{2,1}(Q_T)} + ||q^{(m)}||_{W_p^{1,0}(Q_T)} + << q^{(m)} >>_{G_T}\right)$$
$$+(b_m^0(T))T^{1/p}\sup_{t\le T}||f(x,t)||_{L^p(\mathbb{R}^3)}$$
$$+c_m^{11}(T)\left(\int_0^T |u^{(m)}|^p_\Omega ||Du^{(m)}||^p_{p,\Omega} \, dt\right)^{1/p}$$
$$+c_m^{11}(T)\left(\int_0^T ||Du^{(m)}||_{p,\Omega} \, dt\right)\left(1 + \int_0^T |Du^{(m)}|_\Omega \, dt\right)\left(\int_0^T |u^{(m)}|^p_\Omega |Du^{(m)}|^p_\Omega \, dt\right)^{1/p}\ ]. \tag{5.17}$$

For boundedness we define

$$Z_m := ||u^{(m)}||_{W_p^{2,1}(Q_T)} + ||q^{(m)}||_{W_p^{1,0}(Q_T)} + << q^{(m)} >>_{G_T}. \tag{5.18}$$

Noting that

$$\int_0^T |Du^{(m)}(\cdot,\tau)|_\Omega \, d\tau \le C_{10}T^{1/p'}(\int_0^T ||D^2u^{(m)}(\cdot,\tau)||^p_{p,\Omega} \, d\tau)^{1/p} \le C_{10}T^{1/p'}Z_m \tag{5.19}$$

and

$$\int_0^T ||D^2u^{(m)}(\cdot,\tau)||_{p,\Omega} \, d\tau \le T^{1/p'}(\int_0^T ||D^2u^{(m)}(\cdot,\tau)||^p_{p,\Omega} \, d\tau)^{1/p} \le T^{1/p'}Z_m \tag{5.20}$$

for $(1/p)+(1/p')=1$ we have

$$\begin{aligned}
b_m(T) &\le T^{1/p'}Z_m(1+C_{10}T^{1/p'}Z_m) =: B(T,Z_m), \qquad (5.21)\\
\widehat{b}_m(T) &\le C_{10}(\int_0^T \|D^2u^{(m)}\|_{p,\Omega}^p\,\mathrm{d}\tau)^{1/p}\left\{1+\int_0^T |Du^{(m)}|_\Omega\,\mathrm{d}\tau\left(1+\int_0^T |Du^{(m)}|_\Omega\,\mathrm{d}\tau\right)\right\}^2\\
&\le C_{10}Z_m\left\{1+C_{10}T^{1/p'}Z_m\left(1+C_{10}T^{1/p'}Z_m\right)\right\}^2 = C_{10}Z_m(1+C_{10}B(T,Z_m))^2,\\
\widetilde{b_m}(T) &\le C_{10}(\int_0^T \|D^2u^{(m)}\|_{p,\Omega}^p\,\mathrm{d}\tau)^{1/p}(1+2\int_0^T |Du^{(m)}|_\Omega\,\mathrm{d}\tau)\\
&\le C_{10}Z_m(1+2C_{10}T^{1/p'}Z_m) = 2C_{10}T^{-1/p'}B(T,Z_m).
\end{aligned}$$

Moreover, recalling Lemma 3.2, if we take $T>0$ small so that rather $T^{1/p'}<\frac{1}{2}x_0/\|u^{(m)}\|_{W_p^{2,1}(Q_T)}$ than (3.6), we have

$$b_m^0(t)\le 1/\{1-3(x_0/2)-6(x_0/2)^2-6(x_0/2)^3\} =: C_{11}(x_0),$$

and

$$b_m^1(T)\le 1+C_{11}+CC_{11}B(T,Z_m)\le C_{12}(1+B(T,Z_m)).$$

Similarly, it's easy to see

$$\begin{aligned}
b_m^2(T) &\le C_{13}(1+B(T,Z_m)),\\
c_m^0(T) &\le C_{14}(1+B(T,Z_m)),\\
c_m^1(T) &\le C_{15}(1+B(T,Z_m))^2,\\
c_m^2(T) &\le C_{16}(1+B(T,Z_m)),\\
c_m^3(T) &\le C_{17}(1+B(T,Z_m))^2,\\
c_m^4(T) &\le C_{18}(1+B(T,Z_m))^2,\\
c_m^5(T) &\le C_{19}(1+B(T,Z_m))^2,\\
c_m^{11}(T) &\le C_{20}(1+B(T,Z_m)).
\end{aligned}$$

The above estimates immediately give us

$$\begin{aligned}
c_m^6(T) &\le C_{15}(1+B(T,Z_m))^2+C_{16}(1+B(T,Z_m))\\
&\qquad +C_{12}(1+B(T,Z_m))+C_{11}B(T,Z_m)+C_{19}(1+B(T,Z_m))^2\\
&\le C_{21}(1+B(T,Z_m))^2,\\
b_m^*(T) &\le C_{22}(p)T^{-1/(2p')}B(T,Z_m)+C_{23}(p)T^{1/(2p')}Z_m(1+C_{10}B(T,Z_m))^2.
\end{aligned}$$

By Proposition 2.3 and the Hölder inequality we have

$$
\begin{aligned}
&\left(\int_0^T |u^{(m)}|_\Omega^p \|Du^{(m)}\|_{p,\Omega}^p \, dt\right)^{1/p} \\
\leq\;& c_3\, c_4 \left(\int_0^T \|u^{(m)}\|_{W_p^2(\Omega)}^{3/2} \|u^{(m)}\|_{p,\Omega}^{p-3/2} \|u^{(m)}\|_{W_p^2(\Omega)}^{p/2} \|u^{(m)}\|_{p,\Omega}^{p/2} \, dt\right)^{1/p} \\
=\;& c_3\, c_4 \left(\int_0^T \|u^{(m)}\|_{W_p^2(\Omega)}^{\frac{p}{2}+\frac{3}{2}} \|u^{(m)}\|_{p,\Omega}^{\frac{3}{2}p-\frac{3}{2}} \, dt\right)^{1/p} \\
\leq\;& c_3\, c_4 \left\{\left(\int_0^T \|u^{(m)}\|_{W_p^2(\Omega)}^p \, dt\right)^{\frac{1}{2}+\frac{3}{2p}} \left(\int_0^T \|u^{(m)}\|_{p,\Omega}^{(\frac{3}{2}p-\frac{3}{2})\frac{2p}{p-3}} \, dt\right)^{\frac{1}{2}-\frac{3}{2p}}\right\}^{1/p} \\
\leq\;& c_3\, c_4 \left(\int_0^T \|u^{(m)}\|_{W_p^2(\Omega)}^p \, dt\right)^{\frac{1}{p}(\frac{1}{2}+\frac{3}{2p})} (\sup_{t\leq T} \|u^{(m)}\|_{p,\Omega})^{\frac{3}{2}-\frac{3}{2p}}\, T^{\frac{1}{p}(\frac{1}{2}-\frac{3}{2p})} \\
=\;& c_3\, c_4 \|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{1}{2}+\frac{3}{2p}} (\sup_{t\leq T} \|u^{(m)}\|_{p,\Omega})^{\frac{3}{2}-\frac{3}{2p}}\, T^{\frac{1}{p}(\frac{1}{2}-\frac{3}{2p})} \\
\leq\;& c_3\, c_4 \|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{1}{2}+\frac{3}{2p}} \left(\|u^{(m)}(0)\|_{p,\Omega} + \int_0^T \frac{\partial}{\partial t}\|u^{(m)}\|_{p,\Omega}(t)\, dt\right)^{\frac{3}{2}-\frac{3}{2p}} T^{\frac{1}{p}(\frac{1}{2}-\frac{3}{2p})} \\
=\;& c_3\, c_4 \|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{1}{2}+\frac{3}{2p}} \left(\|v_0\|_{p,\Omega} + \int_0^T \|u_t^{(m)}\|_{p,\Omega}(t)\, dt\right)^{\frac{3}{2}-\frac{3}{2p}} T^{\frac{1}{p}(\frac{1}{2}-\frac{3}{2p})} \\
\leq\;& C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{1}{2}+\frac{3}{2p}} T^{\frac{1}{p}(\frac{1}{2}-\frac{3}{2p})} \left\{\|v_0\|_{p,\Omega}^{\frac{3}{2}-\frac{3}{2p}} + \left(\int_0^T \|u_t^{(m)}\|_{p,\Omega}\, dt\right)^{\frac{3}{2}-\frac{3}{2p}}\right\} \\
=:\;& I + II.
\end{aligned}
$$

Here, we used the Hölder inequality with the pair of indices $(\frac{2p}{p+3}, \frac{2p}{p-3})$ for $p > 3$ in the second inequality. For I we estimate

$$
\begin{aligned}
I \;=\;& C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{1}{2}+\frac{3}{2p}} \cdot T^{\frac{1}{p}(\frac{1}{2}-\frac{3}{2p})} \|v_0\|_{p,\Omega}^{\frac{3}{2}-\frac{3}{2p}} \\
\leq\;& C\left\{\frac{p+3}{2p}(\varepsilon\, \|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{1}{2}+\frac{3}{2p}})^{\frac{2p}{p+3}} + \frac{p-3}{2p}(\frac{1}{\varepsilon}\, T^{\frac{1}{p}(\frac{1}{2}-\frac{3}{2p})} \|v_0\|_{p,\Omega}^{\frac{3}{2}-\frac{3}{2p}})^{\frac{2p}{p-3}}\right\} \\
\leq\;& C\left(\varepsilon^{\frac{2p}{p+3}}\, \|u^{(m)}\|_{W_p^{2,0}(Q_T)} + \varepsilon^{-\frac{2p}{p-3}}\, T^{1/p} \|v_0\|_{p,\Omega}^{3\frac{p-1}{p-3}}\right) \\
\leq\;& C\left(\varepsilon\, \|u^{(m)}\|_{W_p^{2,0}(Q_T)} + \varepsilon^{-\frac{2p}{p-3}}\, T^{1/p} \|v_0\|_{p,\Omega}^{3\frac{p-1}{p-3}}\right).
\end{aligned}
$$

$$
\begin{aligned}
II &= C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{1}{2}+\frac{3}{2p}} T^{\frac{1}{p}(\frac{1}{2}-\frac{3}{2p})} \left(\int_0^T \|u_t^{(m)}\|_{p,\Omega}\, \mathrm{d}t\right)^{\frac{3}{2}-\frac{3}{2p}} \\
&\le C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{1}{2}+\frac{3}{2p}} T^{\frac{1}{p}(\frac{1}{2}-\frac{3}{2p})} \left\{ (\int_0^T \|u_t^{(m)}\|_{p,\Omega}^p\, \mathrm{d}t)^{1/p} (\int_0^T 1^{p'}\, \mathrm{d}t)^{1/p'} \right\}^{\frac{3}{2}-\frac{3}{2p}} \\
&\le C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{1}{2}+\frac{3}{2p}} \|u_t^{(m)}\|_{p,Q_T}^{\frac{3}{2}-\frac{3}{2p}} T^{\frac{3}{2p'}-\frac{1}{p}}.
\end{aligned}
$$

Note that $\frac{3}{2p'} - \frac{1}{p} = \frac{3p-5}{2p} > 0$ for $p > 3$. Hence,

$$
\left(\int_0^T |u^{(m)}|_\Omega^p \|Du^{(m)}\|_{p,\Omega}^p\, \mathrm{d}t\right)^{1/p} \le C\left(\varepsilon\, Z_m + \varepsilon^{-\frac{2p}{p-6}}\, T^{1/p}\|v_0\|_{p,\Omega}^{3\frac{p-1}{p-6}}\right) + C Z_m^2 T^{\frac{3}{2p'}-\frac{1}{p}}.
$$

Similarly, it follows from (2.4) and (2.5) that

$$
\begin{aligned}
&\left(\int_0^T |u^{(m)}|_\Omega^p |Du^{(m)}|_\Omega^p\, \mathrm{d}t\right)^{1/p} \\
\le\;& c_4 c_5 \left(\int_0^T (\|u^{(m)}\|_{W_p^2}^{\frac{3}{2p}} \|u^{(m)}\|_{p,\Omega}^{1-\frac{3}{2p}})^p (\|u^{(m)}\|_{W_p^2}^{\frac{1}{2}+\frac{3}{2p}} \|u^{(m)}\|_{p,\Omega}^{\frac{1}{2}-\frac{3}{2p}})^p\, \mathrm{d}t\right)^{1/p} \\
=\;& C\left(\int_0^T \|u^{(m)}\|_{W_p^2}^{\frac{3}{2}} \|u^{(m)}\|_{p,\Omega}^{p-\frac{3}{2}} \|u^{(m)}\|_{W_p^2}^{\frac{p}{2}+\frac{3}{2}} \|u^{(m)}\|_{p,\Omega}^{\frac{p}{2}-\frac{3}{2}}\, \mathrm{d}t\right)^{1/p} \\
=\;& C\left(\int_0^T \|u^{(m)}\|_{W_p^2}^{\frac{p}{2}+3} \|u^{(m)}\|_{p,\Omega}^{\frac{3}{2}p-3}\, \mathrm{d}t\right)^{1/p} \\
\le\;& C\left\{ (\int_0^T \|u^{(m)}\|_{W_p^2}^p\, \mathrm{d}t)^{\frac{p+6}{2p}} (\int_0^T \|u^{(m)}\|_{p,\Omega}^{\frac{3p(p-2)}{p-6}}\, \mathrm{d}t)^{\frac{p-6}{2p}} \right\}^{1/p} \\
\le\;& C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{p+6}{2p}} (\sup_{t\le T} \|u^{(m)}\|_{p,\Omega})^{\frac{3(p-2)}{2p}}\, T^{\frac{p-6}{2p^2}} \\
\le\;& C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{p+6}{2p}} (\|u^{(m)}(0)\|_{p,\Omega} + \int_0^T \frac{\partial}{\partial t}\|u^{(m)}\|_{p,\Omega}\, \mathrm{d}t)^{\frac{3(p-2)}{2p}}\, T^{\frac{p-6}{2p^2}} \\
\le\;& C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{p+6}{2p}}\, T^{\frac{p-6}{2p^2}} \left\{ \|v_0\|_{p,\Omega}^{\frac{3(p-2)}{2p}} + (\int_0^T \|u_t^{(m)}\|_{p,\Omega}\, \mathrm{d}t)^{\frac{3(p-2)}{2p}} \right\} \\
=:\;& III + IV.
\end{aligned}
$$

Here, we used the Hölder inequality with the pair of indices $(\frac{2p}{p+6}, \frac{2p}{p-6})$ for $p > 6$ in the second inequality. We estimate

$$
\begin{aligned}
III &= C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{p+6}{2p}} \cdot T^{\frac{p-6}{2p^2}} \|v_0\|_{p,\Omega}^{\frac{3(p-2)}{2p}} \\
&\le C\left\{ \frac{p+6}{2p} (\varepsilon^{\frac{2p}{p+6}} \|u^{(m)}\|_{W_p^{2,0}(Q_T)}) + \frac{p-6}{2p} \varepsilon^{-\frac{2p}{p-6}}\, T^{1/p} \|v_0\|_{p,\Omega}^{3\frac{p-2}{p-6}} \right\} \\
&\le C\left(\varepsilon \|u^{(m)}\|_{W_p^{2,0}(Q_T)} + \varepsilon^{-\frac{2p}{p-6}}\, T^{1/p} \|v_0\|_{p,\Omega}^{3\frac{p-2}{p-6}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
IV &= C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{p+6}{2p}}\, T^{\frac{p-6}{2p^2}}\Big(\int_0^T \|u_t^{(m)}\|_{p,\Omega}\, dt\Big)^{\frac{3(p-2)}{2p}} \\
&\le C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{p+6}{2p}}\, T^{\frac{p-6}{2p^2}}\Big\{\Big(\int_0^T \|u_t^{(m)}\|_{p,\Omega}^p\, dt\Big)^{1/p}\Big(\int_0^T 1^{p'}\, dt\Big)^{1/p'}\Big\}^{\frac{3(p-2)}{2p}} \\
&\le C\|u^{(m)}\|_{W_p^{2,0}(Q_T)}^{\frac{p+6}{2p}}\, T^{\frac{3p-8}{2p}}\|u_t^{(m)}\|_{p,Q_T}^{\frac{3(p-2)}{2p}}.
\end{aligned}
$$

Note that $\frac{3p-8}{2p} > 0$ for $p > 6$. Hence,

$$
\left(\int_0^T |u^{(m)}|_\Omega^p |Du^{(m)}|_\Omega^p\, dt\right)^{1/p} \le C\left(\varepsilon\, Z_m + \varepsilon^{-\frac{2p}{p-6}}\, T^{1/p}\|v_0\|_{p,\Omega}^{3\frac{p-2}{p-6}}\right) + CZ_m^2\, T^{\frac{3p-8}{2p}}. \tag{5.22}
$$

We also note that

$$
\int_0^t \|Du^{(m)}\|_{p,\Omega}\, d\tau \le T^{1/p'}\Big(\int_0^t \|Du^{(m)}\|_{p,\Omega}^p\, d\tau\Big)^{1/p} \le T^{1/p'} Z_m
$$

and hence, by (5.19),

$$
\Big(\int_0^t \|Du^{(m)}\|_{p,\Omega}^p\, d\tau\Big)\Big(1 + \int_0^t |Du^{(m)}|_\Omega\, d\tau\Big) \le T^{1/p'} Z_m(1 + C_{10}T^{1/p'} Z_m) \le B(T, Z_m).
$$

Then, it is not difficult to derive from (5.17) that

$$
\begin{aligned}
Z_{m+1} \le C_{30}(T)\ [\ K\ &+\ \Big\{\Big(1 + T^{1/(2p')}Z_m(1 + B(T,Z_m))^2 + T^{1/(-2p')}B(T,Z_m)\Big)(1 + B(T,Z_m))^2\Big\} \\
+\ (1 + B(T,Z_m))\{\ &\varepsilon Z_m(1 + B(T,Z_m)) +\ T^{1/p}(\varepsilon^{-\frac{2p}{p-3}}\|v_0\|_{p,\Omega}^{3\frac{p-1}{p-3}} + \varepsilon^{-\frac{2p}{p-6}}\|v_0\|_{p,\Omega}^{3\frac{p-2}{p-6}}B(T,Z_m)) \\
&+ Z_m^2(T^{\frac{3}{2p'}-\frac{1}{p}} + T^{\frac{3p-8}{2p}}B(T,Z_m))\}\ ]
\end{aligned} \tag{5.23}
$$

$$
=: H(T, Z_m)
$$

holds for any $\varepsilon > 0$, where

$$
K := \|v_0\|_{W_p^{2-2/p}(\Omega)} + (b_m^0(T))T^{1/p}\sup_{t\le T}\|f(x,t)\|_{L^p(\mathbf{R}^3)}.
$$

The right hand side of (5.23) can be written in ascending order of powers of Z_m as

$$
H(T, Z_m) = h_0 + K_1 Z_m + K_2 B(T, Z_m) + H_0(T, Z_m, B(T, Z_m)).
$$

Here,

$h_0(= h_0(T)) = 1 + T^{1/p}\varepsilon^{-\frac{2p}{p-3}}\|v_0\|_{p,\Omega}^\alpha,$
$K_1(= K_1(T)) := T^{1/(2p')} + \varepsilon,$
$K_2(= K_2(T)) := T^{1/(-2p')} + 2 + T^{1/p}(\varepsilon^{-\frac{2p}{p-3}}\|v_0\|_{p,\Omega}^\alpha + \varepsilon^{-\frac{2p}{p-6}}\|v_0\|_{p,\Omega}^\beta),$ and
$H_0(T, Z, B(T, Z))$ is a positive increasing function of both T and Z.

Also, we denote $\alpha := \frac{3(p-1)}{p-3}$, $\beta := \frac{3(p-2)}{p-6}$, which are positive for $p > 6$.

Since $H_0(T,Z,B(T,Z))$ does not have the first order term in Z (It is a polynomial in T with lowest degree $1/p'$ and the highest $17/2$, and in Z with lowest degree 2 and the highest 9), we have the representation of H by splitting $B(T,Z_m)$ in (5.21) into the 2 terms $T^{1/p'}Z_m$ and $C_{10}T^{2/p'}Z_m^2$ as

$$H(T,Z_m) = h_0 + h_1 Z_m + T^{1/(2p')} Z_m H_1(T,Z_m). \tag{5.24}$$

Here,

$$h_1 (= h_1(T)) = K_1 + K_2 T^{1/p'},$$
$$H_1(T,Z) = K_2 C_{10} T^{3/(2p')} Z + T^{-1/(2p')} Z^{-1} H_0(T,Z,B(T,Z)).$$

Take T small enough so that $h_1 < 1$, say,

$$h_1(T) \leq 1/2. \tag{5.25}$$

Now we seek solution Z of the equation $Z = H(T,Z)$. It is easy to see that the equation $Z = H(T,Z)$ has a positive root if the inequality $Z > H(T,Z)$ holds for $Z = h_0\theta$ with some $\theta > 1/(1-h_1)$. In fact, the inequality $h_0\theta > H(T,h_0\theta)$, which is equivalent to

$$(1-h_1)\theta - 1 > T^{1/(2p')}\theta H_1(T,h_0\theta) \tag{5.26}$$

can be seen to have a positive root θ by comparing the graphs of both sides of (5.26) with respect to θ.

First, the condition (5.25) on T is fulfilled if

$$2(T^{1/(2p')} + T^{1/p}) + T(4^{\frac{2p}{p-3}}||v_0||_{p,\Omega}^{\alpha} + 4^{\frac{2p}{p-6}}||v_0||_{p,\Omega}^{\beta}) \leq 1/4. \tag{5.27}$$

Here, we have taken $\varepsilon = 1/4$. Next, for (5.26) it is sufficient for us to impose

$$T^{1/(2p')} 3H_1(T,3h_0) \leq 1/2 \tag{5.28}$$

thanks to the assumption (5.25). Here, we took $\theta = 3$. Under the conditions (5.27) and (5.28) on T, the equation $Z = H(T,Z)$ on Z has one or two positive roots. We denote by z_0 the smaller one. By the form of $H(T,Z)$ ((5.24)) and the choice $\theta = 3$, obviously, $h_0 < z_0 < 3h_0$. Now, suppose $Z_m < z_0$, then $Z_{m+1} \leq H(T,Z_m) \leq H(T,z_0) = z_0$. Thus, we have the boundedness of Z_m as $Z_m < z_0$ for all m.

Acknowledgements.

The author K.I. would like to thank Professor Yoshikazu Giga (The University of Tokyo) and Professor Ben T. Nohara (Automatic Control Laboratory, Musashi Institute of Technology) for their continuous encouragement. He is also grateful to Professor Akio Arimoto (Musashi Institute of Technology), Professor Syuya Kanagawa (Musashi Institute of Technology), Professor Daishi Watabe (Saitama Institute of Technology), Professor Masayuki Hayakawa (Department of Biotechnology, University of Yamanashi), Professor Masahisa Sato (University of Yamanashi), and Professor Takashi Kimura (Department of Information Sciences, Kanagawa University) for kind arrangements for him to get the current part-time teaching positions.

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